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ROBUST GLOBALLY DIVERGENCE-FREE WEAK GALERKIN METHODS FOR STOKES EQUATIONS*

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Abstract

This paper proposes and analyzes a class of robust globally divergence-free weak Galerkin (WG) finite element methods for Stokes equations. The new methods use the P_k/P_{k-1} $(k \geq 1)$ discontinuous finite element combination for velocity and pressure in the interior of elements, and piecewise P_l/P_k $(l = k-1, k)$ for the trace approximations of the velocity and pressure on the inter-element boundaries. Our methods not only yield globally divergence-free velocity solutions, but also have uniform error estimates with respect to the Reynolds number. Numerical experiments are provided to show the robustness of the proposed methods.

Mathematics subject classification: 65M60, 65N30.
Key words: Stokes equations, Weak Galerkin, Globally divergence-free, Uniform error estimates, Local elimination.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a polygonal if $d = 2$ or be a Lipschitz polyhedral domain if $d = 3$. We consider the following Stokes equations: seek the velocity $\boldsymbol{u}$ and the pressure p such that

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\nu = Re^{-1} > 0$ is the fluid viscosity coefficient with Re denoting the Reynolds number, $\boldsymbol{f} \in [L^2(\Omega)]^d$ is the given body force, and the boundary data $\boldsymbol{g} \in [H^{1/2}(\partial\Omega)]^d$ satisfies

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0, \tag{1.2}$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$.

It is well-known that a Galerkin mixed method for (1.1) requires the pair of finite element spaces for the velocity and pressure to satisfy an inf-sup stability condition (see, e.g., [1, 4, 10, 57, 58, 61] and books [12, 31, 33, 41–43]).

Unfortunately, the inf-sup constraint rules out the use of low-order and equal-order elements. In order to circumvent the inf-sup difficulty, many stabilization techniques have been developed to obtain stabilized finite element methods, e.g., Galerkin least-square methods [5, 7, 17, 37], pressure projection methods [8, 14, 18], pressure gradient projection methods [13, 30], and local projection stabilized methods [6, 39, 51].

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Mass conservation is another issue in the numerical solution of incompressible fluid flows. Finite element methods with poor mass conservation, namely not satisfying the incompressibility constraint (at least locally), may lead to undesired instabilities for more complex problems than (1.1) [3, 9, 40, 48, 49, 53].

In literature there are some finite element methods for (1.1) that are inf-sup stable and yield (locally) divergence-free velocity approximations. For stable and divergence-free Stokes elements of conforming $P_k - P_{k-1}$ types (continuous piecewise P_k for velocity and discontinuous piecewise P_{k-1} for pressure), we refer to [56] for a 2D family with any $k \geq 4$ on meshes that do not contain singular vertices, to [2] for a 2D element with $k = 1$ on macro square meshes, to [54] for 2D finite elements with $k = 2, 3$ on macro triangular meshes, and to [68]. In [44] a family of conforming and divergence-free Stokes elements were proposed on general triangular meshes, where the lowest order case consists of enriched piecewise linear polynomials for the velocity and piecewise constant polynomials for the pressure. For stable and locally divergence-free Stokes elements of nonconforming types, we refer to [32] for the Crouzeix-Raviart element method with piecewise constant approximation for the pressure, and to [50, 60, 67] for finite element approximations based on modifying $H(div)$-conforming elements on general triangular/tetrahedral meshes. We also refer to [45, 69, 70] for several stable and (locally) divergence-free Stokes elements on rectangular grids.

In recent years the discontinuous Galerkin (DG) method has become increasingly popular due to its attractive features like local conservation of physical quantities and flexibility in meshing. It has been shown in [25, 62] that DG methods using $H(div)$-conforming elements lead to divergence-free approximations. As pointed out in [35], an inconvenient feature of the DG method is that it may require the penalization parameter to be "sufficiently" large (practically unknown) for stability. This inconvenience was avoided by local discontinuous Galerkin (LDG) methods [15, 16, 24, 29], which have an additional property that fluxes can be eliminated locally, and hybridizable discontinuous Galerkin (HDG) methods [19–23, 26–28, 52], which introduce the numerical trace as an unknown and possess the property of local elimination of unknowns defined in the interior of elements. However, the LDG/HDG methods enforce the incompressibility by using a postprocessing procedure [19, 23, 26–28, 52] or by using element-wise divergence-free spaces for velocity [15, 20, 21]. It has been shown in [26] that globally divergence-free velocity approximations can be obtained when the normal stabilization function τ_n goes to infinity in the HDG methods proposed in [23, 52].

In [65] a family of weak Galerkin (WG) methods were proposed for the Stokes equations, where the P_k/P_{k-1} ($k \geq 1$) discontinuous finite element combination is used for the velocity and pressure, and piecewise P_{k-1} element for the velocity on the interface of the finite element partition. We refer to [66] for another WG scheme for the Stokes model. We note that the velocity approximations in [65, 66] are not (locally) divergence-free. The WG method was first proposed and analyzed to solve second-order elliptic problems [63, 64]. It is designed by using a weakly defined gradient operator over functions with discontinuity, and then allows the use of totally discontinuous functions in the finite element procedure. Similar to the LDG and HDG methods, the WG method is of the property of local elimination of unknowns defined in the interior of elements. The WG method is closely related to the HDG method in the following sense. On one hand, as shown in Remark 2.1 of [47], the WG method for diffusion equations may fall into the HDG framework if introducing the discrete weak gradient as an independent variable. On the other hand, an HDG scheme may be rewritten as a WG scheme by defining some special discrete weak gradient/divergence (cf. (7.9) and (7.10)).