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ANISOTROPIC CROUZEIX-RAVIART TYPE NONCONFORMING FINITE ELEMENT METHODS TO VARIATIONAL INEQUALITY PROBLEM WITH DISPLACEMENT OBSTACLE*

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Abstract

In this paper, anisotropic Crouzeix-Raviart type nonconforming finite element methods are considered for solving the second order variational inequality with displacement obstacle. The convergence analysis is presented and the optimal order error estimates are obtained under the hypothesis of the finite length of the free boundary. Numerical results are provided to illustrate the correctness of theoretical analysis.

Mathematics subject classification: 65N15, 65N30.
Key words: Crouzeix-Raviart type nonconforming finite elements, Anisotropy, Variational inequality, Displacement obstacle, Optimal order error estimates.

1. Introduction

The variational inequality problem with displacement obstacle has been a very interesting subject in many fields, see, e.g., [1,2]. As usual, it reads as: to find $u \in K$, such that

$$a(u, v-u) \geq f(v-u), \quad \forall\, v \in K, \tag{1.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dxdy, \qquad f(v) = \int_\Omega f v dxdy, \tag{1.2a}$$

$$K = \Big\{ \ v \in H_0^1(\Omega): \quad v \geq \chi \ \ a.e. \text{ in } \Omega; \ \ \chi \leq 0 \text{ on } \partial\Omega \ \Big\}, \tag{1.2b}$$

$\Omega \subset R^2$ is bounded convex domain. $f \in L^\infty(\Omega)$ and $\chi \in H^2(\Omega)$ are given functions.

The variational inequality theory was first introduced by Hartman and Stampacchia [3] to study the partial differential equations, and has been playing more and more important role in the contact problem, obstacle problem, elasticity problem, traffic problem, and so on.

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As to the problem shown in (1.1), there have been numerous studies with different finite elements, such as conforming linear triangular element [1,4-6], quadratic element [7], nonconforming Crouzeix-Raviart type linear triangular element and rectangular Wilson element [8-10]. Based on the detailed analysis, the error bound of order $O(h^{3/2-\varepsilon})$, for any $\varepsilon > 0$, was obtained in [4] for the quadratic finite element under the hypothesis of that the free boundary has finite length. Further, [7] derived the same error bound as [4] for the same quadratic element without the above hypothesis. In [8], the Crouzeix-Raviart type nonconforming linear triangular element was used to problem (1.1) and the error bound was estimated with order $O(h)$.

However, to the best knowledge of the authors, all of the above studies on error estimates depend on the essential condition of the discrete meshes, i.e., regular assumption $\frac{h_K}{\rho_K} \leq C$ or quasi-uniform assumption $\frac{h}{h_K} \leq C$, $\forall\ K \in J_h$, where $h_K,\ \rho_K$ denote the diameters of element K and biggest circle contained in K, respectively, $h = \max\limits_{K\in J_h} h_K$, J_h is a subdivision of Ω, C is a positive constant which is independent of h and the function under consideration.

As we know, the domain considered may be narrow or irregular, and the cost of calculation will be very expensive if we employ the regular subdivision on the domain. Naturally, it is an obvious idea to use an anisotropic partition with fewer degrees of freedom for simplicity in the application. But, in this case, some difficulties will arise in the convergence analysis and error estimates of interpolation and consistency errors for nonconforming finite element methods. For example, the Bramble-Hilbert lemma, the traditional interpolation theory in Sobolev spaces, can not be directly applied to the interpolation error estimates for the meshes are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to 0$. On the other hand, when we deal with the consistency error estimate on the longer or the longest edge F of the element K, there will appear a factor $\frac{|F|}{|K|}$, which may tend to infinity and makes the estimate in vain.

In order to overcome the above difficulties, some researches have been devoted to the investigation on the narrow and anisotropic finite elements for the practical problems [11-14]. But there are only a few of articles considering the variational inequality problem with nonconforming finite elements. For example, anisotropic Carey element and Wilson element approximations to the second order obstacle problem were investigated in [15], in which the proofs of the main results are simplified greatly comparing with [8] and [9]. But the techniques used in [15] are only valid to the finite elements when their interpolations can be separated into the conforming part and nonconforming part. Moreover, a class of Crouzeix-Raviart type finite elements were applied to the Signorini variational problem in [16], and [17] extended them to the parabolic variational inequality problem with moving grids.

In [18], a nonconforming rotated Q_1 element was proposed, of which the degrees of freedom are function values of the midpoints of four edges of element K, and the shape function space is spanned by $\{1, x, y, x^2 - y^2\}$. However, it has been proved in [14] that this element can not be applied to anisotropic meshes directly by a counter example. At the same time, [14] also proposed a kind of modified nonconforming finite element with the degrees of freedom of mean-values on the four edges of element K , and the shape function space is spanned by $\{1, x, y, x^2\}$ or $\{1, x, y, y^2\}$, and proved its convergence for the second order problem on a special anisotropic meshes, i.e., the longer edges of all the elements should parallel to x-axis or y-axis, respectively. Obviously, the shape function space of this modification is asymmetrical and the requirement on meshes is too strong.

Recently, there have appeared a lot of studies focusing on the analysis of convergence, supercloseness and supercongvergence for some anisotropic finite element methods (cf. [19-23]). However, the applications of Crouzeix-Raviart type anisotropic nonconforming linear triangular