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EFFICIENT NUMERICAL ALGORITHMS FOR THREE-DIMENSIONAL FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract

This paper detailedly discusses the locally one-dimensional numerical methods for efficiently solving the three-dimensional fractional partial differential equations, including fractional advection diffusion equation and Riesz fractional diffusion equation. The second order finite difference scheme is used to discretize the space fractional derivative and the Crank-Nicolson procedure to the time derivative. We theoretically prove and numerically verify that the presented numerical methods are unconditionally stable and second order convergent in both space and time directions. In particular, for the Riesz fractional diffusion equation, the idea of reducing the splitting error is used to further improve the algorithm, and the unconditional stability and convergency are also strictly proved and numerically verified for the improved scheme.

Mathematics subject classification: 26A33, 65M20.
Key words: Fractional partial differential equations, Numerical stability, Locally one dimensional method, Crank-Nicolson procedure, Alternating direction implicit method.

1. Introduction

The history of fractional calculus can goes back to more than three hundred years ago [12], almost the same as classical calculus. Nowadays it has become more and more popular among various scientific fields, covering anomalous diffusion, materials and mechanical, signal processing and systems identification, control and robotics, rheology, fluid flow, signal processing, and electrical networks et al. [15]. Meanwhile, the diverse fractional partial differential equations (fractional PDEs), as models, appear naturally in the corresponding field.

There are already some important progress for numerically solving the fractional PDEs. The methods used for classical PDEs are well extended to fractional PDEs, for example, the finite difference method [2,18-20,22], finite element method [4,8], and spectral method [14]. However, almost all of them concentrate on one or two dimensional problems. There have been already some useful developments for realizing the operator splitting (locally one dimension) to solve the classical PDEs. This paper focuses on extending the alternating direction implicit (ADI) methods to the three-dimensional fractional PDEs, and improving their efficiency.

The Peaceman and Rachford alternating direction implicit method (PR-ADI) [16] works well for two-dimensional problems. But it can not be extended to higher dimensional problems. Douglas type alternating direction implicit methods (D-ADI) [5-7] are valid for any dimensional

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equations. And PR-ADI and D-ADI are equivalent in two dimensional problems. In this paper, we consider the following three-dimensional fractional advection diffusion equation,

$$\begin{aligned}\frac{\partial u(x,y,z,t)}{\partial t} &= d_1^x \, {}_{x_L}D_x^{\alpha} u(x,y,z,t) + d_2^x \, {}_{x}D_{x_R}^{\alpha} u(x,y,z,t) \\ &\quad + d_1^y \, {}_{y_L}D_y^{\beta} u(x,y,z,t) + d_2^y \, {}_{y}D_{y_R}^{\beta} u(x,y,z,t) \\ &\quad + d_1^z \, {}_{z_L}D_z^{\gamma} u(x,y,z,t) + d_2^z \, {}_{z}D_{z_R}^{\gamma} u(x,y,z,t) + \kappa_x \frac{\partial u(x,y,z,t)}{\partial x} \\ &\quad + \kappa_y \frac{\partial u(x,y,z,t)}{\partial y} + \kappa_z \frac{\partial u(x,y,z,t)}{\partial z} + f(x,y,z,t), \end{aligned} \tag{1.1}$$

and the Riesz fractional diffusion equation

$$\begin{aligned}\frac{\partial u(x,y,z,t)}{\partial t} &= d_1^x \big({}_{x_L}D_x^{\alpha} u(x,y,z,t) + {}_{x}D_{x_R}^{\alpha} u(x,y,z,t) \big) \\ &\quad + d_1^y \big({}_{y_L}D_y^{\beta} u(x,y,z,t) + {}_{y}D_{y_R}^{\beta} u(x,y,z,t) \big) + d_1^z \big({}_{z_L}D_z^{\gamma} u(x,y,z,t) \\ &\quad + {}_{z}D_{z_R}^{\gamma} u(x,y,z,t) \big) + f(x,y,z,t), \end{aligned} \tag{1.1$'$}$$

both with the initial condition

$$u(x,y,z,0) = u_0(x,y,z), \quad \text{for} \quad (x,y,z) \in \Omega, \tag{1.2}$$

and the Dirichlet boundary condition

$$u(x,y,z,t) = 0, \quad \text{for} \quad (x,y,z,t) \in \partial\Omega \times (0,T], \tag{1.3}$$

where $\Omega = (x_L, x_R) \times (y_L, y_R) \times (z_L, z_R) \subset \mathbb{R}^3$, $0 < t \le T$, and the fractional orders $1 < \alpha, \beta, \gamma < 2$; and $f(x,y,z,t)$ is a forcing function; and all the coefficients are non-negative constants. The fractional derivatives used in (1.1) and (1.1′) are defined as, for $1 < \mu < 2$,

$${}_{x_L}D_x^{\mu} u(x) = \frac{1}{\Gamma(2-\mu)} \frac{\partial^2}{\partial x^2} \int_{x_L}^{x} (x-\xi)^{1-\mu} u(\xi) d\xi, \tag{1.4}$$

$${}_{x}D_{x_R}^{\mu} u(x) = \frac{1}{\Gamma(2-\mu)} \frac{\partial^2}{\partial x^2} \int_{x}^{x_R} (\xi - x)^{1-\mu} u(\xi) d\xi. \tag{1.5}$$

From the viewpoint of conversation law, the advection term in the advection diffusion equation should be first order classical derivative, and the fractional derivative corresponding to the diffusion term should be Riemann-Liouville one.

For the two-dimensional case of (1.1)-(1.3), PR-ADI and D-ADI are discussed and we show that they are equivalent for two-dimensional equations. We use D-ADI for the three-dimensional (1.1)-(1.3). The second order finite difference scheme is used to discretize the space fractional derivative and the Crank-Nicolson procedure to the time direction. We theoretically prove and numerically confirm that the given numerical schemes are unconditionally stable and second order convergent in both space and time directions. In general, the ADI methods introduce new error term, called the splitting error, comparing with the original discretizations. Usually the splitting error term does not affect the convergent order, but most of the time it lowers the accuracy seriously. For (1.1′), we use the idea in [7] to reduce the splitting error from $\mathcal{O}(\tau^2)$ to $\mathcal{O}(\tau^3)$ at reasonable computational cost and then recover the accuracy of the original discretization, the improved ADI will be called D-ADI-II. The fractional step (FS) method is also simply discussed to show that, after a minor modification to reduce the splitting error from $\mathcal{O}(\tau)$ to $\mathcal{O}(\tau^3)$, it is equivalent to D-ADI-II.