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Efficient Implementation of a Spectral-Tau Method for a Class of Two-Point Boundary-Value and Initial-Boundary-Value Problems

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Abstract. A simple and efficient spectral method for solving the second, third order and fourth order elliptic equations with variable coefficients and nonlinear differential equations is presented. It is different from spectral-collocation method which leads to dense, ill-conditioned matrices. The spectral method in this paper solves for the coefficients of the solution in a Chebyshev series, leads to discrete systems with special structured matrices which can be factorized and solved efficiently. We also extend the method to boundary value problems in two space dimensions and solve 2-D separable equation with variable coefficients. As an application, we solve Cahn-Hilliard equation iteratively via first-order implicit time discretization scheme. Ample numerical results indicate that the proposed method is extremely accurate and efficient.

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1 Introduction

Spectral methods are now mainstream tools of applied mathematics for solving differential equations (cf. [2,9,21,22,25]). They approximate the numerical solution by global polynomials as the basis functions, provide very accurate approximations with a relatively small number of unknowns, hence, they have gained increasing popularity since

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the 1970s, especially in the field of computational fluid dynamics (see [4,5] and the references therein).

Within spectral methods there is a subdivision between collocation method and coefficient methods which usually include spectral-Galerkin methods and tau methods. Although the collocation method is easy to implement and is convenient for solving problems with variable coefficients and/or nonlinear problems, conventional wisdom states that it leads to dense, ill-conditioned coefficient matrix with a condition number behaving like $\mathrm{O}(N^{2m})$ (m is the order of the differential equation). The spectral-Galerkin methods may lead to well-conditioned linear systems with sparse matrices for problems with constant coefficients by choosing proper basis functions. However, for the general problems with variable coefficients, the Galerkin matrices are usually full, it is not advisable to directly employ the spectral-Galerkin methods, but the Galerkin system for a suitable constant coefficients problems provides an optimal preconditions for solving problems with variable coefficients in [18,21].

Olver and Townsend developed a fast and well-conditioned spectral method for the direct solution of linear ordinary differential equations (ODEs) with variable coefficients recently [16]. The spectral method is generalized to solve the general two-point boundary value problems with the Robin boundary conditions, nonlinear differential equations and higher-order differential equations. We also extend the method to boundary value problems in two space dimensions and solve 2-D separable equation with variable coefficients in this paper.

The spectral method is based on: (1) The unknown function is expanded in the Chebyshev series, and the derivatives are represented in terms of ultraspherical polynomials based on the relation between the Chebyshev polynomials of the first kind and the ultraspherical (or Gegenbauer) polynomials, results in diagonal differentiation matrices. (2) The operators which convert coefficients in the Chebyshev series to those in ultraspherical polynomials are banded. (3) The operators of multiplication for variable coefficients are a Toeplitz matrix plus/minus a Hankel matrix in coefficient space (see [16]). (4) The coefficients of nonlinear terms in the Chebyshev series are computed via the forward/backward discrete Chebyshev transform. (5) For nonlinear equation, we use Newton's method to solve the problem iteratively. (6) We choose a compact combination of Chebyshev polynomials as basis functions which satisfy boundary conditions to deal with boundary condition problems in two-dimension space.

This paper is organized as follows. In the next section, we recall some properties and relations of the Chebyshev polynomials and ultraspherical polynomials, as well as some algorithms. In Section 3, we construct the spectral method for boundary value problems with variable coefficients, as well as nonlinear boundary value problems. Higher-order differential equations are solved by extending the approach in Section 4. In the fifth section, we solve Cahn-Hilliard equation iteratively and extend the method to boundary value problems in two space dimensions. Finally, concluding remarks are presented.