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MODIFIED SPLIT-STEP THETA METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION*

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Abstract

For solving the stochastic differential equations driven by fractional Brownian motion, we present the modified split-step theta method by combining truncated Euler-Maruyama method with split-step theta method. For the problem under a locally Lipschitz condition and a linear growth condition, we analyze the strong convergence and the exponential stability of the proposed method. Moreover, for the stochastic delay differential equations with locally Lipschitz drift condition and globally Lipschitz diffusion condition, we give the order of convergence. Finally, numerical experiments are done to confirm the theoretical conclusions.

Mathematics subject classification: 65L20.
Key words: Stochastic differential equation, Fractional Brownian motion, Split-step theta method, Strong convergence, Exponential stability.

1. Introduction

Recently, stochastic differential equations (SDEs) have been employed to describe many phenomena, such as finance [3], biomedical engineering [2, 12], water resources [16] and so on. Here, we consider the SDE

$$\begin{cases} \mathrm{d}x(t) = \mathscr{X}\big(x(t)\big)\mathrm{d}t + \mathscr{T}\big(x(t)\big)\mathrm{d}B(t), \quad t \in [0, \digamma], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $\mathscr{X}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ and $\mathscr{T}(\cdot) : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ are measurable functions, $B(\cdot)$ is an m-dimensional fractional Brownian motion (fBm) with the Hurst parameter $H \in (1/2, 1)$ and the initial value $x_0 \in \mathbb{R}^m$.

For most SDEs, it is always difficult to give the exact solutions. Thus, many scholars focus their attention on the numerical solutions. In the last decade, great progress has been made in the numerical analysis for SDEs driven by Brownian motion. The split-step theta (SST) scheme in [10] preserved exponential mean square stability for autonomous and non-autonomous

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equations under suitable conditions. The truncated Euler-Maruyama (EM) method in [13] was developed for solving SDEs with locally Lipschitz continuous coefficient. Later on, a series of truncated methods were studied, such as the truncated Milstein method [6], the multilevel Monte Carlo truncated EM method [7], the modified truncated EM method [11] and the full implicit truncated EM method [19].

Compared with the Brownian motion, the lack of independent increments makes it difficult to deal with the fBm. As far as we know, the investigations of numerical methods for SDEs driven by fBm with locally Lipschitz coefficients have produced fewer results than those for SDEs driven by Brownian motion. Recently, [8] applied backward Euler scheme to the CIR problem driven by the fBm and obtained its strong convergence order. In [20], the authors constructed the implicit Euler scheme for the SDEs driven by fBm with locally Lipschitz drift and studied its strong convergence. To our best knowledge, there is few research on the SST method and the truncated EM method for the SDEs with fBm so far. Here, we will combine the SST method with the modified truncated EM method to provide a new modified split-step theta (MSST) method for solving the SDEs the fBm.

We finish this section by presenting its structure of the paper. Section 2 is concerned with some notations on the fBm and the Malliavin derivative, and give some necessary assumptions for the SDE (1.1). In Section 3, we propose the MSST method for this problem and obtain the convergence order. Section 4 analyzes the exponential stability in mean square of the proposed method. Section 5 studies the strong convergence of MSST method for stochastic delay differential equation (SDDE). Finally, the theoretical conclusions are demonstrated by two numerical experiments.

2. Preliminaries

Denote by $(\Omega, \Upsilon, \mathbb{P})$ a complete probability space, $\{\Upsilon_t\}_{t\geq 0}$ is increasing and continuous, $\{\Upsilon_0\}$ contains all $\mathbb{P}$-null sets. Unless otherwise specified, we always use the symbols below. The fBm has the continuous correction (see [1]), that is, for $n \geq 1$,

$$\mathbb{E}|B(w_1) - B(w_2)|^n = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)|w_1 - w_2|^{nH}, \quad \forall w_1, w_2 \in [0, \digamma]. \tag{2.1}$$

Let $L^2_\phi([0,\digamma])$ be an Hilbert space,

$$\phi(w_1, w_2) := H(2H-1)|w_1 - w_2|^{2H-2}$$

and $u : [0,\digamma] \to \mathbb{R}$ be a measurable function. Denote the random variable $\mathcal{F} : \Omega \to \mathbb{R}$ as $\mathcal{F} = \mathcal{K}(B(t_1), B(t_2), \cdots, B(t_n))$, where $\mathcal{K}$ is a smooth function with all bounded derivatives and $0 = t_0 < t_1 < \cdots < t_n = \digamma$. Define the Malliavin derivative (see [17])

$$\mathcal{D}_\varpi \mathcal{F} := \sum_{i=1}^{n} \frac{\partial \mathcal{K}}{\partial x_i}\big(B(t_1), B(t_2), \cdots, B(t_n)\big)1_{[0,t_i]}(\varpi), \quad \varpi \in [0,\digamma].$$

The space $\mathbb{D}^{1,p}$ is the completion of the set of all nonlinear functionals with

$$\|\mathcal{F}\|_{\mathbb{D}^{1,p}} := \big(\mathbb{E}[|\mathcal{F}|^p] + \mathbb{E}[\|\mathcal{D}\mathcal{F}\|^p_\phi]\big)^{\frac{1}{p}}.$$

Denote by δ the adjoint operator of derivative operator $\mathcal{D}$. If there is $\delta(\mathcal{G}) \in L^2_\phi([0,\digamma])$ such that $\mathbb{E}\left[\mathcal{F}\delta(\mathcal{G})\right] = \mathbb{E}\left[\langle \mathcal{G}, \mathcal{D}\mathcal{F}\rangle_\phi\right]$ for any $\mathcal{F} \in \mathbb{D}^{1,2}$, then $\mathcal{G}$ is integrable. Define the Skorohod integral

$$\int_0^F \mathcal{G}(t)\delta B(t) := \delta(\mathcal{G}),$$

which satisfies

$$\mathbb{E}\left(\int_0^F \mathcal{G}(t)\delta B(t)\right) = 0.$$

Denote by $\int_0^F G(t)\mathrm{d}B(t)$ the pathwise integral, then

$$\int_0^F \mathcal{G}(t)\mathrm{d}B(t) = \int_0^F \mathcal{G}(t)\delta B(t) + \int_0^F \int_0^F \phi(t,s)\mathcal{D}_s\mathcal{G}(t)\mathrm{d}t\,\mathrm{d}s.$$

Finally, for later analysis, we assume that the functions $\mathscr{X}(x)$ and $\mathscr{T}(x)$ in (1.1) satisfy locally Lipschitz continuous condition and linear growth condition.

Assumption 2.1. *For any positive constant Λ and $x, \mathring{x} \in \mathbb{R}^m$ with $|x| \vee |\mathring{x}| \le \Lambda$, there is a positive function $L(\Lambda)$ such that*

$$|\mathscr{X}(x) - \mathscr{X}(\mathring{x})| \vee |\mathscr{T}(x) - \mathscr{T}(\mathring{x})| \le L(\Lambda)|x - \mathring{x}|.$$

Assumption 2.2. *For any $x, \mathring{x} \in \mathbb{R}^m$, there is a positive constant K such that*

$$\langle x - \mathring{x}, \mathscr{X}(x) - \mathscr{X}(\mathring{x})\rangle \le K(1 + |x - \mathring{x}|^2), \tag{2.2}$$

$$|\mathscr{T}(x) - \mathscr{T}(\mathring{x})|^2 \le K(1 + |x - \mathring{x}|^2). \tag{2.3}$$

Remark 2.1. For any $x \in \mathbb{R}^m$, there is a positive constant K depending on $\mathscr{X}(0)$ and $\mathscr{T}(0)$ such that $\langle x, \mathscr{X}(x)\rangle + |\mathscr{T}(x)|^2 \le K(1 + |x|^2)$.

In view of (2.2) and (2.3), we can obtain

$$\langle x, \mathscr{X}(x)\rangle = \langle x, \mathscr{X}(x) - \mathscr{X}(0)\rangle + \langle x, \mathscr{X}(0)\rangle \le K + |\mathscr{X}(0)|^2 + \left(K + \frac{1}{2}\right)|x|^2,$$

$$|\mathscr{T}(x)|^2 \le 2|\mathscr{T}(x) - \mathscr{T}(0)|^2 + 2|\mathscr{T}(0)|^2 \le 2K + 2|\mathscr{T}(0)|^2 + 2K|x|^2.$$

It is worth noticing that, when the coefficients in (1.1) satisfy the weaker assumptions than Assumptions 2.1 and 2.2, the existence and boundedness of its exact solutions have been proved in [4].

3. Strong Convergence

Firstly, we introduce the modified truncation functions in [11]. Let Δt^* be a positive number which is sufficiently small, $\omega(\Delta t) : (0, \Delta t^*] \to (0, \infty)$ be a strictly decreasing function such that $L(\omega(\Delta t)) \ge 1$ and

$$\lim_{\Delta t \to 0} \omega(\Delta t) = \infty, \quad \lim_{\Delta t \to 0} L^4\big(\omega(\Delta t)\big) \cdot \Delta t = 0.$$

To simplify the notations, we write $\omega(\Delta t)$ as ω_Δ. Define the modified truncation functions

$$\mathscr{X}_\Delta(x) = \begin{cases} \mathscr{X}(x), & |x| \le \omega_\Delta, \\ \dfrac{|x|}{\omega_\Delta}\mathscr{X}\left(\omega_\Delta \dfrac{x}{|x|}\right), & |x| > \omega_\Delta, \end{cases} \tag{3.1}$$

and

$$\mathscr{T}_\Delta(x)=\begin{cases}\mathscr{T}(x), & |x|\le\omega_\Delta,\\ \dfrac{|x|}{\omega_\Delta}\mathscr{T}\left(\omega_\Delta\dfrac{x}{|x|}\right), & |x|>\omega_\Delta.\end{cases} \tag{3.2}$$

Due to [11, Remark 2.1], the way of defining the modified truncation functions is reasonable, because such barrier functions always exist. It is worth noting that the above modified truncation functions are globally Lipschitz continuous. Moreover, for any $\Delta t\in(0,\Delta t^*]$, the modified truncation functions satisfy the Assumption 2.2 and Remark 2.1.

Lemma 3.1. *Under Assumption* 2.2, *for any* $x,\mathring{x}\in\mathbb{R}^m$, *there exists a positive constant* $\hat{K}$ *such that*

$$\langle x-\mathring{x},\mathscr{X}_\Delta(x)-\mathscr{X}_\Delta(\mathring{x})\rangle\le\hat{K}(1+|x-\mathring{x}|^2), \tag{3.3}$$

$$|\mathscr{T}_\Delta(x)-\mathscr{T}_\Delta(\mathring{x})|^2\le\hat{K}(1+|x-\mathring{x}|^2), \tag{3.4}$$

$$\langle x,\mathscr{X}_\Delta(x)\rangle+|\mathscr{T}_\Delta(x)|^2\le\hat{K}(1+|x|^2). \tag{3.5}$$

Proof. Since the truncation functions are globally Lipschitz continuous, the results (3.3) and (3.4) are straightforward. Thus, we only need to prove (3.5).

If $|x|\le\omega_\Delta$, then $\mathscr{X}_\Delta(x)=\mathscr{X}(x)$ and $\mathscr{T}_\Delta(x)=\mathscr{T}(x)$. Then (3.5) holds.

When $|x|>\omega_\Delta$, according to (3.1) and (3.2), we know

$$\mathscr{X}_\Delta(x)=\frac{|x|}{\omega_\Delta}\mathscr{X}\left(\omega_\Delta\frac{x}{|x|}\right),\quad \mathscr{T}_\Delta(x)=\frac{|x|}{\omega_\Delta}\mathscr{T}\left(\omega_\Delta\frac{x}{|x|}\right).$$

Under Assumption 2.2, we deduce that

$$\begin{aligned}\langle x,\mathscr{X}_\Delta(x)\rangle+|\mathscr{T}_\Delta(x)|^2&=\frac{|x|^2}{\omega_\Delta^2}\omega_\Delta\left\langle\frac{x}{|x|},\mathscr{X}\left(\omega_\Delta\frac{x}{|x|}\right)\right\rangle+\frac{|x|^2}{\omega_\Delta^2}\left|\mathscr{T}\left(\omega_\Delta\frac{x}{|x|}\right)\right|^2\\&\le\frac{|x|^2}{\omega_\Delta^2}K\left(1+\omega_\Delta^2\frac{|x|^2}{|x|^2}\right)\le\hat{K}(1+|x|^2).\end{aligned}$$

Thus, we complete the proof. □

Next, by replacing the coefficients $\mathscr{X}(x)$ and $\mathscr{T}(x)$ with the modified truncation functions $\mathscr{X}_\Delta(x)$ and $\mathscr{T}_\Delta(x)$, we propose MSST method for the SDE (1.1). Let $t_n=n\Delta t$ with $\Delta t=\digamma/N$ and $n=0,1,\dots,N$. Denote the numerical solution of (1.1) by $\mu_n\approx x(t_n)$, then the MSST method can be written as

$$\mathcal{U}_n=\mu_n+\theta\Delta t\mathscr{X}_\Delta(\mathcal{U}_n), \tag{3.6}$$

$$\mu_{n+1}=\mu_n+\Delta t\mathscr{X}_\Delta(\mathcal{U}_n)+\mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n, \tag{3.7}$$

where $\theta\in[1/2,1]$, $\mu_0=x_0$ and $\Delta B_n=B(t_{n+1})-B(t_n)$.

To analyze the convergence of MSST method, we first consider the bounds of μ_n and $\mathcal{U}_n$. In the following, we use C to stand for all the finite positive constants independent of Δt, note that they may be different, even though they appear in the same line.

Lemma 3.2. *Under the Assumptions* 2.1 *and* 2.2, *for any* $0<\Delta t\le\Delta t^*\le 1/(2\theta\hat{K})$ *with* $\theta\in[1/2,1]$ *and* $p\ge 1/H$, *it holds*

$$\mathbb{E}\Big[\sup_{0<\Delta t\le\Delta t^*}\sup_{0\le n\le N}|\mathcal{U}_n|^{2p}\Big]\vee\mathbb{E}\Big[\sup_{0<\Delta t\le\Delta t^*}\sup_{0\le n\le N}|\mu_n|^{2p}\Big]\le C.$$

Proof. From (3.6), due to Lemma 3.1, we can derive

$$\begin{aligned}|\mathcal{U}_n|^2 &= |\mu_n|^2 + \theta^2\Delta t^2 \left|\mathscr{Z}_\Delta(\mathcal{U}_n)\right|^2 + 2\theta\Delta t \left\langle \mathcal{U}_n - \theta\Delta t \mathscr{Z}_\Delta(\mathcal{U}_n), \mathscr{Z}_\Delta(\mathcal{U}_n)\right\rangle \\ &\le |\mu_n|^2 + 2\theta\Delta t \left\langle \mathcal{U}_n, \mathscr{Z}_\Delta(\mathcal{U}_n)\right\rangle \\ &\le |\mu_n|^2 + 2\theta\Delta t\hat{K}(1+|\mathcal{U}_n|^2).\end{aligned}$$

Let $\alpha = 1/(1-2\theta\Delta t\hat{K})$ and $\beta = 2\theta\Delta t\hat{K}/(1-2\theta\Delta t\hat{K})$, then

$$|\mathcal{U}_n|^2 \le \alpha\,|\mu_n|^2 + \beta. \tag{3.8}$$

From (3.7), we get

$$\begin{aligned}|\mu_{n+1}|^2 &= |\mu_n|^2 + \Delta t^2|\mathscr{Z}_\Delta(\mathcal{U}_n)|^2 + |\mathscr{T}_\Delta(\mathcal{U}_n)|^2\cdot|\Delta B_n|^2 + 2\left\langle \mu_n, \Delta t\mathscr{Z}_\Delta(\mathcal{U}_n)\right\rangle \\ &\quad + 2\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle + 2\left\langle \Delta t\mathscr{Z}_\Delta(\mathcal{U}_n), \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle.\end{aligned}$$

With the help of (3.6), we have

$$\begin{aligned}|\mu_{n+1}|^2 &\le |\mu_n|^2 + \Delta t^2|\mathscr{Z}_\Delta(\mathcal{U}_n)|^2 + |\mathscr{T}_\Delta(\mathcal{U}_n)|^2\cdot|\Delta B_n|^2 + 2\left\langle \mathcal{U}_n - \theta\Delta t\mathscr{Z}_\Delta(\mathcal{U}_n), \Delta t\mathscr{Z}_\Delta(\mathcal{U}_n)\right\rangle \\ &\quad + 2\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle + 2\left\langle \frac{\mathcal{U}_n-\mu_n}{\theta}, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle \\ &\le |\mu_n|^2 + |\mathscr{T}_\Delta(\mathcal{U}_n)|^2\cdot|\Delta B_n|^2 + 2\left\langle \mathcal{U}_n, \Delta t\mathscr{Z}_\Delta(\mathcal{U}_n)\right\rangle + 2\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle \\ &\quad + \frac{2}{\theta}\left\langle \mathcal{U}_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle - \frac{2}{\theta}\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle.\end{aligned}$$

By using Lemma 3.1 and (3.8), we arrive at

$$\begin{aligned}|\mu_{n+1}|^2 &\le |\mu_n|^2 + |\mathscr{T}_\Delta(\mathcal{U}_n)|^2\cdot|\Delta B_n|^2 + 2\hat{K}\Delta t(1+\alpha|\mu_n|^2+\beta) \\ &\quad + 2\left(1-\frac{1}{\theta}\right)\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle + \frac{2}{\theta}\left\langle \mathcal{U}_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle \\ &= |\mu_n|^2 + 2\alpha\hat{K}\Delta t|\mu_n|^2 + 2(1+\beta)\hat{K}\Delta t + |\mathscr{T}_\Delta(\mathcal{U}_n)|^2\cdot|\Delta B_n|^2 \\ &\quad + 2\left(1-\frac{1}{\theta}\right)\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle + \frac{2}{\theta}\left\langle \mathcal{U}_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle.\end{aligned}$$

After simple calculations, we derive

$$\begin{aligned}|\mu_{n+1}|^2 &\le |\mu_0|^2 + 2\alpha\hat{K}\Delta t\sum_{\ell=0}^{n}|\mu_\ell|^2 + 2(1+\beta)\hat{K}\digamma + \sum_{\ell=0}^{n}|\mathscr{T}_\Delta(\mathcal{U}_\ell)|^2\cdot|\Delta B_\ell|^2 \\ &\quad + 2\left(1-\frac{1}{\theta}\right)\sum_{\ell=0}^{n}\left\langle \mu_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\right\rangle + \frac{2}{\theta}\sum_{\ell=0}^{n}\left\langle \mathcal{U}_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\right\rangle.\end{aligned} \tag{3.9}$$

Taking the p-th power on both sides of (3.9), we have

$$\begin{aligned}|\mu_n|^{2p} \le 6^{p-1}\Bigg\{ &|\mu_0|^{2p} + (2\alpha\hat{K}\Delta t)^p n^{p-1}\sum_{\ell=0}^{n-1}|\mu_\ell|^{2p} + \big(2(1+\beta)\hat{K}\digamma\big)^p \\ &+ n^{p-1}\sum_{\ell=0}^{n-1}|\mathscr{T}_\Delta(\mathcal{U}_\ell)|^{2p}|\Delta B_\ell|^{2p} + 2^p\left(\frac{1}{\theta}-1\right)^p\left|\sum_{\ell=0}^{n-1}\left\langle \mu_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\right\rangle\right|^p \\ &+ \left(\frac{2}{\theta}\right)^p\left|\sum_{\ell=0}^{n-1}\left\langle \mathcal{U}_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\right\rangle\right|^p\Bigg\}.\end{aligned}$$

For $0 \leq n \leq \kappa$ with κ is an integer and $\kappa \in [0, N)$, we get

$$\begin{aligned}
\mathbb{E}\Big[\max_{0\leq n\leq\kappa} |\mu_n|^{2p}\Big] \leq 6^{p-1}\Bigg\{ & |\mu_0|^{2p} + (2\alpha\hat{K}\Delta t)^p \kappa^{p-1} \sum_{\ell=0}^{\kappa-1} \mathbb{E}|\mu_\ell|^{2p} + (2(1+\beta)\hat{K}F)^p \\
& + \kappa^{p-1} \sum_{\ell=0}^{\kappa-1} \mathbb{E}\left[|\mathscr{T}_\Delta(\mathcal{U}_\ell)|^{2p} \cdot |\Delta B_\ell|^{2p}\right] \\
& + 2^p \left(1 - \frac{1}{\theta}\right)^p \mathbb{E}\left[\max_{0\leq n\leq\kappa} \left| \sum_{\ell=0}^{n-1} \langle \mu_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\rangle \right|^p\right] \\
& + \left(\frac{2}{\theta}\right)^p \mathbb{E}\left[\max_{0\leq n\leq\kappa} \left| \sum_{\ell=0}^{n-1} \langle \mathcal{U}_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\rangle \right|^p\right] \Bigg\}.
\end{aligned} \tag{3.10}$$

Due to the Hölder inequality, (2.1) and (3.8), we obtain

$$\begin{aligned}
& 6^{p-1}\kappa^{p-1} \sum_{\ell=0}^{\kappa-1} \mathbb{E}\left[|\mathscr{T}_\Delta(\mathcal{U}_\ell)|^{2p} \cdot |\Delta B_\ell|^{2p}\right] \\
\leq\ & C\kappa^{p-1}\hat{K}^p \Delta t^{2pH} \sum_{\ell=0}^{\kappa-1} \left((1+\beta)^{2p} + \alpha^{2p}\mathbb{E}|\mu_\ell|^{4p}\right).
\end{aligned} \tag{3.11}$$

Using a similar way to consider other items of (3.10), we arrive at

$$\begin{aligned}
& \mathbb{E}\left[\max_{0\leq n\leq\kappa} \left| \sum_{\ell=0}^{n-1} \langle \mu_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\rangle \right|^p\right] \\
\leq\ & C\hat{K}^{\frac{p}{2}} \Delta t^{pH} \sum_{\ell=0}^{\kappa-1} \left((1+\beta)^{2p} + (1+\alpha^{2p})\mathbb{E}|\mu_\ell|^{4p}\right),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
& \mathbb{E}\left[\max_{0\leq n\leq\kappa} \left| \sum_{\ell=0}^{n-1} \langle \mathcal{U}_\ell, \mathscr{T}_\Delta(\mathcal{U}_\ell)\Delta B_\ell\rangle \right|^p\right] \\
\leq\ & C\hat{K}^{\frac{p}{2}} \Delta t^{pH} \sum_{\ell=0}^{\kappa-1} \left((1+\beta)^{2p} + \beta^{2p} + 2\alpha^{2p}\mathbb{E}|\mu_\ell|^{4p}\right).
\end{aligned} \tag{3.13}$$

Substituting (3.11)-(3.13) into (3.10), we derive

$$\begin{aligned}
\mathbb{E}\Big[\max_{0\leq n\leq\kappa} |\mu_n|^{2p}\Big] \leq\ & 6^{p-1}|\mu_0|^{2p} + C\big((1+\beta)\hat{K}F\big)^p + C\kappa^{p-1}(1+\beta)^{2p}\hat{K}^p F^{2pH-1} \\
& + C\left(\frac{1}{\theta} - 1\right)^p (1+\beta)^{2p}\hat{K}^{\frac{p}{2}} F^{pH-1} \\
& + C\theta^{-p}\big((1+\beta)^{2p} + \beta^{2p}\big)\hat{K}^{\frac{p}{2}} F^{pH-1} \\
& + C(\alpha\hat{K}\Delta t)^p \sum_{\ell=0}^{\kappa-1} \mathbb{E}|\mu_\ell|^{2p} + C\alpha^{2p}\hat{K}^p\Delta t^{2pH} \sum_{\ell=0}^{\kappa-1} \mathbb{E}|\mu_\ell|^{4p} \\
& + C\left(\frac{1}{\theta} - 1\right)^p (1+\alpha^{2p})\hat{K}^{\frac{p}{2}}\Delta t^{pH} \sum_{\ell=0}^{\kappa-1} \mathbb{E}|\mu_\ell|^{4p} \\
& + C\theta^{-p}\alpha^{2p}\hat{K}^{\frac{p}{2}}\Delta t^{pH} \sum_{\ell=0}^{\kappa-1} \mathbb{E}|\mu_\ell|^{4p}
\end{aligned}$$

$$
\begin{aligned}
&\le C + (2\alpha\hat{K}\Delta t)^p \sum_{\ell=0}^{\kappa-1} \mathbb{E}\Big[\max_{0\le i\le \ell} |\mu_i|^{2p}\Big] + C\alpha^{2p}\hat{K}^p\Delta t^{2pH} \sum_{\ell=0}^{\kappa-1} \mathbb{E}\Big[\max_{0\le i\le \ell} |\mu_i|^{4p}\Big] \\
&\quad + C\left(\frac{1}{\theta}-1\right)^p (1+\alpha^{2p})\hat{K}^{\frac{p}{2}}\Delta t^{pH} \sum_{\ell=0}^{\kappa-1} \mathbb{E}\Big[\max_{0\le i\le \ell} |\mu_i|^{4p}\Big] \\
&\quad + C\theta^{-p}\alpha^{2p}\hat{K}^{\frac{p}{2}}\Delta t^{pH} \sum_{\ell=0}^{\kappa-1} \mathbb{E}\Big[\max_{0\le i\le \ell} |\mu_i|^{4p}\Big].
\end{aligned}
$$

By applying the property of expectation and the Willett-Wong inequality (see [18, Theorem 2.3.11]), there exists a positive constant C such that

$$\mathbb{E}\Big[\sup_{0\le \Delta t\le \Delta t^*} \sup_{0\le n\le N} |\mu_n|^{2p}\Big] \le C.$$

Furthermore, by combining with (3.8), we know that $\mathbb{E}[\sup_{0\le\Delta t\le\Delta t^*}\sup_{0\le n\le N}|\mathcal{U}_n|^{2p}]$ is also bounded, which completes the proof. ☐

It is noted that the stochastic integral is a useful tool to simplify the way of dealing with fBm. To analyze the error of the numerical solution, we introduce the continuous version of $\mathcal{U}_n$ and μ_n. Set

$$\mathcal{U}(t) = \sum_{n=0}^{N-1} \mathbf{1}_{\{t_n\le t<t_{n+1}\}}\mathcal{U}_n + \mathbf{1}_{\{t=t_N\}}\mathcal{U}_N, \qquad t\in[0,\mathcal{F}], \tag{3.14}$$

$$\bar{\mu}(t) := \mu_n + (t-t_n)\mathscr{Z}_\Delta(\mathcal{U}_n) + \mathscr{T}_\Delta(\mathcal{U}_n)\big(B(t)-B(t_n)\big), \quad t\in[t_n,t_{n+1}), \tag{3.15}$$

which can be transformed into the integral form

$$\bar{\mu}(t) = \mu_0 + \int_0^t \mathscr{Z}_\Delta\big(\mathcal{U}(s)\big)\mathrm{d}s + \int_0^t \mathscr{T}_\Delta\big(\mathcal{U}(s)\big)\mathrm{d}B(s), \quad t\in[t_n,t_{n+1}). \tag{3.16}$$

Lemma 3.3. *Under Assumptions* 2.1 *and* 2.2*, it holds*

$$\sup_{0\le t\le \mathcal{F}} \mathbb{E}\left[|\bar{\mu}(t)-\mathcal{U}(t)|^2\right] \le CL^2(\omega_\Delta)\Delta t^{2H}, \quad \forall \Delta t\in(0,\Delta t^*]. \tag{3.17}$$

Proof. For any fixed $t\in(0,\mathcal{F}]$, denote by $t_{n(t)}$ the time node such that $t\in(t_{n(t)},t_{n(t)+1}]$, where $n(t)$ is a positive integer and $n(t)+1\le N$. From (3.14)-(3.15), we get

$$\bar{\mu}(t)-\mathcal{U}(t) = (t-t_{n(t)})\mathscr{Z}_\Delta(\mathcal{U}_{n(t)}) + \mathscr{T}_\Delta(\mathcal{U}_{n(t)})\big(B(t)-B(t_{n(t)})\big).$$

With the help of (2.1), [11, Lemma 2.2], Cauchy inequality and Jensen inequality on expectation, we can obtain

$$
\begin{aligned}
\mathbb{E}\left[|\bar{\mu}(t)-\mathcal{U}(t)|^2\right] &\le 2\Delta t^2\mathbb{E}|\mathscr{Z}_\Delta(\mathcal{U}_{n(t)})|^2 + 2\mathbb{E}\left(|\mathscr{T}_\Delta(\mathcal{U}_{n(t)})|^2\cdot|B(t)-B(t_{n(t)})|^2\right) \\
&\le 2\Delta t^2\mathbb{E}\left[L(\omega_\Delta)|\mathcal{U}_{n(t)}|+\mathscr{Z}(0)\right]^2 + 2\Delta t^{2H}\sqrt{\mathbb{E}\left[L(\omega_\Delta)|\mathcal{U}_{n(t)}|+\mathscr{T}(0)\right]^4} \\
&\le 2L^2(\omega_\Delta)\Delta t^2\left[\mathbb{E}|\mathcal{U}_{n(t)}|^2+|\mathscr{Z}(0)|^2\right] + 2L^2(\omega_\Delta)\Delta t^{2H}\sqrt{\mathbb{E}|\mathcal{U}_{n(t)}|^4+|\mathscr{T}(0)|^4}.
\end{aligned}
$$

Here, we get the above formula based on the fact that $L(\omega_\Delta)\ge 1$. Furthermore, we get

$$
\begin{aligned}
\sup_{0\le t\le\mathcal{F}} \mathbb{E}\left[|\bar{\mu}(t)-\mathcal{U}(t)|^2\right] &\le 2L^2(\omega_\Delta)\Delta t^2\Big[\sup_{0\le n(t)\le N}\mathbb{E}|\mathcal{U}_{n(t)}|^2+|\mathscr{Z}(0)|^2\Big] \\
&\quad + 2L^2(\omega_\Delta)\Delta t^{2H}\sqrt{\sup_{0\le n(t)\le N}\mathbb{E}|\mathcal{U}_{n(t)}|^4+|\mathscr{T}(0)|^4}.
\end{aligned}
$$

Combining with the boundness of $\mathbb{E}[\sup_{0\le n\le N}|\mathcal{U}_n|^{2p}]$, we can derive (3.17). ☐

Lemma 3.4. *Under Assumptions* 2.1 *and* 2.2, *we have*

$$\mathbb{E}\Big[\sup_{0\le t\le F}|\bar{\mu}(t)|^2\Big]\le C. \tag{3.18}$$

Proof. By using the notation $n(t)$ in the proof of Lemma 3.3, from (3.6), we get

$$\begin{aligned}\bar{\mu}(t)&=\mu_{n(t)}+(t-t_{n(t)})\mathscr{X}_\Delta(\mathcal{U}_{n(t)})+\mathscr{T}_\Delta(\mathcal{U}_{n(t)})\big(B(t)-B(t_{n(t)})\big)\\&=\mu_{n(t)}+(t-t_{n(t)})\frac{\mathcal{U}_{n(t)}-\mu_{n(t)}}{\theta\Delta t}+\mathscr{T}_\Delta(\mathcal{U}_{n(t)})\big(B(t)-B(t_{n(t)})\big)\\&=\left(1-\frac{t-t_{n(t)}}{\theta\Delta t}\right)\mu_{n(t)}+\frac{t-t_{n(t)}}{\theta\Delta t}\mathcal{U}_{n(t)}+\mathscr{T}_\Delta(\mathcal{U}_{n(t)})\big(B(t)-B(t_{n(t)})\big).\end{aligned}\tag{3.19}$$

Applying the supremum and taking expectation on two sides of (3.19), we arrive at

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le F}|\bar{\mu}(t)|^2\Big]\le 3\bigg\{\left(1+\frac{1}{\theta}\right)^2\mathbb{E}\Big[\sup_{0\le n(t)\le N}|\mu_{n(t)}|^2\Big]+\frac{1}{\theta^2}\mathbb{E}\Big[\sup_{0\le n(t)\le N}|\mathcal{U}_{n(t)}|^2\Big]\\+\mathbb{E}\Big[\sup_{0\le n(t)\le N}|\mathscr{T}_\Delta(\mathcal{U}_{n(t)})\Delta B_{n(t)}|^2\Big]\bigg\}.\end{aligned}\tag{3.20}$$

Here

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le n(t)\le N}|\mathscr{T}_\Delta(\mathcal{U}_{n(t)})\Delta B_{n(t)}|^2\Big]\\\le\,&\mathbb{E}\Big[\sup_{0\le n(t)\le N}|\mathscr{T}_\Delta(\mathcal{U}_{n(t)})|^2\cdot|B(t)-B(t_{n(t)})|^2\Big]\\\le\,&\mathbb{E}\Big[\sup_{0\le n(t)\le N}\hat{K}(1+|\mathcal{U}_{n(t)}|^2)|B(t)-B(t_{n(t)})|^2\Big]\\\le\,&\sum_{j=1}^N\hat{K}\Big(1+\sup_{0\le j\le N}\mathbb{E}|\mathcal{U}_j|^4\Big)\Delta t^{2H}\le C.\end{aligned}$$

Recalling the conclusion in Lemma 3.2, the boundedness of all items on the right-hand of (3.20) guarantees the desired result (3.18). □

To analyze the convergence of the MSST method, we introduce

$$\begin{aligned}&\varphi_\Lambda:=\inf\{t\ge 0:|x(t)|\ge\Lambda\},\\&\varrho_\Lambda:=\inf\{t\ge 0:|\bar{\mu}(t)|\ge\Lambda\text{ or }|\mathcal{U}(t)|\ge\Lambda\},\\&\zeta_\Lambda:=\varphi_\Lambda\wedge\varrho_\Lambda,\quad r(t):=\bar{\mu}(t)-x(t).\end{aligned}$$

Theorem 3.1. *Under the Assumptions* 2.1 *and* 2.2, *for any* $q\ge 2$, $\Delta t\in(0,\Delta t^*]$ *and* $\Lambda\le\omega_\Delta$, *we have*

$$\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q\le CL^{2q}(\omega_\Delta)\Delta t^{qH}. \tag{3.21}$$

Proof. Use the Itô formula (see [15, p. 184]) first, then we apply the relationship between the pathwise integral and Skorohod integral with respect to fBm. For $\zeta<t\wedge\zeta_\Lambda$, it holds

$$\begin{aligned}\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q=q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}&|r(s)|^{q-2}\big\langle r(s),\mathscr{X}_\Delta\big(\mathcal{U}(s)\big)-\mathscr{X}\big(x(s)\big)\big\rangle\mathrm{d}s\\+q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}&|r(s)|^{q-2}\big\langle r(s),\mathscr{T}_\Delta\big(\mathcal{U}(s)\big)-\mathscr{T}\big(x(s)\big)\big\rangle\delta B(s)\\+q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}&\int_0^{t\wedge\zeta_\Lambda}\mathcal{D}_\zeta\Big\{|r(s)|^{q-2}\big\langle r(s),\mathscr{T}_\Delta\big(\mathcal{U}(s)\big)-\mathscr{T}\big(x(s)\big)\big\rangle\Big\}\phi(\zeta,s)\mathrm{d}\zeta\,\mathrm{d}s.\end{aligned}$$

Since the expectation of Skorohod integral equals to zero, then

$$\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q = q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\big\langle r(s),\mathscr{Z}_\Delta\big(\mathcal{U}(s)\big)-\mathscr{Z}\big(x(s)\big)\big\rangle\mathrm{d}s$$
$$+q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}\int_0^{t\wedge\zeta_\Lambda}\mathcal{D}_\zeta\Big\{|r(s)|^{q-2}\big\langle r(s),\mathscr{T}_\Delta\big(\mathcal{U}(s)\big)-\mathscr{T}\big(x(s)\big)\big\rangle\Big\}\phi(\zeta,s)\mathrm{d}\zeta\,\mathrm{d}s.$$

According to the definition of the modified truncation function, it is obvious that when $0\le s\le t\wedge\zeta_\Lambda$, we have $|\mathcal{U}(t)|\le\Lambda\le\omega_\Delta$. To deal with the Malliavin derivative, it is necessary to impose the coercive assumption similar to [9]. Here, for any continuous function $\varrho(x)$, we suppose there exist two constants $\alpha_1>0$, $\beta_1>0$ such that $|\mathcal{D}_\zeta\varrho(x)|\le\alpha_1|\varrho(x)|+\beta_1$. In view of $r(s)=\bar{\mu}(s)-\mathcal{U}(s)+\mathcal{U}(s)-x(s)$, we derive

$$\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q \le q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\big\langle \mathcal{U}(s)-x(s),\mathscr{Z}\big(\mathcal{U}(s)\big)-\mathscr{Z}\big(x(s)\big)\big\rangle\mathrm{d}s$$
$$+q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\big\langle \bar{\mu}(s)-\mathcal{U}(s),\mathscr{Z}\big(\mathcal{U}(s)\big)-\mathscr{Z}\big(x(s)\big)\big\rangle\mathrm{d}s+q\beta_1 F^{2H}$$
$$+\alpha_1 q\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\big\langle \bar{\mu}(s)-x(s),\mathscr{T}\big(\mathcal{U}(s)\big)-\mathscr{T}\big(x(s)\big)\big\rangle\phi(\zeta,s)\mathrm{d}\zeta\,\mathrm{d}s.$$

Due to Assumption 2.2 and Cauchy inequality, we can arrive at

$$\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q \le qK\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\big(1+|\mathcal{U}(s)-x(s)|^2\big)\mathrm{d}s$$
$$+4qL(\omega_\Delta)\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}|\mathcal{U}(s)-x(s)|\cdot|\bar{\mu}(s)-\mathcal{U}(s)|\mathrm{d}s$$
$$+q\alpha_1\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q}\phi(\zeta,s)\mathrm{d}\zeta\,\mathrm{d}s$$
$$+q\alpha_1 K\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\big(1+|\mathcal{U}(s)-x(s)|^2\big)\phi(\zeta,s)\mathrm{d}\zeta\,\mathrm{d}s+q\beta_1 F^{2H}$$
$$\le(1+\alpha_1 F^{2H-1})qK\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}\mathrm{d}s+(1+\alpha_1 F^{2H-1})qK\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q}\mathrm{d}s$$
$$+(1+\alpha_1 F^{2H-1})qK\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}|\bar{\mu}(s)-\mathcal{U}(s)|^2\mathrm{d}s$$
$$+q\alpha_1 F^{2H-1}\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q}\mathrm{d}s+q\beta_1 F^{2H-1}$$
$$+4qL(\omega_\Delta)\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q-2}|\mathcal{U}(s)-x(s)|\cdot|\bar{\mu}(s)-\mathcal{U}(s)|\mathrm{d}s.$$

With the help of Young inequality, we get

$$\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q \le(1+\alpha_1 F^{2H-1})(q-2)K\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q}\mathrm{d}s+2(1+\alpha_1 F^{2H-1})KF$$
$$+q\beta_1 F^{2H-1}+(1+\alpha_1 F^{2H-1})qK\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q}\mathrm{d}s$$
$$+(1+\alpha_1 F^{2H-1})(q-2)K\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|r(s)|^{q}\mathrm{d}s$$

$$
\begin{aligned}
&+ 2(1+\alpha_1 \digamma^{2H-1})K\mathbb{E}\int_0^{t\wedge\zeta_\Lambda} |\bar{\mu}(s)-\mathcal{U}(s)|^q \mathrm{d}s \\
&+ q\alpha_1 \digamma^{2H-1}\mathbb{E}\int_0^{t\wedge\zeta_\Lambda} |r(s)|^q\mathrm{d}s + (q-2)\mathbb{E}\int_0^{t\wedge\zeta_\Lambda} |r(s)|^q\mathrm{d}s \\
&+ 2\big(4L(\omega_\Delta)\big)^{\frac{q}{2}}\mathbb{E}\int_0^{t\wedge\zeta_\Lambda} |\mathcal{U}(s)-x(s)|^{\frac{q}{2}}|\bar{\mu}(s)-\mathcal{U}(s)|^{\frac{q}{2}}\mathrm{d}s.
\end{aligned}
$$

Then, by using

$$
|\mathcal{U}(s)-x(s)|^{\frac{q}{2}} \le 2^{\frac{q}{2}}\big(|\mathcal{U}(s)-\bar{\mu}(s)|^{\frac{q}{2}} + |\bar{\mu}(s)-x(s)|^{\frac{q}{2}}\big),
$$

we arrive at

$$
\begin{aligned}
\mathbb{E}|r(t\wedge\zeta_\Lambda)|^q &\le C\mathbb{E}\int_0^{t\wedge\zeta_\Lambda} |r(s)|^q\mathrm{d}s + C\big(L^q(\omega_\Delta)+L^{\frac{q}{2}}(\omega_\Delta)\big)\mathbb{E}\int_0^{t\wedge\zeta_\Lambda}|\bar{\mu}(s)-\mathcal{U}(s)|^q\mathrm{d}s \\
&\quad + 2(1+\alpha_1\digamma^{2H-1})K\digamma + q\beta_1\digamma^{2H-1} \\
&\le C\mathbb{E}\int_0^{t\wedge\zeta_\Lambda} |r(s)|^q\mathrm{d}s + CL^{2q}(\omega_\Delta)\Delta t^{qH}.
\end{aligned}
$$

Applying the Grönwall inequality, we derive (3.21). □

Now, we are in the position to give the convergence order of MSST method at the terminal time $\digamma$.

Theorem 3.2. *Under Assumptions* 2.1 *and* 2.2, *if there exist* $p>q\ge 2$, $\Delta t\in(0,\Delta t^*]$ *such that*

$$
L^{-\frac{pq}{p-q}}(\omega_\Delta)\Delta t^{-\frac{pqH}{2(p-q)}} \le \omega_\Delta,
$$

then $\mathbb{E}|r(\digamma)|^q \le CL^{2q}(\omega_\Delta)\Delta t^{qH}$.

Proof. With the help of Young inequality, for any $\delta>0$ and $p>q\ge 2$, the q-th moment of the error for MSST method (3.6)-(3.7) at $\digamma$ satisfies

$$
\begin{aligned}
\mathbb{E}\left(|r(\digamma)|^q\right) &\le \mathbb{E}\left(|r(\digamma)|^q\mathbf{1}_{\{\zeta_\Lambda>\digamma\}}\right) + \mathbb{E}\left(|r(\digamma)|^q\mathbf{1}_{\{\zeta_\Lambda\le\digamma\}}\right) \\
&\le \mathbb{E}\left(|r(\digamma)|^q\mathbf{1}_{\{\zeta_\Lambda>\digamma\}}\right) + \frac{q\delta}{p}\mathbb{E}|r(\digamma)|^p + \frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}(\zeta_\Lambda\le\digamma).
\end{aligned}
\tag{3.22}
$$

Applying the Cauchy inequality to $|r(\digamma)|^p$, by using Lemma 3.4 and the boundness of the moment of $x(\digamma)$, we obtain

$$
\mathbb{E}|r(\digamma)|^p \le C\mathbb{E}|\bar{\mu}(\digamma)|^p + C\mathbb{E}|x(\digamma)|^p \le C.
$$

According to the definition of ζ_Λ,

$$
\mathbb{P}(\zeta_\Lambda\le\digamma) \le \mathbb{P}(\varphi_\Lambda\le\digamma) + \mathbb{P}(\varrho_\Lambda\le\digamma).
$$

Since $\mathbb{P}(\varrho_\Lambda\le\digamma)\le C/\Lambda^2$ and $\mathbb{P}(\varphi_\Lambda\le\digamma)\le C/\Lambda^2$, (3.22) can be rewritten as

$$
\mathbb{E}|r(\digamma)|^q \le \mathbb{E}\left(|r(\digamma\wedge\zeta_\Lambda)|^q\right) + \frac{Cq\delta}{p} + \frac{C(p-q)}{p\Lambda^2\delta^{q/(p-q)}}.
$$

If we choose

$$
\delta = L^{2q}(\omega_\Delta)\Delta t^{qH}, \quad \Lambda = \left(L^q(\omega_\Delta)\Delta t^{\frac{(qH)}{2}}\right)^{-\frac{p}{p-q}},
$$

then

$$\mathbb{E}|r(\digamma)|^q \leq \mathbb{E}\left(|r(\digamma \wedge \zeta_\Lambda)|^q\right) + \frac{CqL^{2q}(\omega_\Delta)\Delta t^{qH}}{p} + \frac{C(p-q)}{p\left(L^{2q}(\omega_\Delta)\Delta t^{qH}\right)^{-p/(p-q)}\left(L^{2q}(\omega_\Delta)\Delta t^{qH}\right)^{q/(p-q)}}.$$

Due to Theorem 3.1, we derive

$$\mathbb{E}|r(\digamma)|^q \leq \mathbb{E}\left(|r(\digamma \wedge \zeta_R)|^q\right) + CL^{2q}(\omega_\Delta)\Delta t^{qH} \leq CL^{2q}(\omega_\Delta)\Delta t^{qH}.$$

The proof is complete. □

4. Exponential Stability

Here, we consider the exponential stability of MSST method for SDE (1.1). Here, we select $\Delta t > 0$ as an arbitrary fixed stepsize, $t_n = n\Delta t$.

Definition 4.1 ([14, Definition 2.3]). *Let $p > 0$, for a given $\Delta t > 0$, a numerical solution is said to be exponential stable in the p-th moment for the SDE (1.1) with the initial value $x_0 \in \mathbb{R}^m$, if there exist positive constants ϱ_1 and λ_1 such that*

$$\mathbb{E}|\mu_n|^p \leq \varrho_1 |x_0|^p r^{-\lambda_1 t_n}.$$

To analyze the exponential stability of MSST method, we need the following condition.

Assumption 4.1. *There exist positive constants K_1 and K_2 such that for any $x \in \mathbb{R}^m$, $\mathscr{X}(x)$ and $\mathscr{T}(x)$ satisfy $\langle x, \mathscr{X}(x)\rangle \leq -K_1|x|^2$ and $|x| \cdot |\mathscr{T}(x)\Delta B| + |\mathscr{T}(x)\Delta B|^2 \leq K_2|x|^2$, where $\Delta B = B(t_1) - B(t_2)$, $\forall t_1, t_2 \in [0, \digamma]$, K_1 and K_2 satisfy*

$$2(1-\theta)\Delta t K_1 - \big(3 + (1-\theta)\Delta t\big)K_2 > 16(1-\theta)\Delta t L^2(\omega_\Delta) + 16\Delta t^2 L^2(\omega_\Delta).$$

Theorem 4.1. *Let the coefficients $\mathscr{X}(x)$, $\mathscr{T}(x)$ satisfy Assumptions 2.1, 2.2 and 4.1. Then the MSST method (3.6), (3.7) is exponential stable in mean square.*

Proof. Squaring on both sides of (3.7) yields

$$|\mu_{n+1}|^2 = |\mu_n|^2 + \Delta t^2|\mathscr{X}_\Delta(\mathcal{U}_n)|^2 + |\mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n|^2 + 2\Delta t\left\langle \mu_n, \mathscr{X}_\Delta(\mathcal{U}_n)\right\rangle + 2\left\langle \mu_n, \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle + 2\Delta t\left\langle \mathscr{X}_\Delta(\mathcal{U}_n), \mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n\right\rangle.$$

Because of $\mu_n = \mathcal{U}_n - \theta\Delta t\mathscr{X}_\Delta(\mathcal{U}_n)$, then

$$|\mu_{n+1}|^2 \leq |\mu_n|^2 + \Delta t^2|\mathscr{X}_\Delta(\mathcal{U}_n)|^2 + |\mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n|^2 - 2\Delta t\left\langle \mathcal{U}_n, \mathscr{X}_\Delta(\mathcal{U}_n)\right\rangle + 2|\mathcal{U}_n|\,|\mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n| + (1-\theta)\Delta t|\mathscr{X}_\Delta(\mathcal{U}_n)|^2 + (1-\theta)\Delta t|\mathscr{T}_\Delta(\mathcal{U}_n)\Delta B_n|^2.$$

Under Assumption 4.1, we arrive at

$$\mathbb{E}|\mu_{n+1}|^2 \leq \mathbb{E}|\mu_n|^2 + 16\Delta t^2 L^2(\omega_\Delta)\mathbb{E}|\mathcal{U}_n|^2 - 2K_1\Delta t\mathbb{E}|\mathcal{U}_n|^2 + \big(3 + (1-\theta)\Delta t\big)K_2\mathbb{E}|\mathcal{U}_n|^2 + 16(1-\theta)\Delta t L^2(\omega_\Delta)\mathbb{E}|\mathcal{U}_n|^2. \tag{4.1}$$

By squaring on both sides of (3.6), in view of the Assumption 4.1, we have

$$|\mathcal{U}_n|^2 \le \langle \mathcal{U}_n, \mu_n\rangle - \theta K_1 \Delta \frac{|\mathcal{U}_n|^2}{2} + \frac{|\mu_n|^2}{2} - \theta K_1 \Delta t |\mathcal{U}_n|^2.$$

Thus, we can obtain $|\mathcal{U}_n|^2 \le \alpha_2^{-1}|\mu_n|^2$, where $\alpha_2 = 1 + 2\theta K_1 \Delta t$. Inserting it into (4.1) leads to

$$\begin{aligned}\alpha_2 \mathbb{E}|\mu_{n+1}|^2 \le \big[&\alpha_2 + 16\Delta t^2 L^2(\omega_\Delta) + \big(3 + (1-\theta)\Delta t\big)K_2 \\ &+ 16(1-\theta)\Delta t L^2(\omega_\Delta) - 2K_1\Delta t\big]\mathbb{E}|\mu_n|^2.\end{aligned}$$

Multiplying $\eta^{(n+1)\Delta t}$ with a constant $\eta > 1$ on both sides leads to

$$\begin{aligned}&\alpha_2 \eta^{(n+1)\Delta t}\mathbb{E}|\mu_{n+1}|^2 - \alpha_2 \eta^{n\Delta t}\mathbb{E}|\mu_n|^2 \\ \le\ &\big[\big(\alpha_2 + 16\Delta t^2 L^2(\omega_\Delta) + \big(3 + (1-\theta)\Delta t\big)K_2 \\ &\quad + 16(1-\theta)\Delta t L^2(\omega_\Delta) - 2K_1\Delta t\big)\eta^{\Delta t} - 1\big]\eta^{n\Delta t}\mathbb{E}|\mu_n|^2.\end{aligned}$$

Choose λ_1 such that

$$0 < \lambda_1 \le \frac{1}{\Delta t}\ln\Big(1+2\theta K_1\Delta t+16\Delta t^2 L^2(\omega_\Delta)+\big(3+(1-\theta)\Delta t\big)K_2+16(1-\theta)\Delta t L^2(\omega_\Delta)-2K_1\Delta t\Big)^{-1},$$

and $\eta = e^{\lambda_1}$, then

$$\begin{aligned}&(1+2\theta K_1\Delta t) r^{\lambda_1 (n+1)\Delta t}\mathbb{E}|\mu_{n+1}|^2 \\ \le\ &(1+2\theta K_1\Delta t)|x_0|^2 + \big[\big(1+2\theta K_1\Delta t + 16\Delta t^2 L^2(\omega_\Delta) + \big(3+(1-\theta)\Delta t\big)K_2 \\ &\qquad + 16(1-\theta)\Delta t L^2(\omega_\Delta) - 2K_1\Delta t\big) r^{\lambda_1\Delta t} - 1\big]\sum_{\ell=0}^{n} r^{\lambda_1 \ell\Delta t}\mathbb{E}|\mu_\ell|^2.\end{aligned}$$

Thus, there is a positive constant ϱ_1 such that $r^{\lambda_1 t_{n+1}}\mathbb{E}|\mu_{n+1}|^2 \le \varrho_1 |x_0|^2$, which completes the proof. $\square$

5. SDE with a Delay Term

Consider the SDDE

$$\begin{cases} \mathrm{d}y(t) = \mathscr{F}\big(y(t)\big)\mathrm{d}t + \mathscr{G}\big(y(t), y(t-\tau)\big)\mathrm{d}B(t), & t \in [0, \digamma], \\ y(t) = \xi(t), & t \in [-\tau, 0], \end{cases} \tag{5.1}$$

where $\tau > 0$, $\mathscr{F} : \mathbb{R}^m \to \mathbb{R}^m$ and $\mathscr{G} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ are measurable functions, $B(\cdot)$ is an m-dimensional fBm with $H \in (1/2, 1)$, $\xi : [-\tau, 0] \to \mathbb{R}^m$ is Υ_0 measurable function satisfying $\sup_{-\tau\le t\le 0}\mathbb{E}|\xi(t)|^2 \le M$ with M being a positive constant. The existence and boundedness of its exact solutions are given in [5]. To study the strong convergence of the MSST method for the problem (5.1), we first give the following assumptions.

Assumption 5.1. *For any positive constant $\varLambda$, there is a positive function $L(\varLambda)$ with respect to $\varLambda$ such that for any y, $\mathring{y} \in \mathbb{R}^m$, $|y| \vee |\mathring{y}| \le \varLambda$, $\mathscr{F}(\cdot)$ satisfies*

$$|\mathscr{F}(y) - \mathscr{F}(\mathring{y})| \le L(\varLambda)|y - \mathring{y}|.$$

Assumption 5.2. *There is a positive constant K such that, for any y, $\mathring{y} \in \mathbb{R}^m$, $\mathscr{F}(\cdot)$ satisfies*

$$\langle y-\mathring{y}, \mathscr{F}(y)-\mathscr{F}(\mathring{y})\rangle \leq K(1+|y-\mathring{y}|^2).$$

Remark 5.1. There is a positive constant K with respect to $\mathscr{F}(0)$, such that for any $y \in \mathbb{R}^m$,

$$\langle y, \mathscr{F}(y)\rangle \leq K(1+|y|^2).$$

Define $\mathscr{F}_\Delta(y)$ by the similar way as (3.1) and (3.2), that is

$$\mathscr{F}_\Delta(y) = \begin{cases} \mathscr{F}(y), & |y| \leq \omega_\Delta, \\ \dfrac{|y|}{\omega_\Delta}\mathscr{F}\left(\omega_\Delta \dfrac{y}{|y|}\right), & |y| > \omega_\Delta. \end{cases}$$

It is obvious that the modified truncation function $\mathscr{F}_\Delta(y)$ is globally Lipschitz continuous. In addition, for any $\Delta t \in (0, \Delta t^*]$, $\mathscr{F}_\Delta(\cdot)$ satisfies Assumption 5.2 and Remark 5.1. Then, we give the following conclusion.

Lemma 5.1. *Under Assumption* 5.2, *for any y, $\mathring{y} \in \mathbb{R}^m$, $\Delta t \in (0, \Delta t^*]$, there is a positive constant $\hat{K}$ such that*

$$\langle y-\mathring{y}, \mathscr{F}_\Delta(y)-\mathscr{F}_\Delta(\mathring{y})\rangle \leq \hat{K}(1+|y-\mathring{y}|^2), \quad \langle y, \mathscr{F}_\Delta(y)\rangle \leq \hat{K}(1+|y|^2).$$

For the coefficient $\mathscr{G}(\cdot,\cdot)$, we assume it satisfies the globally Lipschitz continuous condition and the polynomial growth condition.

Assumption 5.3. *For any $\lambda \in (0,1]$, y, $\mathring{y}$, z, $\mathring{z} \in \mathbb{R}^m$ with $t \in [0, \digamma]$, there exist two positive constants M_0 and K_0 such that*

$$\begin{aligned} &|\mathscr{G}(y,z)-\mathscr{G}(\mathring{y},\mathring{z})| \leq M_0(|y-\mathring{y}|+|z-\mathring{z}|), \\ &|\mathscr{G}(y,z)|^2 \leq K_0(1+|y|^{2\lambda}+|z|^{2\lambda}). \end{aligned}$$

Here, we suppose that there is $m \in \mathbb{Z}$ such that $\tau = m\Delta t$. Applying the MSST method to SDDE (5.1), we obtain the numerical scheme

$$\Psi_n = \Psi_n + \theta\Delta t \mathscr{F}_\Delta(\Psi_n), \tag{5.2}$$

$$\psi_{n+1} = \Psi_n + \Delta t \mathscr{F}_\Delta(\Psi_n) + \mathscr{G}(\Psi_n, \Psi_{n-m})\Delta B_n, \tag{5.3}$$

where $\Psi_n = \xi(n\Delta t)$ for $n = -m, -m+1, \ldots, 0$ and ψ_n is the numerical solution, $\psi_0 = \xi(0)$.

To analyze the convergence of MSST method for the SDDE (5.1), we give the following bounds for ψ_n and Ψ_n.

Lemma 5.2. *Under the Assumptions* 5.1-5.3, *for any $0 < \Delta t \leq \Delta t^* \leq 1/(2\theta\hat{K})$, $\theta \in [1/2, 1]$ and $p \geq 1/H$, the following moment property holds:*

$$\mathbb{E}\Big[\sup_{0<\Delta t\leq\Delta t^*}\sup_{0\leq n\leq N}|\Psi_n|^{2p}\Big] \vee \mathbb{E}\Big[\sup_{0<\Delta t\leq\Delta t^*}\sup_{0\leq n\leq N}|\psi_n|^{2p}\Big] \leq C. \tag{5.4}$$

Proof. Similar to (3.8), we have $|\Psi_n|^2 \le \alpha|\psi_n|^2 + \beta$. By squaring on both sides of (5.3), inserting $\mathscr{F}_\Delta(\Psi_n) = (\Psi_n - \psi_n)/(\theta\Delta t)$ and applying Lemma 5.1, we derive

$$\begin{aligned}|\psi_n|^2 \le |\psi_0|^2 &+ 2\alpha\hat{K}\Delta t\sum_{\ell=0}^{n-1}\mathbb{E}|\psi_\ell|^2 + 2(1+\beta)\hat{K}F \\ &+ \sum_{\ell=0}^{n-1}\mathbb{E}\left[|\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})|^2\cdot|\Delta B_\ell|^2\right] \\ &+ 2\left(\frac{1}{\theta}-1\right)\mathbb{E}\left[\left|\sum_{\ell=0}^{n-1}\langle\psi_\ell,\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})\Delta B_\ell\rangle\right|\right] \\ &+ \frac{2}{\theta}\mathbb{E}\left[\left|\sum_{\ell=0}^{n-1}\langle\Psi_\ell,\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})\Delta B_\ell\rangle\right|\right].\end{aligned}$$

For $0 \le n \le \kappa$ with κ is an integer and $\kappa \in [0, N)$, by taking the p-th power and the mathematical expectation on both sides, we have

$$\begin{aligned}\mathbb{E}\Big[\max_{0\le n\le\kappa}|\psi_n|^{2p}\Big] \le 6^{p-1}\Bigg\{&|\psi_0|^{2p} + (2\alpha\hat{K}\Delta t)^p\kappa^{p-1}\sum_{\ell=0}^{\kappa-1}\mathbb{E}|\psi_\ell|^{2p} + \big(2(1+\beta)\hat{K}F\big)^p \\ &+ \kappa^{p-1}\sum_{\ell=0}^{\kappa-1}\mathbb{E}\left[|\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})|^{2p}\cdot|\Delta B_\ell|^{2p}\right] \\ &+ 2^p\left(\frac{1}{\theta}-1\right)^p\mathbb{E}\left[\max_{0\le n\le\kappa}\left|\sum_{\ell=0}^{n-1}\langle\psi_\ell,\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})\Delta B_\ell\rangle\right|^p\right] \\ &+ \left(\frac{2}{\theta}\right)^p\mathbb{E}\left[\max_{0\le n\le\kappa}\left|\sum_{\ell=0}^{n-1}\langle\Psi_\ell,\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})\Delta B_\ell\rangle\right|^p\right]\Bigg\}. \qquad (5.5)\end{aligned}$$

Similar to (3.11)-(3.13), we get

$$\begin{aligned}&\sum_{\ell=0}^{\kappa-1}\mathbb{E}\left[|\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})|^{2p}\cdot|\Delta B_\ell|^{2p}\right] \\ &\le CK_0^p\Delta t^{2pH}\sum_{\ell=0}^{\kappa-1}\big(1+M^{2\lambda p}+\beta^{2\lambda p}+\alpha^{2\lambda p}\mathbb{E}|\psi_\ell|^{4\lambda p}\big),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left[\max_{0\le n\le\kappa}\left|\sum_{\ell=0}^{n-1}\langle\psi_\ell,\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})\Delta B_\ell\rangle\right|^p\right] \\ &\le CK_0^{\frac{p}{2}}\Delta t^{pH}\sum_{\ell=0}^{\kappa-1}\big(1+M^{2\lambda p}+\beta^{2\lambda p}+\mathbb{E}|\psi_\ell|^{4p}+\alpha^{2\lambda p}\mathbb{E}|\psi_\ell|^{4\lambda p}\big), \\ &\mathbb{E}\left[\max_{0\le n\le\kappa}\left|\sum_{\ell=0}^{n-1}\langle\Psi_\ell,\mathscr{G}(\Psi_\ell,\Psi_{\ell-m})\Delta B_\ell\rangle\right|^p\right] \\ &\le CK_0^{\frac{p}{2}}\Delta t^{pH}\sum_{\ell=0}^{n-1}\big(1+M^{2\lambda p}+\beta^{2p}+\beta^{2\lambda p}+\alpha^{2p}\mathbb{E}|\psi_\ell|^{4p}+\alpha^{2\lambda p}\mathbb{E}|\psi_\ell|^{4\lambda p}\big).\end{aligned}$$

We insert the above three inequalities into (5.5), then

$$
\begin{aligned}
\mathbb{E}\Big[\max_{0\le n\le\kappa}|\psi_n|^{2p}\Big] \le{}& 6^{p-1}|\psi_0|^{2p} + C\big((1+\beta)\hat{K}F\big)^p + CK_0^p\big(1+M^{2\lambda}+\beta^{2\lambda}\big)F^{2pH-2} \\
&+ C\left(\frac{1}{\theta}-1\right)^p K_0^{\frac{p}{2}}\big(1+M^{2\lambda p}+\beta^{2\lambda p}\big)F^{pH-1} \\
&+ C\theta^{-p}K_0^{\frac{p}{2}}\big(1+M^{2\lambda p}+\beta^{2p}\big)F^{pH-1} + C(\alpha\hat{K}\Delta t)^p\sum_{\ell=0}^{\kappa-1}\mathbb{E}|\psi_\ell|^{2p} \\
&+ C\alpha^{2\lambda}K_0^p\Delta t^{2pH}\sum_{\ell=0}^{\kappa-1}\mathbb{E}|\psi_\ell|^{4\lambda p} \\
&+ C\left(\frac{1}{\theta}-1\right)^p K_0^{\frac{p}{2}}\Delta t^{pH}\sum_{\ell=0}^{\kappa-1}\big(\mathbb{E}|\psi_\ell|^{4p}+\alpha^{2\lambda}\mathbb{E}|\psi_\ell|^{4\lambda p}\big) \\
&+ C\theta^{-p}K_0^{\frac{p}{2}}\Delta t^{pH}\sum_{\ell=0}^{\kappa-1}\big(\alpha^{2p}\mathbb{E}|\psi_\ell|^{4p}+\alpha^{2\lambda p}\mathbb{E}|\psi_\ell|^{4\lambda p}\big) \\
\le{}& C + C(\alpha\hat{K}\Delta t)^p\sum_{\ell=0}^{\kappa-1}\mathbb{E}\Big[\max_{0\le i\le\ell}|\psi_i|^{2p}\Big] + C\alpha^{2\lambda}K_0^p\Delta t^{2pH}\sum_{\ell=0}^{\kappa-1}\mathbb{E}\Big[\max_{0\le i\le\ell}|\psi_i|^{4p}\Big] \\
&+ C(1+\alpha^{2\lambda})\left(\frac{1}{\theta}-1\right)^p K_0^{\frac{p}{2}}\Delta t^{pH}\sum_{\ell=0}^{\kappa-1}\mathbb{E}\Big[\max_{0\le i\le\ell}|\psi_i|^{4p}\Big] \\
&+ C(\alpha^{2p}+\alpha^{2\lambda p})\theta^{-p}K_0^{\frac{p}{2}}\Delta t^{pH}\sum_{\ell=0}^{\kappa-1}\mathbb{E}\Big[\max_{0\le i\le\ell}|\psi_i|^{4p}\Big] \\
\le{}& C + C\Delta t^p\sum_{\ell=0}^{\kappa-1}\mathbb{E}\Big[\max_{0\le i\le\ell}|\psi_i|^{2p}\Big] + C\big(\Delta t^{pH}+\Delta t^{2pH}\big)\sum_{\ell=0}^{\kappa-1}\mathbb{E}\Big[\max_{0\le i\le\ell}|\psi_i|^{4p}\Big].
\end{aligned}
$$

Using the property of expectation and Willett-Wong inequality again, we deduce that

$$
\mathbb{E}\Big[\sup_{0\le\Delta t\le\Delta t^*}\sup_{0\le n\le N}|\psi_n|^{2p}\Big] \le C.
$$

Due to $|\Psi_n|^2 \le \alpha|\psi_n|^2+\beta$, we can obtain that

$$
\mathbb{E}\Big[\sup_{0\le\Delta t\le\Delta t^*}\sup_{0\le n\le N}|\Psi_n|^{2p}\Big] \le C.
$$

The proof is complete. □

Denote

$$
\begin{aligned}
\Psi(t) &:= \sum_{n=0}^{N-1}1_{\{t_n\le t<t_{n+1}\}}\Psi_n + 1_{\{t=t_N\}}\Psi_N, \\
\bar{\psi}(t) &:= \sum_{n=0}^{N-1}1_{\{t_n\le t<t_{n+1}\}}\psi_n + 1_{\{t=t_N\}}\psi_N,
\end{aligned}
$$

where $\Psi(t)=\xi(t)$ for $t\in[-\tau,0]$. Then

$$
\bar{\psi}(t) = \psi_0 + \int_0^t \mathscr{F}_\Delta\big(\Psi(s)\big)\mathrm{d}s + \int_0^t \mathscr{G}\big(\Psi(s),\Psi(s-\tau)\big)\mathrm{d}B(s), \quad t\in[t_n,t_{n+1}).
$$

Lemma 5.3. *Under Assumptions* 5.1-5.3*, it holds*

$$\sup_{0\le t\le F} \mathbb{E}\left[|\bar{\psi}(t)-\Psi(t)|^2\right] \le CL^2(\omega_\Delta)\Delta t^{2H}. \tag{5.6}$$

Proof. It is similar to Lemma 3.3, so we omit it here. □

Lemma 5.4. *Under the Assumptions* 5.1-5.3*, we have*

$$\mathbb{E}\Big[\sup_{0\le t\le F}|\bar{\psi}(t)|^2\Big] \le C. \tag{5.7}$$

Proof. By using a similar process as (3.19) to deal with $\bar{\psi}(t)$, we can obtain

$$\bar{\psi}(t)|^2 \le 3\left\{\left(1+\frac{1}{\theta}\right)^2|\psi_{n(t)}|^2+\frac{1}{\theta^2}|\Psi_{n(t)}|^2+\left|\mathscr{G}_\Delta(\Psi_{n(t)},\Psi_{n(t)-m})\Delta B_{n(t)}\right|^2\right\}.$$

Here, the last item satisfies

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le n(t)\le N}\left|\mathscr{G}_\Delta(\Psi_{n(t)},\Psi_{n(t)-m})\Delta B_{n(t)}\right|^2\Big]\\
&\le \mathbb{E}\Big[\sup_{0\le n(t)\le N} K_0^2\big(1+|\Psi_{n(t)}|^{2\lambda}+|\Psi_{n(t)-m}|^{2\lambda}\big)\big|B(t)-B(t_{n(t)})\big|^2\Big]\\
&\le \sum_{j=1}^{N} CK_0\Big(1+M^{2\lambda}+\sup_{0\le j\le N}\mathbb{E}|\Psi_j|^{4\lambda}\Big)\Delta t^{2H}\le C.
\end{aligned}$$

Using a similar way to obtain (3.18), we get (5.7). □

Define

$$\begin{aligned}
&\bar{\varphi}_\Lambda = \inf\{t\ge 0: |y(t)|\vee|y(t-\tau)|\ge\Lambda\},\\
&\bar{\varrho}_\Lambda = \inf\{t\ge 0: |\bar{\psi}(t)|\ge\Lambda \text{ or } |\Psi(t)|\ge\Lambda \text{ or } |\Psi_{t-\tau}|\ge\Lambda\},\\
&\bar{\zeta}_\Lambda = \bar{\varphi}_\Lambda\wedge\bar{\varrho}_\Lambda,\\
&\bar{r}(t) = \bar{\psi}(t)-y(t).
\end{aligned}$$

Theorem 5.1. *Under Assumptions* 5.1-5.3*, for any $q\ge 2$, $\Delta t\in(0,\Delta t^*]$ and $\Lambda\le\omega_\Delta$, we have*

$$\mathbb{E}|\bar{r}(t\wedge\bar{\zeta}_\Lambda)|^q \le CL^{2q}(\omega_\Delta)\,\Delta t^{qH}.$$

Proof. When $0\le s\le t\wedge\bar{\zeta}_\Lambda$, one has $|\Psi(t)|\le\Lambda\le\omega_\Delta$. It is obvious that

$$\begin{aligned}
\mathbb{E}|\bar{r}(t\wedge\bar{\zeta}_\Lambda)|^q \le\ & q\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q-2}\left\langle\Psi(s)-y(s),\mathscr{F}\big(\Psi(s)\big)-\mathscr{F}\big(y(s)\big)\right\rangle\mathrm{d}s\\
&+q\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q-2}\left\langle\bar{\psi}(s)-\Psi(s),\mathscr{F}\big(\Psi(s)\big)-\mathscr{F}\big(y(s)\big)\right\rangle\mathrm{d}s+q\beta_1F^{2H}\\
&+\alpha_1 q\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q-2}\big\langle\bar{\psi}(s)-y(s),\mathscr{G}\big(\Psi(s),\Psi(s-\tau)\big)\\
&\qquad-\mathscr{G}\big(y(s),y(s-\tau)\big)\big\rangle\phi(\zeta,s)\mathrm{d}\zeta\,\mathrm{d}s.
\end{aligned}$$

From Assumption 5.2, Cauchy inequality and Young inequality, we derive

$$\begin{aligned}\mathbb{E}|\bar{r}(t\wedge\bar{\zeta}_\Lambda)|^q &\leq q\big(\hat{K}+2\alpha_1 M_0 F^{2H-1}\big)\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q-2}\mathrm{d}s\\&\quad+q\big(\hat{K}+2\alpha_1 M_0 F^{2H-1}\big)\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q}\mathrm{d}s\\&\quad+q\big(\hat{K}+2\alpha_1 M_0 F^{2H-1}\big)\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q-2}|\bar{\psi}(s)-\Psi(s)|^2\mathrm{d}s\\&\quad+q\alpha_1 F^{2H-1}\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q}\mathrm{d}s+q\beta_1 F^{2H-1}\\&\quad+4qL(\omega_\Delta)\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q-2}\left\langle\Psi(s)-y(s)\right\rangle\cdot|\bar{\psi}(s)-\Psi(s)|\mathrm{d}s.\end{aligned}$$

Similar to the analysis in Theorem 3.1, we derive

$$\begin{aligned}\mathbb{E}|\bar{r}(t\wedge\bar{\zeta}_\Lambda)|^q &\leq C\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q}\mathrm{d}s+C\big(L^q(\omega_\Delta)+L^{\frac{q}{2}}(\omega_\Delta)\big)\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{\psi}(s)-\Psi(s)|^q\mathrm{d}s\\&\quad+2\big(\hat{K}+2\alpha_1 M_0 F^{2H-1}\big)+q\beta_1 F^{2H-1}\\&\leq C\mathbb{E}\int_0^{t\wedge\bar{\zeta}_\Lambda}|\bar{r}(s)|^{q}\mathrm{d}s+CL^{2q}(\omega_\Delta)\Delta t^{qH}.\end{aligned}$$

By using the Grönwall inequality, we completes the proof. □

Theorem 5.2. *Under Assumptions* 5.1-5.3, *if there exist* $p>q\geq 2$ *and* $\Delta t\in(0,\Delta t^*]$ *such that*

$$L^{-\frac{pq}{p-q}}(\omega_\Delta)\Delta t^{-\frac{pqH}{2(p-q)}}\leq\omega_\Delta,$$

then

$$\mathbb{E}|\bar{r}(F)|^q\leq CL^{2q}(\omega_\Delta)\Delta t^{qH}.$$

Proof. It is similar to Theorem 3.2, so we omit it here. □

6. Numerical Experiments

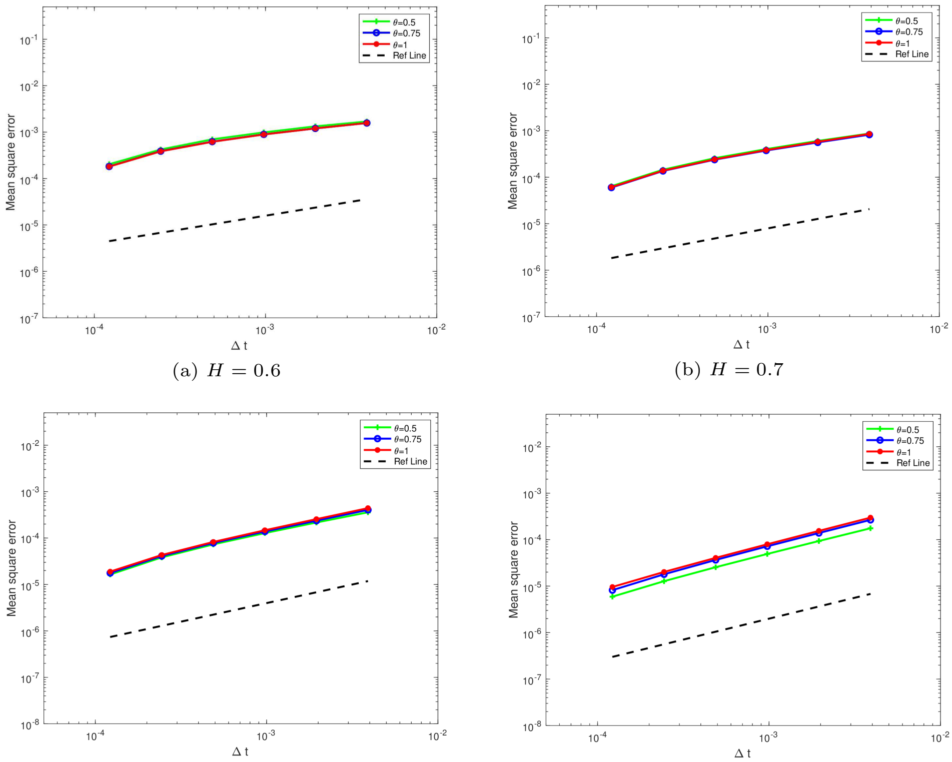
To demonstrate our theoretical conclusions, we make two experiments in this section. Here, we do $M=5000$ times independent tests and show the mathematical expectation by the way of calculating mean value. At the terminal time F,

$$\epsilon_N(F)=\left(\frac{1}{M}\sum_{\ell=1}^{M}\left|x^\ell(F)-x_N^\ell\right|^2\right)^{\frac{1}{2}}.$$

Example 6.1. Consider the one-dimension SDE

$$\mathrm{d}x(t)=\big(-0.3x(t)-0.1x^3(t)\big)\mathrm{d}t+\big(-0.1x(t)+0.9x^3(t)\big)\mathrm{d}B(t),\quad t\in[0,1],\tag{6.1}$$

where $x(0)=-0.5$.



(a) $H = 0.6$ (b) $H = 0.7$

(c) $H = 0.8$ (d) $H = 0.9$

Fig. 6.1. The mean square errors versus stepsize for SDE (6.1).

Here,

$$\mathscr{L}(x) = -0.3x - 0.1x^3, \quad \mathscr{T}(x) = -0.1x + 0.9x^3$$

are locally Lipschitz continuous for $|x| \leq \varLambda$ with $L(\varLambda) = -0.4 + 3\varLambda^2$, where $\varLambda$ is a bounded positive constant. It can be proved that the coefficients satisfy Assumption 2.2.

Let

$$\omega_\Delta = \sqrt{\frac{\Delta t^{-\frac{\upsilon}{2}} + 0.4}{3}} \quad \text{with} \quad \upsilon \in (0, 1).$$

We use the numerical result generated by backward EM method with $\Delta t = 2^{-14}$ as the reference solution. Fig. 6.1 shows the mean square errors for our method with $\Delta t = 2^{-8}, 2^{-9}, \cdots, 2^{-13}$. The slope of the referenced dotted line is H. By an observation of four subfigures, we find that the convergence order is close to H, which keeps consistence with Theorem 3.2.

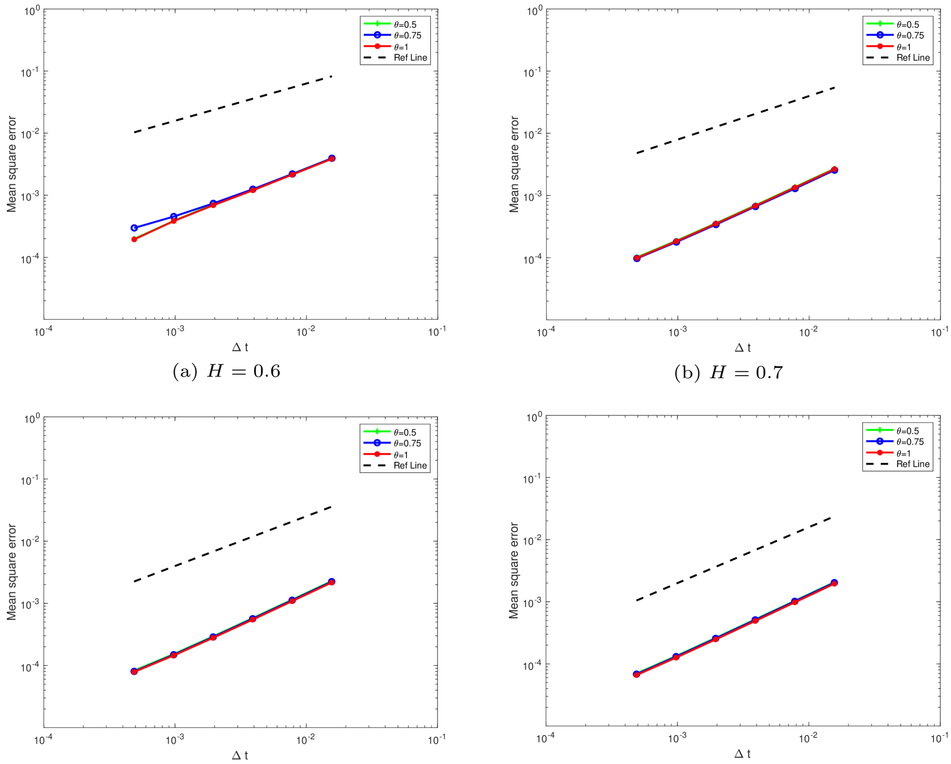
Example 6.2. Consider the one-dimension SDDE

$$\begin{cases} dx(t) = \big(-0.05 - 0.01x^5(t)\big)\,dt - 0.8x(t-1)\,dB(t), & t \in [0, 4], \\ x(t) = t + 1, & t \in [-1, 0]. \end{cases} \tag{6.2}$$

Set

$$\omega_\Delta = \sqrt[4]{\frac{\Delta t^{-\frac{\upsilon}{5}}}{0.05}} \quad \text{with} \quad \upsilon \in (0, 1).$$

Here, we use the numerical solution of backward EM method with $\Delta t = 2^{-12}$ as the reference solution. In Fig. 6.2, all the black dotted reference lines are with the slope of H and we plot



(a) $H = 0.6$ (b) $H = 0.7$

(c) $H = 0.8$ (d) $H = 0.9$

Fig. 6.2. The mean square errors versus stepsize for SDDE (6.2).

the mean square errors for our method with $\Delta t = 2^{-6}, 2^{-7}, \cdots, 2^{-11}$ respectively. It can be seen that, in agreement with Theorem 5.2, the convergence order of MSST method is also H for SDDE.

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