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NUMERICAL ANALYSIS FOR STOCHASTIC TIME-SPACE FRACTIONAL DIFFUSION EQUATION DRIVEN BY FRACTIONAL GAUSSIAN NOISE*

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Abstract

In this paper, we consider the strong convergence of the time-space fractional diffusion equation driven by fractional Gaussian noise with Hurst index $H \in (1/2, 1)$. A sharp regularity estimate of the mild solution and the numerical scheme constructed by finite element method for integral fractional Laplacian and backward Euler convolution quadrature for Riemann-Liouville time fractional derivative are proposed. With the help of inverse Laplace transform and fractional Ritz projection, we obtain the accurate error estimates in time and space. Finally, our theoretical results are accompanied by numerical experiments.

Mathematics subject classification: 65M12, 65M60, 35R11, 35R60.
Key words: Fractional Laplacian, Stochastic fractional diffusion equation, Fractional Gaussian noise, Finite element, Convolution quadrature, Error analysis.

1. Introduction

In the framework of uncoupled continuous time random walk, if both the second moment of the jump length and the mean waiting time diverge, the model describes competition between subdiffusion and Lévy flights [32]. The equivalent microscopic model is based on the subordinated Langevin equation with stable noise. The probability density function of the position of the particle motion is governed by the fractional Fokker-Planck equation with temporal and spatial fractional derivatives [8]. If the system is influenced by external fluctuating source term, e.g. fractional Gaussian noise, it has the form (1.1), which is the equation we focus on in this paper.

Let $\mathbb{D} \subset \mathbb{R}^d, d = 1, 2, 3$, be a bounded domain with smooth boundary and $\psi(x,t)$ the solution of

$$\begin{cases} \partial_t \psi(x,t) + {}_0\partial_t^{1-\alpha} \mathcal{A}^s \psi(x,t) = \dot{W}_Q^H(x,t), & (x,t) \in \mathbb{D} \times (0,T], \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{D}, \\ \psi(x,t) = 0, & (x,t) \in \mathbb{D}^c \times [0,T], \end{cases} \tag{1.1}$$

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where $\mathcal{A}^s(=(-\Delta)^s)$ is defined by [10]

$$\mathcal{A}^s\psi = c_{d,s}\,\mathrm{P.V.}\int_{\mathbb{R}^d}\frac{\psi(x)-\psi(y)}{|x-y|^{d+2s}}dy, \quad s\in(0,1)$$

with

$$c_{d,s}=\frac{2^{2s}s\Gamma(d/2+s)}{\pi^{d/2}\Gamma(1-s)},$$

$\mathbb{D}^c$ means the complement of $\mathbb{D}$, T denotes a fixed terminal time, ∂_t is the first-order derivative in t, ${}_0\partial_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative, defined by [36]

$${}_0\partial_t^{1-\alpha}\psi=\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t(t-\xi)^{\alpha-1}\psi(\xi)d\xi, \quad \alpha\in(0,1), \tag{1.2}$$

$\dot{W}_Q^H$ denotes fractional Gaussian noise, W_Q^H is fractional Gaussian process with Hurst index $H\in(1/2,1)$ and covariance operator Q on a complete filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq0})$ and it can be written as

$$W_Q^H(x,t)=\sum_{k=1}^{\infty}\sqrt{\Lambda_k}\phi_k(x)W_k^H(t),$$

where $\{(\Lambda_k,\phi_k)\}_{k=1}^{\infty}$ are eigenvalues and orthonormal eigenfunctions of the self-adjoint, non-negative linear operator Q on $\mathbb{H}=L^2(\mathbb{D}), W_k^H, k=1,2,\ldots,$ are independent one-dimensional fractional Brownian motion (fBm) process with Hurst index H, which are determined by covariance function [37]

$$\mathrm{Cov}(t,r)=\mathbb{E}\left[W_k^H(t)W_k^H(r)\right]=\frac{1}{2}\left(t^{2H}+r^{2H}-|t-r|^{2H}\right), \quad t,r\geq0,$$

where $\mathbb{E}$ means the expectation operator. In this paper, we assume that $A^{-\rho}Q^{1/2}$ is a Hilbert-Schmidt operator on $\mathbb{H}$, where ρ is a real number, A denotes the classical Laplace operator $-\Delta$ with a zero Dirichlet boundary condition, and its domain $\mathcal{D}(A)=H_0^1(\mathbb{D})\cap H^2(\mathbb{D})$.

Obviously, the problem (1.1) can be divided into the following two problems, i.e. a deterministic problem

$$\begin{cases}\partial_t v+{}_0\partial_t^{1-\alpha}\mathcal{A}^s v=0, & (x,t)\in\mathbb{D}\times(0,T],\\ v(\cdot,0)=\psi_0, & x\in\mathbb{D},\\ v=0, & (x,t)\in\mathbb{D}^c\times[0,T],\end{cases} \tag{1.3}$$

and a stochastic problem

$$\begin{cases}\partial_t u+{}_0\partial_t^{1-\alpha}\mathcal{A}^s u=\dot{W}_Q^H, & (x,t)\in\mathbb{D}\times(0,T],\\ u(\cdot,0)=0, & x\in\mathbb{D},\\ u=0, & (x,t)\in\mathbb{D}^c\times[0,T].\end{cases} \tag{1.4}$$

Extensive numerical schemes for the deterministic fractional diffusion equation (1.3) have been proposed in [2, 5, 34]. Also, there have been many works for numerically solving stochastic partial differential equations (PDEs) involving Laplace and spectral fractional Laplacian, one can refer to [7, 18, 19, 25, 26, 28, 35, 39]. But for stochastic PDEs involving integral fractional Laplacian, the related researches are still few. In this paper, we provide a numerical scheme for stochastic PDE (1.4) based on backward Euler convolution quadrature for Riemann-Liouville fractional derivative and finite element method for integral fractional Laplacian.

Different from the Laplace and spectral fractional Laplacian, the eigenfunctions of $\mathcal{A}^s$ are unknown, so how to well characterize the influence of the noise on the regularity of the solution for (1.4) is a challenge. Here with the help of the representation of the mild solution, we provide the sharp regularity of the mild solution of (1.4) by building the resolvent estimate of integral fractional Laplacian and using the equivalence of the fractional Sobolev spaces, i.e. for $s \in [0,1]$, the spaces $\hat{H}^s(\mathbb{D})$ and $\dot{H}^s(\mathbb{D})$ are equivalent (refer to Section 2 for the definitions of $\hat{H}^s(\mathbb{D})$ and $\dot{H}^s(\mathbb{D})$). Then based on the Laplace transform of mild solution, we use backward Euler convolution quadrature to construct the temporal semi-discrete scheme, and thanks to the properties of generating function of backward Euler method, the Laplace transform of piecewise constant functions and Burkholder-Davis-Gundy inequality [16, 37], we obtain error estimate for time semi-discrete scheme by transforming the Wiener integral with respect to fBm into the one of Brownian motion. Finally, finite element method is used to discretize the integral fractional Laplacian and we introduce the fractional Ritz projection to build the estimate $\|(\mathcal{A}_h^s)^{-1/2}P_hA^{s/2}\| \leq C$ (see Section 5) to get the accurate spatial error estimate.

The paper is organized as follows. In Section 2, some preliminaries about fBm and fractional Sobolev spaces are introduced. In Section 3, we provide the spatial regularity estimate and Hölder regularity estimate about the mild solution of Eq. (1.4). In Section 4, the Riemann-Liouville fractional derivative is approximated by backward Euler convolution quadrature method and we provide the error estimates for the semidiscrete scheme. In Section 5, finite element method is used to discretize integral fractional Laplacian and error estimates for the fully discrete scheme are provided. In Section 6, extensive numerical examples verify the theoretically predicted convergence order. We conclude the paper with some discussions in the last section.

Throughout this paper, C denotes a generic positive constant, whose value may differ at each occurrence. The notation "$\tilde{\ }$" means taking Laplace transform and let $\epsilon > 0$ be arbitrarily small.

2. Preliminaries

We provide some facts on fBm and fractional Sobolev spaces, which can refer to [17, 33, 37]. Introduce $\mathbb{H}_0 = Q^{1/2}(\mathbb{H})$, whose inner product is $(\mu, \nu)_{\mathbb{H}_0} = (Q^{-1/2}\mu, Q^{-1/2}\nu)$ for $\mu, \nu \in \mathbb{H}_0$. Denote all the bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ and the ones from $\mathbb{H}_0$ to $\mathbb{H}$ by $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}(\mathbb{H}_0, \mathbb{H})$, respectively. The subspaces of $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}(\mathbb{H}_0, \mathbb{H})$ consisting of Hilbert-Schmidt operators are defined by $\mathcal{L}_2$ and $\mathcal{L}_2^0$ with norms, respectively, given by

$$\|\mathcal{S}\|_{\mathcal{L}_2}^2 = \langle \mathcal{S}, \mathcal{S} \rangle_{\mathcal{L}_2} = \sum_{j \in \mathbb{N}} (\mathcal{S}\eta_j, \mathcal{S}\eta_j)_{\mathbb{H}}, \quad \mathcal{S} \in \mathcal{L}_2,$$
$$\|\mathcal{T}\|_{\mathcal{L}_2^0}^2 = \langle \mathcal{T}, \mathcal{T} \rangle_{\mathcal{L}_2^0} = \sum_{j \in \mathbb{N}} (\mathcal{T}\bar{\eta}_j, \mathcal{T}\bar{\eta}_j)_{\mathbb{H}}, \quad \mathcal{T} \in \mathcal{L}_2^0,$$

which are independent of the specific choice of orthonormal basis $\{\eta_j\}_{j \in \mathbb{N}}$ in $\mathbb{H}$ and $\{\bar{\eta}_j\}_{j \in \mathbb{N}}$ in $\mathbb{H}_0$. We denote $W_Q^H(x,t)$ as $W_Q^H(t)$ and $\mathbb{E}$ as expectation operator in the following. Define $\mathbb{H} = L^2(\mathbb{D})$ with inner product $(\cdot,\cdot)$ and abbreviate $\|\cdot\|_{\mathcal{L}(\mathbb{H})}$ as $\|\cdot\|$. And define the space of $\mathbb{H}$-valued q-integrable random variables ($q > 0$) by

$$L^q(\Omega, \mathbb{H}) = \{v, \mathbb{E}\|v\|_{\mathbb{H}}^q < \infty\}$$

with norm $\|\cdot\|_{L^q(\Omega,\mathbb{H})} = \mathbb{E}(\|\cdot\|_{\mathbb{H}}^q)^{1/q}$.

For any $q \geq -1$, denote the space $\hat{H}^q(\mathbb{D}) = \mathcal{D}(A^{q/2})$ [38] with the norm given by

$$|\mu|^2_{\hat{H}^q(\mathbb{D})} = \left\| A^{\frac{q}{2}} \mu \right\|_{L^2(\mathbb{D})} = \left(\sum_{j=1}^{\infty} \varkappa_j^q (\mu, \varphi_j)^2 \right)^{\frac{1}{2}},$$

where $\{(\varkappa_j, \varphi_j)\}_{j=1}^{\infty}$ are A's eigenvalues ordered non-decreasingly and the corresponding eigenfunctions normalized in the $\mathbb{H}$ norm. The eigenvalues of Laplace operator satisfy the following estimates.

Lemma 2.1 ([20, 24]). *Let $\mathbb{D}$ be a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with volume $|\mathbb{D}|$. Denote $\varkappa_j$ as the j-th eigenvalue of the Dirichlet boundary problem for minus Laplace operator $-\Delta$ in $\mathbb{D}$. There is, for all $j \geq 1$,*

$$\varkappa_j \geq \frac{C_d d}{d+2} j^{\frac{2}{d}} |\mathbb{D}|^{-\frac{2}{d}},$$

where $C_d = (2\pi)^2 B_d^{-2/d}$ and B_d denotes the volume of the unit d-dimensional ball.

Then we recall some fractional Sobolev spaces [1–4, 6, 11, 27]. For a given open set $\mathbb{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, the fractional Sobolev spaces $H^s(\mathbb{D})$ with $s > 0$ are defined by

$$H^s(\mathbb{D}) = \left\{ w \in H^{\lfloor s \rfloor}(\mathbb{D}) : |w|^2_{H^s(\mathbb{D})} = \int\int_{\mathbb{D}^2} \frac{|D^{\lfloor s \rfloor} w(x) - D^{\lfloor s \rfloor} w(y)|^2}{|x-y|^{d+2(s-\lfloor s \rfloor)}} dx\, dy < \infty \right\}$$

with the norm

$$\|w\|_{H^s(\mathbb{D})} = \left(\|w\|^2_{H^{\lfloor s \rfloor}(\mathbb{D})} + |w|^2_{H^s(\mathbb{D})} \right)^{\frac{1}{2}},$$

where $\lfloor s \rfloor$ means the biggest integer not larger than s and $D^{\lfloor s \rfloor}$ is $\lfloor s \rfloor$-th order derivative. Introduce the subspace of $H^s(\mathbb{R}^d)$, consisting of the functions supported in $\mathbb{D}$ and $s \in (0, 1)$ by [2, 6]

$$\dot{H}^s(\mathbb{D}) = \{ w \in H^s(\mathbb{R}^d) : \operatorname{supp} w \subset \mathbb{D} \},$$

which can also be defined by complex interpolation when $\mathbb{D}$ is a Lipschitz domain [2, 6], i.e.

$$\dot{H}^s(\mathbb{D}) = \left[L^2(\mathbb{D}), H_0^1(\mathbb{D}) \right]_s.$$

The dual space of $\dot{H}^s(\mathbb{D})$ is denoted as $H^{-s}(\mathbb{D})$.

Remark 2.1. According to [2], for $s \in (0, 1)$, the norm of $\dot{H}^s(\mathbb{D})$ is induced by inner product, i.e.

$$\langle u, w \rangle_s := \frac{c_{d,s}}{2} \int\int_{(\mathbb{R}^d \times \mathbb{R}^d) \setminus (\mathbb{D}^c \times \mathbb{D}^c)} \frac{\big(u(x) - u(y)\big)\big(w(x) - w(y)\big)}{|x-y|^{d+2s}} dy\, dx \tag{2.1}$$

is a multiple of the $H^s(\mathbb{R}^d)$-seminorm. We can get that the $H^s(\mathbb{R}^d)$-seminorm is equivalent to the full $H^s(\mathbb{R}^d)$-norm on this space by using the fractional Poincaré-type inequality [11].

Remark 2.2. It is well-known that $\hat{H}^0(\mathbb{D}) = L^2(\mathbb{D}), \hat{H}^1(\mathbb{D}) = H_0^1(\mathbb{D})$ and $\hat{H}^2(\mathbb{D}) = H^2(\mathbb{D}) \bigcap H_0^1\ (\mathbb{D})$. According to [38], when $\mathbb{D}$ is a Lipschitz domain, we have $\hat{H}^{-s}(\mathbb{D}) = H^{-s}(\mathbb{D})$ with $s \in (0, 1)$ and $\hat{H}^s(\mathbb{D}) = \dot{H}^s(\mathbb{D})$ with $s \in [0, 1]$. Moreover, $\hat{H}^s(\mathbb{D}) = \dot{H}^s(\mathbb{D}) = H^s(\mathbb{D})$ for $s \in (0, 1/2)$.

Next, we recall the elliptic regularity of fractional Laplacian $\mathcal{A}^s$ in [2,13].

Theorem 2.1 ([13]). *Let $u \in \hat{H}^s(\mathbb{D})$ be the solution of the Dirichlet problem*

$$\begin{cases} (-\Delta)^s u = g, & \text{in } \mathbb{D}, \\ u = 0, & \text{in } \mathbb{D}^c, \end{cases} \tag{2.2}$$

where $\mathbb{D} \subset \mathbb{R}^d$ is a bounded domain with smooth boundary and $g \in H^\sigma(\mathbb{D})$ for some $\sigma \geq -s$ and $s \in (0,1)$. Then, there exists a constant C such that

$$|u|_{H^{s+\gamma}(\mathbb{R}^d)} \leq C\|g\|_{H^\sigma(\mathbb{D})},$$

where $\gamma = \min(s+\sigma, 1/2-\epsilon)$ with $\epsilon > 0$ arbitrarily small.

As for one-dimensional fBm process, we have the following fact [28,33]:

$$\int_0^t f(s)dW^H(s) = C_H\left(H-\frac{1}{2}\right)\int_0^t\int_s^t f(r)(r-s)^{H-\frac{3}{2}}\left(\frac{s}{r}\right)^{\frac{1}{2}-H} dr\, dW(s), \quad t \in [0,T], \tag{2.3}$$

where

$$C_H = \left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}\right)^{\frac{1}{2}}$$

and $W(s)$ is Brownian process.

Lastly, we define two sectors Σ_θ and $\Sigma_{\theta,\kappa}$ with $\kappa > 0$ and $\pi/2 < \theta < \pi$ as

$$\Sigma_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| \leq \theta\},$$
$$\Sigma_{\theta,\kappa} = \{z \in \mathbb{C} : |z| > \kappa, |\arg z| \leq \theta\},$$

and the contour $\Gamma_{\theta,\kappa}$ is given by

$$\Gamma_{\theta,\kappa} = \left\{\varrho e^{\pm \mathbf{i}\theta} : \varrho \geq \kappa\right\} \bigcup \left\{\kappa e^{\mathbf{i}\phi} : |\phi| \leq \theta\right\}, \quad \frac{\pi}{2} < \theta < \pi, \quad \kappa > 0,$$

where the circular arc is oriented counterclockwise and the two rays are oriented with an increasing imaginary part and $\mathbf{i}^2 = -1$.

3. A Priori Estimate of the Solution for (1.4)

With the help of Laplace transform and inverse Laplace transform, we write the mild solution of Eq. (1.4) as

$$u = \int_0^t \mathcal{R}(t-r)dW_Q^H(r), \tag{3.1}$$

where the operator $\mathcal{R}(t)$ is defined by

$$\mathcal{R}(t) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}} e^{zt}z^{\alpha-1}(z^\alpha + \mathcal{A}^s)^{-1}dz. \tag{3.2}$$

Lemma 3.1. *Let $\mathcal{A}^s$ be the fractional Laplacian with a zero Dirichlet boundary condition and $s \in (0,1)$. Assume $\alpha \in (0,1)$ and $z \in \Sigma_{\theta,\kappa}$ with $\pi/2 < \theta < \pi$. Then it follows that*

$$\|(z^\alpha + \mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D}) \to \hat{H}^{\sigma+2\mu s}(\mathbb{D})} \leq C|z|^{(\mu-1)\alpha},$$

where $\mu \in [0, \min(1, (1-\sigma)/(2s))]$ and $\sigma \in [-s, 1/2-s)$. When $\sigma \in [1/2-s, 0]$ with $s \in [1/2, 1)$, we have

$$\|(z^\alpha + \mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D}) \to L^2(\mathbb{D})} \leq C|z|^{(-\frac{\sigma}{2s}-1)\alpha}.$$

Proof. Let $(z^\alpha+\mathcal{A}^s)u=f$ in $\mathbb{D}$ with $z\in\Sigma_{\theta,\kappa}$, $u=0$ in $\mathbb{D}^c$ and $f\in\hat{H}^{-s}(\mathbb{D})$. Then it holds

$$|z^\alpha(u,u)+\langle u,u\rangle_s|=|(f,u)|\leq\|f\|_{\hat{H}^{-s}(\mathbb{D})}\|u\|_{\hat{H}^s(\mathbb{D})}.$$

Using the fact $a|z|+b\leq C|az+b|$ with $a,b\geq 0$ and $z\in\Sigma_\theta$ [12], one has

$$|z^\alpha(u,u)+\langle u,u\rangle_s|=\left|z^\alpha\|u\|^2_{L^2(\mathbb{D})}+\|u\|^2_{\hat{H}^s(\mathbb{D})}\right|\geq C|z|^\alpha\|u\|^2_{L^2(\mathbb{D})}+C\|u\|^2_{\hat{H}^s(\mathbb{D})},$$

which leads to

$$\|u\|_{\hat{H}^s(\mathbb{D})}\leq C\|f\|_{\hat{H}^{-s}(\mathbb{D})},\quad |z|^{\frac{\alpha}{2}}\|u\|_{L^2(\mathbb{D})}\leq C\|f\|_{\hat{H}^{-s}(\mathbb{D})}.$$

Thus

$$\begin{aligned}&\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^{-s}(\mathbb{D})\to\hat{H}^s(\mathbb{D})}\leq C,\\&\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^{-s}(\mathbb{D})\to L^2(\mathbb{D})}\leq C|z|^{-\frac{\alpha}{2}}.\end{aligned}$$

Due to the operator $\mathcal{A}^s:\ H^l(\mathbb{R}^d)\to H^{l-2s}(\mathbb{R}^d)$ is a bounded and invertible operator [2] and $H^{-s}(\mathbb{R}^d)\subset\hat{H}^{-s}(\mathbb{D})$, there holds

$$\|\mathcal{A}^s(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^{-s}(\mathbb{D})\to\hat{H}^{-s}(\mathbb{D})}\leq C.$$

Combining the fact

$$\mathcal{A}^s(z^\alpha+\mathcal{A}^s)^{-1}=I-z^\alpha(z^\alpha+\mathcal{A}^s)^{-1}$$

with I being an identity operator, we see

$$\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^{-s}(\mathbb{D})\to\hat{H}^{-s}(\mathbb{D})}\leq C|z|^{-\alpha}. \tag{3.3}$$

By interpolation property and the fact that

$$\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^\varsigma(\mathbb{D})\to\hat{H}^\varsigma(\mathbb{D})}\leq C|z|^{-\alpha}$$

for $\varsigma=\max(0,1/2-s)$ (see [2,34]), we obtain for $\sigma\in[-s,\max(0,1/2-s))$ and $s\in(0,1)$,

$$\begin{aligned}&\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D})\to\hat{H}^\sigma(\mathbb{D})}\leq C|z|^{-\alpha},\\&\|\mathcal{A}^s(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D})\to\hat{H}^\sigma(\mathbb{D})}\leq C,\end{aligned}$$

and for $s\in[1/2,1)$ and $\sigma\in[1/2-s,0]$, it holds

$$\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D})\to L^2(\mathbb{D})}\leq C|z|^{(-\frac{\sigma}{2s}-1)\alpha}.$$

Combining Theorem 2.1, the definition of $\dot{H}^s(\mathbb{D})$, and Remarks 2.1 and 2.2, we have for $s\in(0,1)$ and $\sigma\in[-s,1/2-s)$,

$$\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D})\to H^{\sigma+2s}(\mathbb{R}^d)}\leq C,$$

and

$$\|(z^\alpha+\mathcal{A}^s)^{-1}\|_{\hat{H}^\sigma(\mathbb{D})\to\hat{H}^{\sigma+2s}(\mathbb{D})}\leq C\quad\text{for}\quad\sigma+2s\in[0,1].$$

Then interpolation properties give the desired results. □

Further we can obtain the spatial regularity estimate of u.

Theorem 3.1. *Let u be the mild solution of Eq.* (1.4) *and* $\|A^{-\rho}\|_{\mathcal{L}_2^0} < \infty$ *with* $\rho \in (s/2 - 1/4, \min(s/2, sH/\alpha - \epsilon)]$. *Then there exists a positive constant C such that*

$$\|A^\sigma u(t)\|^2_{L^q(\Omega,\mathbb{H})} \le C,$$

where $q \ge 2$ *and* $2\sigma \in [0, \min(2s - 2\rho, 2sH/\alpha - 2\rho - \epsilon, 1)]$ *with* $\epsilon > 0$ *arbitrarily small.*

Proof. Equation (3.1) and Burkholder-Davis-Gundy inequality (cf. [37, Lemma 7.2] or [16, Lemma 2.1]) give

$$\begin{aligned}
\|A^\sigma u(t)\|^2_{L^q(\Omega,\mathbb{H})} &= \left\|\int_0^t A^\sigma \mathcal{R}(t-r)dW_Q^H(r)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
&\le C\left\|\int_0^t\int_r^t A^\sigma\mathcal{R}(t-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
&\le C\int_0^t\left\|\int_r^t A^\sigma\mathcal{R}(t-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'\right\|^2_{\mathcal{L}_2^0}dr\\
&\le Ct^{2H-1}\int_0^t\left(\int_r^t\|A^\sigma\mathcal{R}(t-r')\|_{\mathcal{L}_2^0}(r'-r)^{H-\frac{3}{2}}dr'\right)^2 r^{1-2H}dr.
\end{aligned}$$

Using $H \in (1/2, 1)$, the condition $\|A^{-\rho}\|_{\mathcal{L}_2^0} < \infty$, Eq. (3.2), and Lemma 3.1, we find that

$$\|A^\sigma\mathcal{R}(t-r')\|_{\mathcal{L}_2^0} \le C\|A^\sigma\mathcal{R}(t-r')A^\rho\| \le C(t-r')^{-\frac{\rho+\sigma}{s}\alpha},$$

which yields

$$\begin{aligned}
\|A^\sigma u(t)\|^2_{L^q(\Omega,\mathbb{H})} &\le Ct^{2H-1}\int_0^t\left(\int_r^t(t-r')^{-\frac{\rho+\sigma}{s}\alpha}(r'-r)^{H-\frac{3}{2}}dr'\right)^2 r^{1-2H}dr\\
&\le Ct^{2H-1}\int_0^t(t-r)^{-2\frac{\rho+\sigma}{s}\alpha+2H-1}r^{1-2H}dr,
\end{aligned}$$

where it is required that $-2\rho \in [-s, 1/2 - s)$ and $2\sigma \le \min(2s - 2\rho, 1)$. Moreover, to preserve the boundness of $\|A^\sigma u\|^2_{L^q(\Omega,\mathbb{H})}$, we need $H - (\rho+\sigma)\alpha/s > 0$, i.e. $0 \le \sigma < sH/\alpha - \rho$. Hence, the proof is complete. □

Lastly, we provide the Hölder regularity of the mild solution u.

Theorem 3.2. *Let u be the mild solution of Eq.* (1.4) *and* $\|A^{-\rho}\|_{\mathcal{L}_2^0} < \infty$ *with* $\rho \in [0, \min(s/2, sH/\alpha - \epsilon)]$. *Then u satisfies*

$$\left\|A^\sigma\big(u(t) - u(t-\tau)\big)\right\|^2_{L^q(\Omega,\mathbb{H})} \le C\tau^{2\gamma},$$

where $q \ge 2, 2\sigma \in [0, \min(2s - 2\rho, 2sH/\alpha - 2\rho - \epsilon, 1)]$ *and* $\gamma \in (0, H - (\sigma+\rho)\alpha/s)$.

Proof. Here we just give the proof for $\sigma = 0$, as for $\sigma \ne 0$, the corresponding estimates can be got similarly. Using (2.3), we arrive at

$$\left\|\frac{u(t)-u(t-\tau)}{\tau^\gamma}\right\|^2_{L^q(\Omega,\mathbb{H})} = \left\|\frac{1}{\tau^\gamma}\left(\int_0^t\mathcal{R}(t-r)dW_Q^H(r) - \int_0^{t-\tau}\mathcal{R}(t-\tau-r)dW_Q^H(r)\right)\right\|^2_{L^q(\Omega,\mathbb{H})}$$

$$
\begin{aligned}
&\leq C\Bigg\|\frac{1}{\tau^\gamma}\Bigg(\int_0^t\int_r^t \mathcal{R}(t-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\\
&\qquad -\int_0^{t-\tau}\int_r^{t-\tau}\mathcal{R}(t-\tau-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\Bigg)\Bigg\|^2_{L^q(\Omega,\mathbb{H})}\\
&\leq C\Bigg\|\frac{1}{\tau^\gamma}\int_0^{t-\tau}\int_r^{t-\tau}(\mathcal{R}(t-r')-\mathcal{R}(t-\tau-r'))(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\Bigg\|^2_{L^q(\Omega,\mathbb{H})}\\
&\quad +C\Bigg\|\frac{1}{\tau^\gamma}\int_{t-\tau}^{t}\int_r^{t}\mathcal{R}(t-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\Bigg\|^2_{L^q(\Omega,\mathbb{H})}\\
&\quad +C\Bigg\|\frac{1}{\tau^\gamma}\int_0^{t-\tau}\int_{t-\tau}^{t}\mathcal{R}(t-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\Bigg\|^2_{L^q(\Omega,\mathbb{H})}\\
&\leq I+II+III.
\end{aligned}
$$

Using Burkholder-Davis-Gundy inequality, one can obtain that

$$
I\leq C(t-\tau)^{2H-1}\int_0^{t-\tau}\left(\int_r^{t-\tau}\left\|\frac{\mathcal{R}(t-r')-\mathcal{R}(t-\tau-r')}{\tau^\gamma}\right\|_{\mathcal{L}_2^0}(r'-r)^{H-\frac{3}{2}}dr'\right)^2 r^{1-2H}dr.
$$

Lemma 3.1 and the fact that $|(e^{z\tau}-1)/\tau^\gamma|\leq|z|^\gamma$ with $z\in\Gamma_{\theta,\kappa}$ and $\gamma\in[0,1]$ [14] show that

$$
\begin{aligned}
&\left\|\frac{\mathcal{R}(t-r')-\mathcal{R}(t-\tau-r')}{\tau^\gamma}\right\|_{\mathcal{L}_2^0}\\
&\leq C\left\|\int_{\Gamma_{\theta,\kappa}}e^{z(t-\tau-r')}\frac{e^{z\tau}-1}{\tau^\gamma}z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}A^\rho dz\right\|\\
&\leq C\int_{\Gamma_{\theta,\kappa}}|e^{z(t-\tau-r')}|\,|z|^{\frac{\rho\alpha}{s}-1+\gamma}|dz|\\
&\leq C(t-\tau-r')^{-\gamma-\frac{\rho\alpha}{s}},
\end{aligned}
$$

which leads to

$$
I\leq C(t-\tau)^{2H-1}\int_0^{t-\tau}(t-\tau-r)^{2(H-\frac{1}{2}-\gamma-\frac{\rho\alpha}{s})}r^{1-2H}dr,
$$

where the last inequality holds when $-2\rho\in[-s,0]$ and $\gamma>0$. To preserve the boundness of I, it also needs that $\gamma\in(0,H-\rho\alpha/s)$.

Using Burkholder-Davis-Gundy inequality and Lemma 3.1 again, we get

$$
\begin{aligned}
II&\leq C\frac{1}{\tau^{2\gamma}}t^{2H-1}\int_{t-\tau}^t\left(\int_r^t\|\mathcal{R}(t-r')\|_{\mathcal{L}_2^0}(r'-r)^{H-\frac{3}{2}}dr'\right)^2 r^{1-2H}dr\\
&\leq C\frac{t^{2H-1}}{\tau^{2\gamma}}\int_{t-\tau}^t\left(\int_r^t\left\|\int_{\Gamma_{\theta,\kappa}}e^{z(t-r')}z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}A^\rho dz\right\|(r'-r)^{H-\frac{3}{2}}dr'\right)^2 r^{1-2H}dr\\
&\leq C\frac{1}{\tau^{2\gamma}}t^{2H-1}\int_{t-\tau}^t(t-r)^{2(H-\frac{1}{2}-\frac{\rho\alpha}{s})}r^{1-2H}dr,
\end{aligned}
$$

where the last inequality holds when $-2\rho\in[-s,0]$, and we require $\gamma\in(0,H-\rho\alpha/s)$ to ensure the boundness of II.

Similarly, for III, we obtain

$$\begin{aligned}
III \leq & C\frac{1}{\tau^{2\gamma}}\int_0^{t-\tau}\left(\int_{t-\tau}^{t}\|\mathcal{R}(t-r')A^{\rho}\|(r'-r)^{H-\frac{3}{2}}r'^{H-\frac{1}{2}}dr'\right)^2 r^{1-2H}dr\\
\leq & C\frac{1}{\tau^{2\gamma}}t^{2H-1}\int_0^{t-\tau}(t-\tau-r)^{-1+2\epsilon}r^{1-2H}\left(\int_{t-\tau}^{t}\|\mathcal{R}(t-r')A^{\rho}\|\big(r'-(t-\tau)\big)^{H-1-\epsilon}dr'\right)^2 dr\\
\leq & C\frac{1}{\tau^{2\gamma}}\tau^{2(H-\epsilon-\frac{\rho\alpha}{s})},
\end{aligned}$$

where the above inequalities hold when $-2\rho\in[-s,0]$ and $\gamma\in(0,1]$. Also, we need to require $\gamma\in(0,H-\rho\alpha/s)$ to preserve the boundness of III. Thus, combining I, II, and III leads to the desired estimate. □

4. Time Discretization and Error Analysis

In this section, we turn to the discretization in time. Backward Euler convolution quadrature introduced in [29–31] is used to discretize the Riemann-Liouville fractional derivative, and with the help of Burkholder-Davis-Gundy inequality, we obtain the corresponding error estimates of the temporal semi-discrete scheme.

Denote time step size $\tau=T/N$ ($N\in\mathbb{N}^+$) and $t_i=i\tau, i=0,1,\ldots,N$, where T is a fixed terminal time. Using backward Euler convolution quadrature method, we have the following semi-discrete scheme of (1.4):

$$\frac{u^n-u^{n-1}}{\tau}+\sum_{i=0}^{n-1}d_i^{(1-\alpha)}\mathcal{A}^s u^{n-i}=\bar{\partial}_\tau W_Q^H(t_n), \tag{4.1}$$

where $\{d_i^{(\alpha)}\}_{i=0}^{\infty}$ can be obtained by

$$\big(\delta_\tau(\zeta)\big)^\alpha=\left(\frac{1-\zeta}{\tau}\right)^\alpha=\sum_{i=0}^{\infty}d_i^{(\alpha)}\zeta^i,\quad \alpha\in(0,1), \tag{4.2}$$

and

$$\bar{\partial}_\tau W_Q^H(t)=\begin{cases}0, & t=t_0,\\ \frac{1}{\tau}\big(W_Q^H(t_j)-W_Q^H(t_{j-1})\big), & t\in(t_{j-1},t_j],\\ 0, & t>t_N.\end{cases} \tag{4.3}$$

Introduce the notation "$\sim$" as Laplace transform. The fact [14]

$$\sum_{n=1}^{\infty}\bar{\partial}_\tau W_Q^H(t_n)e^{-zt_n}=\frac{z}{e^{z\tau}-1}\widetilde{\bar{\partial}_\tau W_Q^H},$$

and simple calculations (which can refer to [14, 35]) lead to that the solution of Eq. (4.1) can be represented by

$$u^n=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}^{\tau}}e^{zt_n}\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\Big(\big(\delta_\tau(e^{-z\tau})\big)^\alpha+\mathcal{A}^s\Big)^{-1}\frac{z\tau}{e^{z\tau}-1}\widetilde{\bar{\partial}_\tau W_Q^H}dz,$$

where

$$\Gamma^{\tau}_{\theta,\kappa}=\left\{z\in\mathbb{C}:\kappa\leq|z|\leq\frac{\pi}{\tau\sin(\theta)},\,|\arg z|=\theta\right\}\cup\{z\in\mathbb{C}:|z|=\kappa,\,|\arg z|\leq\theta\}.$$

Let

$$\bar{\mathcal{R}}(t)=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^{\tau}_{\theta,\kappa}}e^{zt}\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^{\alpha}+\mathcal{A}^s\big)^{-1}\frac{z\tau}{e^{z\tau}-1}dz,$$

and the convolution property of Laplace transform gives

$$u^n=\int_0^{t_n}\bar{\mathcal{R}}(t_n-s)\bar{\partial}_\tau W_Q^H(s)ds. \tag{4.4}$$

Then we recall a lemma in [14], which is used to estimate $\|u(t_n)-u^n\|^2_{L^q(\Omega,\mathbb{H})}$.

Lemma 4.1 ([14]). *Let δ_τ be defined in (4.2),*

$$\Gamma^{\tau}_{\xi}=\left\{z=-\frac{\ln\xi}{\tau}+\mathbf{i}y:y\in\mathbb{R}\ \text{ and }\ |y|\leq\frac{\pi}{\tau}\right\}$$

with a fixed $\xi\in(0,1)$, and $\theta\in(\pi/2,\operatorname{arccot}(-2/\pi))$, where arccot denotes the inverse function of cot. If z lies in the region enclosed by Γ^{τ}_{ξ}, $\Gamma^{\tau}_{\theta,\kappa}$, and the two lines $\mathbb{R}\pm\mathbf{i}\pi/\tau$ with $0<\kappa\leq\min(1/T,-\ln(\xi)/\tau)$, then we have:

1. *$\delta_\tau(e^{-z\tau})$ and $(\delta_\tau(e^{-z\tau})+\mathcal{A}^s)^{-1}$ are both analytic.*
2. *There exist positive constants C_0,C_1 and C such that*

$$\begin{aligned}
&\delta_\tau\left(e^{-z\tau}\right)\in\Sigma_\theta, && \forall z\in\Gamma^{\tau}_{\theta,\kappa},\\
&C_0|z|\leq\left|\delta_\tau\left(e^{-z\tau}\right)\right|\leq C_1|z|, && \forall z\in\Gamma^{\tau}_{\theta,\kappa},\\
&\left|\delta_\tau\left(e^{-z\tau}\right)-z\right|\leq C\tau|z|^2, && \forall z\in\Gamma^{\tau}_{\theta,\kappa},\\
&\left|\delta_\tau\left(e^{-z\tau}\right)^{\alpha}-z^{\alpha}\right|\leq C\tau|z|^{\alpha+1}, && \forall z\in\Gamma^{\tau}_{\theta,\kappa},
\end{aligned}$$

where $\alpha\in(0,1),\kappa\in(0,\min(1/T,-\ln(\xi)/\tau))$. Here C_0,C_1 and C are independent of τ.

In the following proofs, we need to take $\kappa\leq\pi/(t_n|\sin(\theta)|)$. First, we provide a lemma, which is important in our error analyses.

Lemma 4.2. *For $\beta<1$ and $\gamma<0$, the following estimates hold:*

$$\begin{aligned}
&\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^{\tau}_{\theta,\kappa}}|e^{2z(t_n-r)}|\,|z|^{\gamma}|dz|r^{-\beta}dr\\
\leq{}& C\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^{\tau}_{\theta,\kappa}}|z|^{\gamma+\beta-1}e^{\cos(\theta)|z|t_n}|dz|+Ct_n^{1-\beta}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^{\tau}_{\theta,\kappa}}e^{C\cos(\theta)|z|t_n}|z|^{\gamma}|dz|\leq Ct_n^{-\beta-\gamma},\\
&\int_0^{t_n}\int_{\Gamma^{\tau}_{\theta,\kappa}}|e^{2z(t_n-r)}|\,|z|^{\gamma}|dz|r^{-\beta}dr\\
\leq{}& C\int_{\Gamma^{\tau}_{\theta,\kappa}}|z|^{\gamma+\beta-1}e^{\cos(\theta)|z|t_n}|dz|+Ct_n^{1-\beta}\int_{\Gamma^{\tau}_{\theta,\kappa}}e^{C\cos(\theta)|z|t_n}|z|^{\gamma}|dz|\leq Ct_n^{-\beta-\gamma}.
\end{aligned}$$

Proof. Using the mean value theorem, we have

$$\begin{aligned}
&\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|e^{2z(t_n-r)}|\,|z|^\gamma|dz|r^{-\beta}dr\\
\leq&\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{2\cos(\theta)|z|t_n}|z|^\gamma\int_0^{\frac{t_n}{4}}e^{-2\cos(\theta)|z|r}r^{-\beta}dr|dz|\\
&+\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{2\cos(\theta)|z|t_n}|z|^\gamma\int_{\frac{t_n}{4}}^{t_n}e^{-2\cos(\theta)|z|r}r^{-\beta}dr|dz|\\
\leq& C\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{\cos(\theta)|z|t_n}|z|^\gamma\int_0^{\frac{t_n}{4}}e^{\cos(\theta)|z|r}r^{-\beta}dr|dz|\\
&+Ct_n^{-\beta}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{2\cos(\theta)|z|t_n}|z|^\gamma\int_{\frac{t_n}{4}}^{t_n}e^{-2\cos(\theta)|z|r}dr|dz|\\
\leq& C\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{\gamma+\beta-1}e^{\cos(\theta)|z|t_n}|dz|\\
&+Ct_n^{1-\beta}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{C\cos(\theta)|z|t_n}|z|^\gamma|dz|\leq Ct_n^{-\beta-\gamma}.
\end{aligned}$$

Similarly, there is

$$\begin{aligned}
&\int_0^{t_n}\int_{\Gamma^\tau_{\theta,\kappa}}|e^{2z(t_n-r)}|\,|z|^\gamma|dz|r^{-\beta}dr\\
\leq&\int_{\Gamma^\tau_{\theta,\kappa}}e^{2\cos(\theta)|z|t_n}|z|^\gamma\int_0^{\frac{t_n}{4}}e^{-2\cos(\theta)|z|r}r^{-\beta}dr|dz|\\
&+\int_{\Gamma^\tau_{\theta,\kappa}}e^{2\cos(\theta)|z|t_n}|z|^\gamma\int_{\frac{t_n}{4}}^{t_n}e^{-2\cos(\theta)|z|r}r^{-\beta}dr|dz|\\
\leq& C\int_{\Gamma^\tau_{\theta,\kappa}}e^{\cos(\theta)|z|t_n}|z|^\gamma\int_0^{\frac{t_n}{4}}e^{\cos(\theta)|z|r}r^{-\beta}dr|dz|\\
&+Ct_n^{-\beta}\int_{\Gamma^\tau_{\theta,\kappa}}e^{2\cos(\theta)|z|t_n}|z|^\gamma\int_{\frac{t_n}{4}}^{t_n}e^{-2\cos(\theta)|z|r}dr|dz|\\
\leq& C\int_{\Gamma^\tau_{\theta,\kappa}}|z|^{\gamma+\beta-1}e^{\cos(\theta)|z|t_n}|dz|+Ct_n^{1-\beta}\int_{\Gamma^\tau_{\theta,\kappa}}e^{C\cos(\theta)|z|t_n}|z|^\gamma|dz|\leq Ct_n^{-\beta-\gamma}.
\end{aligned}$$

This ends the proof. □

Next, we provide the error estimates for the time semi-discrete scheme (4.1).

Theorem 4.1. *Let $u(t)$ and u^n be the solutions of Eqs.* (1.4) *and* (4.1)*, respectively. Assume $\|A^{-\rho}\|_{\mathcal{L}_2^0}<\infty$ with $\rho\in[0,\min(s/2,sH/\alpha)]$, $s\in(0,1)$, $\alpha\in(0,1)$ and $H\in(1/2,1)$. Then it holds*

$$\|u(t_n)-u^n\|^2_{L^q(\Omega,\mathbb{H})}\leq C\tau^{2H-\frac{2\rho\alpha}{s}}$$

with $q\geq 2$.

Proof. Subtracting (4.4) from (3.1) yields

$$\|u(t_n)-u^n\|^2_{L^q(\Omega,\mathbb{H})}=\left\|\int_0^{t_n}\mathcal{R}(t_n-r)dW^H_Q(r)-\int_0^{t_n}\bar{\mathcal{R}}(t_n-r)\big(\bar\partial_\tau W^H_Q(r)\big)dr\right\|^2_{L^q(\Omega,\mathbb{H})}$$

$$\begin{aligned}
&\le C\left\|\int_0^{t_n}\big(\mathcal{R}(t_n-r)-\bar{\mathcal{R}}(t_n-r)\big)dW_Q^H(r)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
&\quad+C\left\|\int_0^{t_n}\bar{\mathcal{R}}(t_n-r)\big(dW_Q^H(r)-\bar{\partial}_\tau W_Q^H(r)dr\big)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
&=\vartheta_1+\vartheta_2.
\end{aligned}\tag{4.5}$$

Here, using (2.3) and Burkholder-Davis-Gundy inequality, we separate ϑ_1 into two parts

$$\begin{aligned}
\vartheta_1\le{}& C\left\|\int_0^{t_n}\int_r^{t_n}\big(\mathcal{R}(t_n-r')-\bar{\mathcal{R}}(t_n-r')\big)(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
\le{}& Ct_n^{2H-1}\int_0^{t_n}\left\|\int_r^{t_n}\big(\mathcal{R}(t_n-r')-\bar{\mathcal{R}}(t_n-r')\big)(r'-r)^{H-\frac{3}{2}}dr'\right\|^2_{\mathcal{L}_2^0}r^{1-2H}dr\\
\le{}& Ct_n^{2H-1}\int_0^{t_n}\left(\int_r^{t_n}\left\|\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-r')}z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}A^\rho dz\right\|(r'-r)^{H-\frac{3}{2}}dr'\right)^2r^{1-2H}dr\\
&+Ct_n^{2H-1}\int_0^{t_n}\left(\int_r^{t_n}\left\|\int_{\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-r')}\left(z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}\right.\right.\right.\\
&\left.\left.\qquad-\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^\alpha+\mathcal{A}^s\big)^{-1}\frac{z\tau}{e^{z\tau}-1}\right)A^\rho dz\right\|\\
&\left.\qquad\times(r'-r)^{H-\frac{3}{2}}dr'\right)^2r^{1-2H}dr\\
\le{}&\vartheta_{1,1}+\vartheta_{1,2}.
\end{aligned}$$

By using of $\rho\in[0,\min(s/2,sH/\alpha)]$ and Lemmas 3.1, 4.2, one can get

$$\begin{aligned}
&\int_r^{t_n}\left\|\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-r')}z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}A^\rho dz\right\|(r'-r)^{H-\frac{3}{2}}dr'\\
\le{}& C\int_r^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|e^{z(t_n-r')}|\,|z|^{\frac{\rho\alpha}{s}-1}|dz|(r'-r)^{H-\frac{3}{2}}dr'\\
\le{}& C\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{\frac{\rho\alpha}{s}-H-\frac{1}{2}}e^{\cos(\theta)|z|(t_n-r)}|dz|\\
&+C(t_n-r)^{H-\frac{1}{2}}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{\frac{\rho\alpha}{s}-1}e^{C\cos(\theta)|z|(t_n-r)}|dz|.
\end{aligned}$$

Using Cauchy-Schwarz inequality leads to

$$\begin{aligned}
\vartheta_{1,1}\le{}& Ct_n^{2H-1}\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{\frac{2\rho\alpha}{s}-2H-1+\epsilon}|dz|\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{-\epsilon}e^{2\cos(\theta)|z|(t_n-r)}|dz|r^{1-2H}dr\\
&+Ct_n^{2H-1}\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{\frac{2\rho\alpha}{s}-2+\epsilon}e^{C\cos(\theta)|z|(t_n-r)}|dz|\\
&\times\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{-\epsilon}e^{C\cos(\theta)|z|(t_n-r)}|dz|(t_n-r)^{2H-1}r^{1-2H}dr\\
\le{}& C\tau^{2H-\frac{2\rho\alpha}{s}-\epsilon}t_n^{2H-1}\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{-\epsilon}e^{2\cos(\theta)|z|(t_n-r)}|dz|r^{1-2H}dr
\end{aligned}$$

$$
\begin{aligned}
&+C\tau^{2H-\frac{2\rho\alpha}{s}-\epsilon}t_n^{2H-1}\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma_{\theta,\kappa}^\tau}|z|^{2H-2}e^{C\cos(\theta)|z|(t_n-r)}|dz|\\
&\times\int_{\Gamma_{\theta,\kappa}\backslash\Gamma_{\theta,\kappa}^\tau}|z|^{-\epsilon}e^{C\cos(\theta)|z|(t_n-r)}|dz|(t_n-r)^{2H-1}r^{1-2H}dr\leq C\tau^{2H-\frac{2\rho\alpha}{s}}.
\end{aligned}
$$

By Lemmas 3.1 and 4.1, we have

$$
\begin{aligned}
&\left\|\left(z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}-\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1}\left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha+\mathcal{A}^s\right)^{-1}\frac{z\tau}{e^{z\tau}-1}\right)A^\rho\right\|\\
\leq&\left\|\left(z^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}-\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}\right)A^\rho\right\|\\
&+\left\|\left(\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1}(z^\alpha+\mathcal{A}^s)^{-1}-\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1}\left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha+\mathcal{A}^s\right)^{-1}\right)A^\rho\right\|\\
&+\left\|\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1}\left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha+\mathcal{A}^s\right)^{-1}\left(1-\frac{z\tau}{e^{z\tau}-1}\right)A^\rho\right\|\leq C\tau|z|^{\frac{\rho\alpha}{s}}.
\end{aligned}
$$

The above estimate and Lemma 4.2 lead to

$$
\begin{aligned}
\vartheta_{1,2}\leq& C\tau^2t_n^{2H-1}\int_0^{t_n}\left(\int_r^{t_n}\int_{\Gamma_{\theta,\kappa}^\tau}|e^{z(t_n-r')}|\,|z|^{\frac{\rho\alpha}{s}}|dz|(r'-r)^{H-\frac{3}{2}}dr'\right)^2r^{1-2H}dr\\
\leq& C\tau^2t_n^{2H-1}\int_0^{t_n}\left(\int_{\Gamma_{\theta,\kappa}^\tau}e^{\cos(\theta)|z|(t_n-r)}|z|^{\frac{\rho\alpha}{s}+\frac{1}{2}-H}|dz|\right.\\
&\left.\qquad+(t_n-r)^{H-\frac{1}{2}}\int_{\Gamma_{\theta,\kappa}^\tau}e^{C|z|\cos(\theta)t_n}|z|^{\frac{\rho\alpha}{s}}|dz|\right)^2r^{1-2H}dr\\
\leq& C\tau^2t_n^{2H}\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{\frac{2\rho\alpha}{s}+1-2H-\epsilon}|dz|\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{\epsilon}e^{2|z|\cos(\theta)(t_n-r)}|dz|r^{1-2H}dr\\
&+C\tau^2t_n^{2H-1}\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{\frac{2\rho\alpha}{s}-\epsilon}e^{C|z|\cos(\theta)(t_n-r)}|dz|\\
&\times\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{\epsilon}e^{C|z|\cos(\theta)(t_n-r)}|dz|(t_n-r)^{2H-1}r^{1-2H}dr.
\end{aligned}\tag{4.6}
$$

Simple calculations give

$$
\begin{aligned}
&\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{\frac{2\rho\alpha}{s}-\epsilon}e^{C|z|\cos(\theta)(t_n-r)}|dz|\\
\leq& C\left(\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{\frac{4\rho\alpha}{s}+3-4H-\epsilon}|dz|\right)^{\frac{1}{2}}\left(\int_{\Gamma_{\theta,\kappa}^\tau}|z|^{4H-3}e^{C|z|\cos(\theta)(t_n-r)}|dz|\right)^{\frac{1}{2}}\\
\leq& C\tau^{2H-2-\frac{2\rho\alpha}{s}+\epsilon}(t_n-r)^{1-2H}.
\end{aligned}
$$

Thus

$$
\vartheta_{1,2}\leq C\tau^{2H-\frac{2\rho\alpha}{s}}.
$$

As for ϑ_2, by the definition of $\bar{\partial}_\tau W_Q^H(r)$, we have

$$
\vartheta_2\leq C\left\|\int_0^{t_n}\bar{\mathcal{R}}(t_n-r)\left(dW_Q^H(r)-\bar{\partial}_\tau W_Q^H(r)dr\right)\right\|_{L^q(\Omega,\mathbb{H})}^2
$$

$$
\begin{aligned}
&\leq C\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{1}{\tau}\int_{t_{i-1}}^{t_i}\big(\bar{\mathcal{R}}(t_n-r)-\bar{\mathcal{R}}(t_n-\xi)\big)d\xi\,dW_Q^H(r)\right\|_{L^q(\Omega,\mathbb{H})}^2\\
&\leq C\left\|\int_0^{t_n}\frac{1}{\tau}\sum_{i=1}^{n}\chi_{(t_{i-1},t_i]}(r)\int_{t_{i-1}}^{t_i}\big(\bar{\mathcal{R}}(t_n-r)-\bar{\mathcal{R}}(t_n-\xi)\big)d\xi\,dW_Q^H(r)\right\|_{L^q(\Omega,\mathbb{H})}^2,
\end{aligned}
$$

where $\chi_{(a,b)}(r)$ means the characteristic function on (a,b). Introduce

$$
\mathcal{G}(t_n-r)=\frac{1}{\tau}\sum_{i=1}^{n}\chi_{(t_{i-1},t_i]}(r)\int_{t_{i-1}}^{t_i}\big(\bar{\mathcal{R}}(t_n-r)-\bar{\mathcal{R}}(t_n-\xi)\big)d\xi.
$$

After simple calculations, there holds

$$
\begin{aligned}
\mathcal{G}(t_n-r)=&\frac{1}{2\pi\mathbf{i}\tau}\sum_{i=1}^{n}\chi_{(t_{i-1},t_i]}(r)\int_{t_{i-1}}^{t_i}\int_{\Gamma_{\theta,\kappa}^\tau}\big(e^{z(t_n-r)}-e^{z(t_n-\xi)}\big)\\
&\qquad\times\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^{\alpha}+\mathcal{A}^s\big)^{-1}\frac{z\tau}{e^{z\tau}-1}dz\,d\xi\\
=&\frac{1}{2\pi\mathbf{i}}\sum_{i=1}^{n}\chi_{(t_{i-1},t_i]}(r)\Bigg(\int_{\Gamma_{\theta,\kappa}^\tau}e^{z(t_n-r)}\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^{\alpha}+\mathcal{A}^s\big)^{-1}\frac{z\tau}{e^{z\tau}-1}dz\\
&\qquad-\int_{\Gamma_{\theta,\kappa}^\tau}e^{z(t_n-r)}e^{z(r-t_i)}\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^{\alpha}+\mathcal{A}^s\big)^{-1}dz\Bigg).
\end{aligned}
$$

Simple calculations, (2.3), and $\|A^{-\rho}\|_{\mathcal{L}_2^0}<\infty$ give

$$
\begin{aligned}
\vartheta_2\leq& C\left\|\int_0^{t_n}\mathcal{G}(t_n-r)dW_Q^H(r)\right\|_{L^q(\Omega,\mathbb{H})}^2\\
\leq& C\left\|\int_0^{t_n}\int_r^{t_n}\mathcal{G}(t_n-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\right\|_{L^q(\Omega,\mathbb{H})}^2\\
\leq& C\int_0^{t_n}\left\|\int_r^{t_n}\mathcal{G}(t_n-r')(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'\right\|_{\mathcal{L}_2^0}^2dr\\
\leq& Ct_n^{2H-1}\int_0^{t_n}\left(\int_r^{t_n}\|\mathcal{G}(t_n-r')\|_{\hat{H}^{-2\rho}(\mathbb{D})\to L^2(\mathbb{D})}(r'-r)^{H-\frac{3}{2}}dr'\right)^2r^{1-2H}dr.
\end{aligned}
$$

By Lemmas 3.1 and 4.1, there holds

$$
\begin{aligned}
&\|\mathcal{G}(t_n-r)\|_{\hat{H}^{-2\rho}(\mathbb{D})\to L^2(\mathbb{D})}\\
\leq& C\sum_{i=1}^{n}\chi_{(t_{i-1},t_i]}(r)\int_{\Gamma_{\theta,\kappa}^\tau}|e^{z(t_n-r)}|\left|1-\frac{z\tau}{e^{z\tau}-1}\right|\\
&\qquad\times\big\|\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^{\alpha}+\mathcal{A}^s\big)^{-1}\big\|_{\hat{H}^{-2\rho}(\mathbb{D})\to L^2(\mathbb{D})}|dz|\\
&+C\sum_{i=1}^{n}\chi_{(t_{i-1},t_i]}(r)\int_{\Gamma_{\theta,\kappa}^\tau}|e^{z(t_n-r)}|\left|1-e^{z(r-t_i)}\right|\\
&\qquad\times\big\|\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big(\big(\delta_\tau(e^{-z\tau})\big)^{\alpha}+\mathcal{A}^s\big)^{-1}\big\|_{\hat{H}^{-2\rho}(\mathbb{D})\to L^2(\mathbb{D})}|dz|\\
\leq& C\tau\int_{\Gamma_{\theta,\kappa}^\tau}|e^{z(t_n-r)}|\,|z|^{\frac{\rho\alpha}{s}}|dz|.
\end{aligned}
$$

Thus similar to the derivation of (4.6), we have

$$\vartheta_2 \leq C\tau^{2H-\frac{2\rho\alpha}{s}}.$$

Collecting above estimates about ϑ_1 and ϑ_2 leads to the desired result. □

5. Spatial Discretization and Error Analysis

Now we begin by using the finite element method to discretize the integral fractional Laplacian operator, and then the error estimates for the fully discrete scheme of Eq. (1.4) are also provided.

Denote X_h as piecewise linear finite element space

$$X_h = \left\{\vartheta_h \in C(\bar{\mathbb{D}}) : \vartheta_h|_{\mathbf{T}} \in \mathcal{P}^1,\ \forall \mathbf{T} \in \mathcal{T}_h,\ \vartheta_h|_{\partial\mathbb{D}} = 0\right\},$$

where $\mathcal{T}_h$ is a shape regular quasi-uniform partition of the domain $\mathbb{D}$, h is the maximum diameter, and $\mathcal{P}^1(\mathbf{T})$ denotes the set of polynomials on the element $\mathbf{T}$ with degree no more than 1. Introduce the L^2-orthogonal projection $P_h : \mathbb{H} \to X_h$ [38] by

$$(P_h u, \vartheta_h) = (u, \vartheta_h), \quad \forall \vartheta_h \in X_h,$$

and $\mathcal{A}_h^s$ is defined by $(\mathcal{A}_h^s u_h, \vartheta_h) = \langle u_h, \vartheta_h\rangle_s$ for $u_h, \vartheta_h \in X_h$. The fully discrete Galerkin scheme for Eq. (1.4) reads: For every $t \in (0, T]$, find $u_h^n \in X_h$ such that

$$\begin{cases} \left(\dfrac{u_h^n - u_h^{n-1}}{\tau}, \vartheta_h\right) + \sum\limits_{i=0}^{n-1} d_i^{(1-\alpha)} \left\langle u_h^{n-i}, \vartheta_h\right\rangle_s \\ = \left(\dfrac{W_Q^H(t_n) - W_Q^H(t_{n-1})}{\tau}, \vartheta_h\right), \quad \forall \vartheta_h \in X_h, \\ u_h^0 = 0. \end{cases} \tag{5.1}$$

Also, (5.1) can be written as

$$\frac{u_h^n - u_h^{n-1}}{\tau} + \sum_{i=0}^{n-1} d_i^{(1-\alpha)} \mathcal{A}_h^s u_h^{n-i} = P_h \frac{W_Q^H(t_n) - W_Q^H(t_{n-1})}{\tau}. \tag{5.2}$$

Following the derivations in [14, 15, 35], the solution of (5.1) can be written as

$$u_h^n = \int_0^{t_n} \frac{1}{\tau} \sum_{i=1}^{n} \chi_{(t_{i-1}, t_i]}(s) \int_{t_{i-1}}^{t_i} \bar{\mathcal{R}}_h(t_n - \xi) P_h d\xi \, dW_Q^H(s), \tag{5.3}$$

where

$$\bar{\mathcal{R}}_h(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt} \left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1} \left(\left(\delta_\tau(e^{-z\tau})\right)^{\alpha} + \mathcal{A}_h^s\right)^{-1} \frac{z\tau}{e^{z\tau} - 1} dz.$$

Similarly, the solution of (4.1) can be represented as

$$u^n = \int_0^{t_n} \frac{1}{\tau} \sum_{i=1}^{n} \chi_{(t_{i-1}, t_i]}(s) \int_{t_{i-1}}^{t_i} \bar{\mathcal{R}}(t_n - \xi) d\xi \, dW_Q^H(s). \tag{5.4}$$

Lemma 5.1. *Let $\mathcal{A}^s = (-\Delta)^s$ with homogeneous Dirichlet boundary condition, $s \in (0,1)$, and $z \in \Gamma_{\theta,\kappa}$ with $\theta \in (\pi/2, \pi)$. Assume $v \in \hat{H}^\sigma(\mathbb{D})$ with $\sigma \in [-s, 1/2-s)$. Denote $w = (z^\alpha + \mathcal{A}^s)^{-1} v$ and $w_h = (z^\alpha + \mathcal{A}_h^s)^{-1} P_h v$. Then one has*

$$\|w - w_h\|_{\mathbb{H}} + h^{\min(s, \frac{1}{2}-\epsilon)} \|w - w_h\|_{\hat{H}^s(\mathbb{D})} \le C h^\gamma \|v\|_{\hat{H}^\sigma(\mathbb{D})},$$

where

$$\gamma = \begin{cases} 2s + \sigma, & s \in (0, 1/2), \\ s + \sigma + 1/2 - \epsilon, & s \in [1/2, 1). \end{cases}$$

Proof. Let $\epsilon > 0$ be arbitrarily small. According to the definitions of w and w_h, there hold

$$\begin{aligned} z^\alpha (w, \chi) + \langle w, \chi \rangle_s &= (v, \chi), \qquad \forall \chi \in \hat{H}^s(\mathbb{D}), \\ z^\alpha (w_h, \chi_h) + \langle w_h, \chi_h \rangle_s &= (v, \chi_h), \quad \forall \chi_h \in X_h. \end{aligned}$$

Thus

$$z^\alpha (e, \chi_h) + \langle e, \chi_h \rangle_s = 0, \quad \forall \chi_h \in X_h,$$

where $e = w - w_h$. Then one has

$$\begin{aligned} |z|^\alpha \|e\|^2_{\mathbb{H}} + \|e\|^2_{\hat{H}^s(\mathbb{D})} &\le C \big| |z|^\alpha \|e\|^2_{\mathbb{H}} + \|e\|^2_{\hat{H}^s(\mathbb{D})} \big| \\ &= C \big| |z|^\alpha (e, w - \chi) + \langle e, (w - \chi) \rangle_s \big|. \end{aligned}$$

Taking $\chi_h = \pi_h w$ as the suitable quasi-interpolation [2,3] of w and using the Cauchy-Schwarz inequality, we obtain

$$|z|^\alpha \|e\|^2_{\mathbb{H}} + \|e\|^2_{\hat{H}^s(\mathbb{D})} \le C h^{s+\sigma} |z|^\alpha \|e\|_{\mathbb{H}} \|w\|_{\hat{H}^{s+\sigma}(\mathbb{D})} + C h^{s+\sigma} \|e\|_{\hat{H}^s(\mathbb{D})} \|w\|_{\hat{H}^{2s+\sigma}(\mathbb{D})}.$$

According to Lemma 3.1, one has

$$|z|^\alpha \|e\|^2_{\mathbb{H}} + \|e\|^2_{\hat{H}^s(\mathbb{D})} \le C h^{s+\sigma} \|v\|_{\hat{H}^\sigma(\mathbb{D})} \left(|z|^{\frac{\alpha}{2}} \|e\|_{\mathbb{H}} + \|e\|_{\hat{H}^s(\mathbb{D})} \right).$$

So

$$|z|^{\frac{\alpha}{2}} \|e\|_{\mathbb{H}} + \|e\|_{\hat{H}^s(\mathbb{D})} \le C h^{s+\sigma} \|v\|_{\hat{H}^\sigma(\mathbb{D})}.$$

Similarly, for $\phi \in \mathbb{H}$, we set

$$\varphi = (z^\alpha + A)^{-1} \phi, \quad \varphi_h = (z^\alpha + A_h)^{-1} P_h \phi.$$

By a duality argument, one has

$$\|e\|_{\mathbb{H}} = \sup_{\phi \in \mathbb{H}} \frac{|(e, \phi)|}{\|\phi\|_{\mathbb{H}}} = \sup_{\phi \in \mathbb{H}} \frac{|z^\alpha (e, \varphi) + \langle e, \varphi \rangle_s|}{\|\phi\|_{\mathbb{H}}}.$$

Then, using the fact that [2]

$$|z|^{\frac{\alpha}{2}} \|\varphi - \varphi_h\|_{\mathbb{H}} + \|\varphi - \varphi_h\|_{\hat{H}^s(\mathbb{D})} \le C h^{\min(s, \frac{1}{2}-\epsilon)} \|\phi\|_{\mathbb{H}},$$

we have

$$\begin{aligned} \big| z^\alpha (e, \varphi) + \langle e, \varphi \rangle_s \big| &= \big| |z|^\alpha (e, \varphi - \varphi_h) + \langle e, (\varphi - \varphi_h) \rangle_s \big| \\ &\le |z|^{\frac{\alpha}{2}} \|e\|_{\mathbb{H}} |z|^{\frac{\alpha}{2}} \|\varphi - \varphi_h\|_{\mathbb{H}} + \|e\|_{\hat{H}^s(\mathbb{D})} \|\varphi - \varphi_h\|_{\hat{H}^s(\mathbb{D})} \\ &\le C h^\gamma \|v\|_{\hat{H}^\sigma(\mathbb{D})} \|\phi\|_{\mathbb{H}}, \end{aligned}$$

where

$$\gamma = \begin{cases} 2s + \sigma, & s \in (0, 1/2), \\ s + \sigma + 1/2 - \epsilon, & s \in [1/2, 1). \end{cases}$$

So we complete the proof. □

Besides, for $s \in (0,1)$, introduce the fractional Ritz projection $R_h^s : \hat{H}^s(\mathbb{D}) \to X_h$ defined by

$$\big(\mathcal{A}^s(u - R_h^s u), \vartheta_h\big) = \big\langle u - R_h^s u, \vartheta_h \big\rangle_s = 0, \quad \forall \vartheta_h \in X_h,$$

and it has the following properties.

Lemma 5.2. *Assume $u \in \hat{H}^\gamma(\mathbb{D})$ with $\gamma \in [s, s+1/2)$. Then we have*

$$\begin{aligned}
&\big\|R_h^s u\big\|_{\hat{H}^s(\mathbb{D})} \le C\|u\|_{\hat{H}^s(\mathbb{D})},\\
&\big\|R_h^s u - u\big\|_{\mathbb{H}} + h^{\min(s,\frac{1}{2}-\epsilon)}\big\|R_h^s u - u\big\|_{\hat{H}^s(\mathbb{D})} \le C h^{\gamma - s + \min(s,\frac{1}{2}-\epsilon)}\|u\|_{\hat{H}^\gamma(\mathbb{D})}.
\end{aligned}$$

Proof. Simple calculations lead to

$$\begin{aligned}
\big\|R_h^s u\big\|_{\hat{H}^s(\mathbb{D})} &\le C\sqrt{\big(\mathcal{A}^s R_h^s u, R_h^s u\big)} \le C \sup_{v_h \in X_h} \frac{\big(\mathcal{A}^s R_h^s u, v_h\big)}{\|v_h\|_{\hat{H}^s(\mathbb{D})}}\\
&\le C \sup_{v_h \in X_h} \frac{\big(\mathcal{A}^s u, v_h\big)}{\|v_h\|_{\hat{H}^s(\mathbb{D})}} \le C \sup_{v \in \hat{H}^s(\mathbb{D})} \frac{\big(\mathcal{A}^s u, v\big)}{\|v\|_{\hat{H}^s(\mathbb{D})}} \le C\|u\|_{\hat{H}^s(\mathbb{D})}.
\end{aligned}$$

Similar to the proof of Lemma 5.1, we can obtain the second estimate. □

Combining the definitions of $\mathcal{A}^s, \mathcal{A}_h^s$, and P_h results in

$$\big(\mathcal{A}_h^s R_h^s u, v_h\big) = \big\langle R_h^s u, v_h \big\rangle_s = \langle u, v_h\rangle_s = \big(\mathcal{A}^s u, v_h\big) = \big(P_h \mathcal{A}^s u, v_h\big),$$

which leads to

$$\mathcal{A}_h^s R_h^s = P_h \mathcal{A}^s.$$

Taking $\mathcal{A}^s \varphi = \phi$ and $\mathcal{A}_h^s R_h^s \varphi = P_h \phi$ with $\phi \in \mathbb{H}$ and $s \in (0,1)$, yields

$$\begin{aligned}
\big\|\big(\mathcal{A}_h^s\big)^{-\frac{1}{2}} P_h \phi\big\|_{\mathbb{H}} &= \Big(\big(\mathcal{A}_h^s\big)^{-\frac{1}{2}} P_h \phi, \big(\mathcal{A}_h^s\big)^{-\frac{1}{2}} P_h \phi\Big)^{\frac{1}{2}}\\
&= \big(P_h \phi, \big(\mathcal{A}_h^s\big)^{-1} P_h \phi\big)^{\frac{1}{2}}\\
&= \big(\mathcal{A}_h^s R_h^s \varphi, R_h^s \varphi\big)^{\frac{1}{2}} = \big\|R_h^s \varphi\big\|_{\hat{H}^s(\mathbb{D})}\\
&\le C\|\varphi\|_{\hat{H}^s(\mathbb{D})} \le C\big\|A^{-\frac{s}{2}}\phi\big\|_{\mathbb{H}}.
\end{aligned}$$

By the stability of L^2 projection and interpolation theory [4], we have for $\phi \in \mathbb{H}$ and $s \in (0,1)$,

$$\big\|(\mathcal{A}_h^s)^{-\frac{\sigma}{2}} P_h \phi\big\|_{\mathbb{H}} \le C\big\|A^{-\frac{s\sigma}{2}}\phi\big\|_{\mathbb{H}}, \quad \sigma \in [0,1],$$

leading to

$$\begin{aligned}
&\big\|(z^\alpha + \mathcal{A}^s)^{-1} A^\sigma - (z^\alpha + \mathcal{A}_h^s)^{-1} P_h A^\sigma\big\|\\
\le{}& \|(z^\alpha + \mathcal{A}^s)^{-1} A^\sigma\| + \big\|(z^\alpha + \mathcal{A}_h^s)^{-1} P_h A^\sigma\big\|\\
\le{}& C|z|^{(\frac{\sigma}{s}-1)\alpha} + \big\|(z^\alpha + \mathcal{A}_h^s)^{-1}(\mathcal{A}_h^s)^{\frac{\sigma}{s}}(\mathcal{A}_h^s)^{-\frac{\sigma}{s}} P_h A^\sigma\big\|\\
\le{}& C|z|^{(\frac{\sigma}{s}-1)\alpha}, \quad \sigma \in [0, s/2],
\end{aligned} \tag{5.5}$$

where we use Lemma 3.1. For $s \in (0,1/2)$ and $\phi \in \hat{H}^{2s}(\mathbb{D})$, by Lemma 5.2, we obtain

$$\begin{aligned}
&\big\|\big((z^\alpha + \mathcal{A}^s)^{-1} - \big(z^\alpha + \mathcal{A}_h^s\big)^{-1} R_h^s\big)\phi\big\|_{\mathbb{H}}\\
={}& \big\|z^{-\alpha}\big(I - \mathcal{A}^s(z^\alpha + \mathcal{A}^s)^{-1}\big)\phi - z^{-\alpha}\big(R_h^s - \mathcal{A}_h^s\big(z^\alpha + \mathcal{A}_h^s\big)^{-1} R_h^s\big)\phi\big\|_{\mathbb{H}}\\
\le{}& |z|^{-\alpha}\big\|(I - R_h^s)\phi\big\|_{\mathbb{H}} + |z|^{-\alpha}\big\|\big((z^\alpha + \mathcal{A}^s)^{-1} - \big(z^\alpha + \mathcal{A}_h^s\big)^{-1} P_h\big)\mathcal{A}^s \phi\big\|_{\mathbb{H}}\\
\le{}& C|z|^{-\alpha} h^{2s}\|\phi\|_{\hat{H}^{2s}(\mathbb{D})},
\end{aligned}$$

which gives

$$\begin{aligned}
&\left\|\left((z^\alpha+\mathcal{A}^s)^{-1}-\left(z^\alpha+\mathcal{A}_h^s\right)^{-1}P_h\right)\phi\right\|_{\mathbb{H}}\\
\leq&\left\|\left((z^\alpha+\mathcal{A}^s)^{-1}-\left(z^\alpha+\mathcal{A}_h^s\right)^{-1}R_h^s\right)\phi\right\|_{\mathbb{H}}\\
&+\left\|\left((z^\alpha+\mathcal{A}_h^s)^{-1}R_h^s-\left(z^\alpha+\mathcal{A}_h^s\right)^{-1}P_h\right)\phi\right\|_{\mathbb{H}}\\
\leq& Ch^{2s}|z|^{-\alpha}\|\phi\|_{\hat{H}^{2s}(\mathbb{D})}.
\end{aligned}\tag{5.6}$$

Theorem 5.1. *Let u^n and u_h^n be the solutions of (4.1) and (5.1), respectively. For $\rho\in[\max(s/2-1/4+\epsilon,-s),\min(s/2,sH/\alpha-\epsilon)]$ with $s\in(0,1),\alpha\in(0,1)$, and $H\in(1/2,1)$, we have*

$$\left\|u^n-u_h^n\right\|^2_{L^q(\Omega,\mathbb{H})}\leq\begin{cases}Ch^{\min\left(\frac{(4\rho-1-2s)(Hs-\alpha\rho)}{\alpha(\rho-s)}-\epsilon,2s-4\rho+1-2\epsilon\right)}, & s\in[1/2,1),\\ Ch^{\min\left(\frac{4sH}{\alpha}-4\rho-\epsilon,4s-4\rho\right)}, & s\in(0,1/2),\end{cases}$$

where $q\geq 2$.

Proof. Subtracting (5.4) from (5.3) yields

$$\begin{aligned}
\|u^n-u_h^n\|^2_{L^q(\Omega,\mathbb{H})}=&\left\|\int_0^{t_n}\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}(t_n-\xi)-\bar{\mathcal{R}}_h(t_n-\xi)P_h d\xi\,dW_Q^H(r)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
=&\left\|\int_0^{t_n}\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r)\int_{t_{i-1}}^{t_i}\mathcal{E}(t_n-\xi)d\xi\,dW_Q^H(r)\right\|^2_{L^q(\Omega,\mathbb{H})},
\end{aligned}$$

where

$$\mathcal{E}(t_n-\xi)=\bar{\mathcal{R}}(t_n-\xi)-\bar{\mathcal{R}}_h(t_n-\xi)P_h.$$

By Burkholder-Davis-Gundy inequality, one has

$$\begin{aligned}
&\left\|u^n-u_h^n\right\|^2_{L^q(\Omega,\mathbb{H})}\\
\leq& C\left\|\int_0^{t_n}\int_r^{t_n}\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r')\int_{t_{i-1}}^{t_i}\mathcal{E}(t_n-\xi)d\xi(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'dW_Q(r)\right\|^2_{L^q(\Omega,\mathbb{H})}\\
\leq& C\int_0^{t_n}\left\|\int_r^{t_n}\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r')\int_{t_{i-1}}^{t_i}\mathcal{E}(t_n-\xi)d\xi(r'-r)^{H-\frac{3}{2}}\left(\frac{r}{r'}\right)^{\frac{1}{2}-H}dr'\right\|^2_{\mathcal{L}_2^0}dr\\
\leq& Ct_n^{2H-1}\int_0^{t_n}\left(\int_r^{t_n}\left\|\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r')\int_{t_{i-1}}^{t_i}\mathcal{E}(t_n-\xi)A^\rho d\xi\right\|(r'-r)^{H-\frac{3}{2}}dr'\right)^2r^{1-2H}dr.
\end{aligned}$$

After simple calculations, we have

$$\begin{aligned}
&\left\|\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r')\int_{t_{i-1}}^{t_i}\mathcal{E}(t_n-\xi)A^\rho d\xi\right\|\\
\leq& C\left\|\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(r')\int_{t_{i-1}}^{t_i}\int_{\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-r')}e^{z(r'-\xi)}\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1}\frac{z\tau}{e^{z\tau}-1}\right.\\
&\left.\times\left(\left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha+\mathcal{A}^s\right)^{-1}-\left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha+\mathcal{A}_h^s\right)^{-1}P_h\right)A^\rho dz\,d\xi\right\|
\end{aligned}$$

$$\leq C\left\| \int_{\Gamma^\tau_{\theta,\kappa}} e^{z(t_n-r')}\left(\delta_\tau(e^{-z\tau})\right)^{\alpha-1} \times \left(\left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha + \mathcal{A}^s \right)^{-1} - \left(\left(\delta_\tau(e^{-z\tau})\right)^\alpha + \mathcal{A}^s_h \right)^{-1} P_h \right) A^\rho dz \right\|.$$

For $\rho \in [\max(s/2 - 1/4 + \epsilon, 0), s/2)$, using (5.5), Lemma 5.1, and the interpolation properties, we find

$$\left\| \frac{1}{\tau}\sum_{i=1}^n \chi_{(t_{i-1},t_i]}(r') \int_{t_{i-1}}^{t_i} \mathcal{E}(t_n-\xi)A^\rho d\xi \right\| \leq Ch^{(1-\beta)\gamma} \int_{\Gamma^\tau_{\theta,\kappa}} |e^{z(t_n-r')}|\,|z|^{\alpha-1+\beta(\frac{\rho}{s}-1)\alpha}|dz|$$

with $\beta \in [0,1]$ and

$$\gamma = \begin{cases} 2s-2\rho, & s\in(0,1/2),\\ s-2\rho+1/2-\epsilon, & s\in[1/2,1). \end{cases}$$

Thus

$$\|u^n - u^n_h\|^2_{L^q(\Omega,\mathbb{H})} \leq Ch^{2(1-\beta)\gamma} t_n^{2H-1} \int_0^{t_n} \left(\int_r^{t_n} \int_{\Gamma^\tau_{\theta,\kappa}} |e^{z(t_n-r')}|\,|z|^{\alpha-1+\beta(\frac{\rho}{s}-1)\alpha}|dz|(r'-r)^{H-\frac{3}{2}}dr' \right)^2 r^{1-2H}dr.$$

Using Lemma 4.2 leads to

$$\int_r^{t_n} \int_{\Gamma^\tau_{\theta,\kappa}} |e^{z(t_n-r')}|\,|z|^{\alpha-1+\beta(\frac{\rho}{s}-1)\alpha}|dz|(r'-r)^{H-\frac{3}{2}}dr' \leq C(t_n-r)^{H-\frac{1}{2}+\beta(1-\frac{\rho}{s})\alpha-\alpha}.$$

To preserve the boundness of $\|u^n - u^n_h\|^2_{L^q(\Omega,\mathbb{H})}$, we need to require

$$H - \frac{1}{2} + \beta\left(1-\frac{\rho}{s}\right)\alpha - \alpha > -\frac{1}{2},$$

i.e.

$$\beta \geq \max\left(\frac{s\alpha - sH}{(s-\rho)\alpha} + \epsilon, 0 \right).$$

Thus

$$\|u^n - u^n_h\|^2_{L^q(\Omega,\mathbb{H})} \leq \begin{cases} Ch^{\min(\frac{(4\rho-1-2s)(Hs-\alpha\rho)}{\alpha(\rho-s)}-\epsilon, 2s-4\rho+1-2\epsilon)}, & s\in[1/2,1),\\ Ch^{\min(\frac{4sH}{\alpha}-4\rho-\epsilon, 4s-4\rho)}, & s\in(0,1/2). \end{cases}$$

For $\rho < 0$, we have $s \in (0, 1/2)$. Combining Lemma 5.1, (5.6), and the interpolation properties results in

$$\left\| \frac{1}{\tau}\sum_{i=1}^n \chi_{(t_{i-1},t_i]}(r') \int_{t_{i-1}}^{t_i} \mathcal{E}(t_n-\xi)A^\rho d\xi \right\| \leq Ch^{2\beta s-2\rho} \int_{\Gamma^\tau_{\theta,\kappa}} |e^{z(t_n-r')}|\,|z|^{\beta\alpha-1}|dz|$$

with $\beta \in [0, 1]$. Similarly, there holds

$$\int_r^{t_n} \int_{\Gamma^\tau_{\theta,\kappa}} |e^{z(t_n-r')}|\,|z|^{\beta\alpha-1}|dz|(r'-r)^{H-\frac{3}{2}}dr' \leq C(t_n-r)^{H-\frac{1}{2}-\beta\alpha}.$$

To preserve the boundness of $\|u^n - u_h^n\|^2_{L^q(\Omega,\mathbb{H})}$, we need to require $H - 1/2 - \beta\alpha > -1/2$, i.e. $\beta \le \min(H/\alpha - \epsilon, 1)$. So we get

$$\left\|u^n - u_h^n\right\|^2_{L^q(\Omega,\mathbb{H})} \le Ch^{\min(\frac{4sH}{\alpha} - 4\rho - \epsilon, 4s - 4\rho)}.$$

The proof is complete. □

Similarly, using the same way to discretize Eq. (1.3), we have the following fully discrete scheme:

$$\begin{cases} \left(\dfrac{v_h^n - v_h^{n-1}}{\tau}, \vartheta_h\right) + \sum\limits_{i=0}^{n-1} d_i^{(1-\alpha)} \left\langle v_h^{n-i}, \vartheta_h \right\rangle_s = 0, \quad \forall \vartheta_h \in X_h, \\ v_h^0 = P_h \psi_0, \end{cases} \tag{5.7}$$

where v_h^n is the numerical solution of Eq. (1.3) at t_n.

Then according to Theorems 4.1, 5.1 and the convergence analyses of (5.7) provided in [2,34], the following error estimates for numerically solving Eq. (1.1) are obtained.

Theorem 5.2. *Let $\psi = u + v$ and $\psi_h^n = u_h^n + v_h^n$ be exact and numerical solutions of Eq.* (1.1)*, where u_h^n and v_h^n are the solutions of* (5.1) *and* (5.7)*, respectively. Assume $\|A^{-\rho}\|_{\mathcal{L}_2^0} < \infty$ and $\psi_0 \in \dot{H}^\sigma(\mathbb{D})$ with $\rho \in [0, \min(s/2, sH/\alpha - \epsilon)]$ and $\sigma \ge 0$. Then it holds*

$$\begin{aligned} \left\|\psi - \psi_h^n\right\|^2_{L^2(\Omega,\mathbb{H})} \le\ & C\tau^{2H - \frac{2\rho\alpha}{s}} + C\tau^2 t_n^{-2} \|\psi_0\|^2_{L^2(\mathbb{D})} \\ & + Ch^{2\min(s + \min(s+\sigma, \frac{1}{2} - \epsilon), 1)} t^{-2\alpha} \|\psi_0\|^2_{\dot{H}^\sigma(\mathbb{D})} \\ & + \begin{cases} Ch^{\min(\frac{(4\rho - 1 - 2s)(Hs - \alpha\rho)}{\alpha(\rho - s)} - \epsilon, 2s - 4\rho + 1 - 2\epsilon)}, & s \in [1/2, 1), \\ Ch^{\min(\frac{4sH}{\alpha} - 4\rho - \epsilon, 4s - 4\rho)}, & s \in (0, 1/2). \end{cases} \end{aligned}$$

Remark 5.1. Here, we just provide the convergence rates of fully discrete scheme of Eq. (1.1) when the initial data ψ_0 belongs to $\dot{H}^\sigma(\mathbb{D})$. As for the initial data in $L^\infty(\mathbb{D})$ or Dirac delta measure, we will consider them in the future work with the help of the technique provided in [9, 21–23].

6. Numerical Experiments

In this part, we present some examples to verify the theoretical results in Theorems 4.1 and 5.1 with different s, α, H, and ρ. Suppose that the covariance operator Q shares the eigenfunctions with the operator A and denote its eigenvalues as $\Lambda_k = k^m$, where $k = 1, 2, \ldots,$ and $m \le 0$. By the assumption $\|A^{-\rho}\|_{\mathcal{L}_2^0} < \infty$ and Lemma 2.1, we have that ρ is close to $(1 + m)d/4$ and d is the dimension of the space.

For convenience, we choose the domain $\mathbb{D} = (0, 1)$ and approximate $W_Q^H(x, t)$ by the truncated series

$$W_Q^H(x, t) \approx \sum_{k=1}^{M} \sqrt{\Lambda_k}\phi_k(x) W_k^H(t),$$

where

$$\phi_k(x) = \sqrt{2}\sin(k\pi x).$$

Here it is sufficient to take $M = 5000$ to preserve the desired convergence in the truncation. We use 500 trajectories to compute the solution of Eq. (1.4). Due to that the exact solution u is unknown, we calculate

$$e_h = \left(\frac{1}{500} \sum_{i=1}^{500} \left\| u_h^N(\omega_i) - u_{h/2}^N(\omega_i) \right\|_{\mathbb{H}}^2 \right)^{\frac{1}{2}},$$

$$e_\tau = \left(\frac{1}{500} \sum_{i=1}^{500} \| u_\tau(\omega_i) - u_{\tau/2}(\omega_i) \|_{\mathbb{H}}^2 \right)^{\frac{1}{2}}$$

to measure the spatial and temporal errors, where the $u_h^N(\omega_i)$ $(u_\tau(\omega_i))$ means the numerical solution of u at t_N with mesh size h (time step size τ) and trajectory ω_j, so the spatial and temporal convergence rates can be, respectively, tested by

$$\text{Rate} = \frac{\ln(e_h/e_{h/2})}{\ln(2)}, \quad \text{Rate} = \frac{\ln(e_\tau/e_{\tau/2})}{\ln(2)}.$$

Example 6.1. In this example, we solve Eq. (1.4) numerically with terminal time $T = 1$ by numerical scheme (5.1) to validate the temporal convergence rates. Here, we take $h = 1/256$ to make the error incurred by spatial discretization negligible. The numerical results with different m, α, s, H are presented in Table 6.1. All the results agree with the predicted theoretical convergence rates (the numbers in the bracket in the last column) by Theorem 4.1.

Table 6.1: Temporal errors and convergence rates.

m	$(\alpha, s, H)\backslash\tau$	1/32	1/64	1/128	1/256	Rates
0	$(0.7, 0.6, 0.85)$	6.235E-03	4.194E-03	2.819E-03	1.896E-03	≈0.5723 (0.5583)
	$(0.8, 0.7, 0.85)$	6.098E-03	4.179E-03	2.822E-03	1.887E-03	≈0.564 (0.5643)
0.5	$(0.4, 0.3, 0.8)$	4.768E-03	2.970E-03	1.840E-03	1.119E-03	≈0.697 (0.6333)
	$(0.6, 0.8, 0.6)$	9.666E-03	6.728E-03	4.719E-03	3.402E-03	≈0.5022 (0.5063)
1	$(0.3, 0.4, 0.8)$	1.604E-03	9.357E-04	5.568E-04	3.212E-04	≈0.7734 (0.8)
	$(0.8, 0.6, 0.6)$	1.295E-02	8.586E-03	5.741E-03	3.739E-03	≈0.5974 (0.6)

Example 6.2. Here, we perform some numerical experiments to validate the spatial convergence rates. We take $\tau = T/512$ to eliminate the influence from temporal discretization. We choose different m, α, s, H, and the corresponding numerical results with $s \in (0, 1/2)$, $T = 0.1$ and $s \in [1/2, 1)$, $T = 0.01$ are presented in Tables 6.2 and 6.3, respectively, which verify the results of Theorem 5.1.

Table 6.2: Spatial errors and convergence rates with $s \in (0, 1/2)$.

m	$(\alpha, s, H)\backslash h$	1/128	1/256	1/512	1/1024	Rates
0.5	$(0.3, 0.3, 0.7)$	2.909E-02	2.334E-02	1.840E-02	1.453E-02	≈0.3338 (0.35)
	$(0.3, 0.4, 0.8)$	6.834E-03	4.678E-03	3.199E-03	2.187E-03	≈0.5479 (0.55)
1	$(0.5, 0.3, 0.7)$	1.376E-02	9.383E-03	6.336E-03	4.239E-03	≈0.5661 (0.6)
	$(0.5, 0.4, 0.8)$	3.307E-03	1.902E-03	1.103E-03	6.337E-04	≈0.7946 (0.8)
1.5	$(0.9, 0.2, 0.6)$	3.034E-02	2.189E-02	1.578E-02	1.119E-02	≈0.4796 (0.5167)

Table 6.3: Spatial errors and convergence rates with $s \in [1/2, 1)$.

m	$(\alpha, s, H)\backslash h$	1/128	1/256	1/512	1/1024	Rates
-0.2	$(0.5, 0.6, 0.6)$	3.097E-03	1.811E-03	1.040E-03	5.985E-04	≈0.7905 (0.7)
	$(0.5, 0.7, 0.8)$	3.147E-04	1.580E-04	7.865E-05	3.917E-05	≈1.002 (0.8)
-0.4	$(0.7, 0.6, 0.6)$	6.537E-03	3.884E-03	2.238E-03	1.253E-03	≈0.7944 (0.6476)
	$(0.7, 0.7, 0.8)$	5.487E-04	2.591E-04	1.220E-04	5.722E-05	≈1.0872 (0.9)
-0.6	$(0.9, 0.6, 0.6)$	1.181E-02	7.612E-03	4.872E-03	2.985E-03	≈0.6615 (0.54)

7. Conclusions

The macroscopic descriptions for the competition between subdiffusion and Lévy flights are governed by the fractional Fokker-Planck equation with temporal and spatial fractional derivatives. We do the numerical analyses for the stochastic version of the model, which are driven by the external fractional Gaussian noise. The backward Euler convolution quadrature and finite element method are, respectively, used to approximate the time and spatial operators. The complete error analyses are provided, and the numerical experiments verify the effectiveness of the presented numerical scheme.

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