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WONG-ZAKAI APPROXIMATIONS FOR STOCHASTIC VOLTERRA EQUATIONS*

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Abstract

In this paper, we shall prove a Wong-Zakai approximation for stochastic Volterra equations under appropriate assumptions. We may apply it to a class of stochastic differential equations with the kernel of fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and subfractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. As far as we know, this is the first result on stochastic Volterra equations in this topic.

Mathematics subject classification: 60H10, 60H07.
Key words: Stochastic Volterra equations, Wong-Zakai approximations, Fractional Brownian motion, Subfractional Brownian motion, Quadratic mean convergence.

1. Introduction and Main Results

Consider the following stochastic Volterra equations:

$$X_t = \xi + \int_0^t b(t, s, X_s)\mathrm{d}s + \int_0^t \sigma(t, s, X_s)\mathrm{d}W_s, \tag{1.1}$$

where $\xi \in \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable functions, and $\{W_t\}_{t\geq 0}$ is an m-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F})_t, \mathbb{P})$. Here the stochastic integral is the usual Itô's integral.

Stochastic Volterra equations arise in many applications such as mathematical finance, biology, etc. There is a big amount of literature devoted to the study of stochastic Volterra equations. Let us mention a few of them. When the coefficients $\sigma(t, s, x)$ and $b(t, s, x)$ are Lipschitz continuous in x and uniformly with respect to t, s, the existence and uniqueness of the strong solutions to Eq. (1.1) were first studied by Berger and Mizel [6, 7]. Later, the existence and uniqueness as well as the continuity of the solution to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients were considered in [42]. Meanwhile, Euler schemes and large deviations for stochastic Volterra equations with singular kernels were established by Zhang [49].

Note that an important task in applications is to realize stochastic differential equations (abbreviated SDEs) on computers, that is, to construct a discretized approximation. The

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Wong-Zakai approximation of SDEs (a.s. or in mean square) by random differential equations is considered by Wong-Zakai [43, 44], Ikeda-Watanabe [23], Karatzas-Shreve [25]. It is well known that if we replace the Brownian motion in SDEs by some smooth approximations (such as linear interpolation, mollifier, etc.), then the solution of the approximating equation converges (a.s. or q.s. or in mean square) to the Stratonovich form of the original equation (e.g. [2, 3, 5, 9, 15, 18–22, 27, 31, 34–41, 45–48, 50]).

However, to the best of our knowledge, the Wong-Zakai approximation for stochastic Volterra equations has not been established. It is natural to ask whether the Wong-Zakai continues to hold for stochastic Volterra equations. We remark that the stochastic Volterra equations is in fact an anticipating SDEs, which is much difficult to study. Since the solution of stochastic Volterra equations is neither Markovian, nor a semimartingale, Itô's formula usually used in the studies of SDEs is not available in this case. In this paper, we shall prove the Wong-Zakai approximation for stochastic Volterra equations, which is first paper to study the problem.

Here and below, C will denote a positive constant that is not depending on n and may have different values from one place to another one. For simplicity, we use $|\cdot|$ to denote both the Euclidean norm for a vector in $\mathbb{R}^d$ and the Hilbert-Schmidt norm for a matrix in $\mathbb{R}^{d\times m}$.

In the present paper, we shall restrict our discussion to time interval $[0,1]$ and make the following assumptions:

(H1) The function $b(t,s,x)$ is differentiable with respect to the first variable, and the function $\sigma(t,s,x)$ is differentiable with respect to the first and the third variable. Also there are binary functions $g_i(t,s)\geq 0, i=1,2,3,4$, and $\lambda_1,\lambda_2\in(0,1/2)$ such that for any $t,s\in[0,1], 0\leq u\leq v\leq 1$ and $x\in\mathbb{R}^d$,

$$|b_1(t,s,x)|+|\sigma_1(t,s,x)|+|\sigma_{13}(t,s,x)|+|\sigma_{31}(t,s,x)|\leq Cg_1(t,s), \tag{1.2}$$
$$|b(t,s,x)|\leq Cg_2(t,s)(1+|x|^{\lambda_1}), \tag{1.3}$$
$$|\sigma(t,s,x)|\leq Cg_3(t,s)(1+|x|^{\lambda_2}), \tag{1.4}$$
$$|\sigma_3(t,s,x)|\leq Cg_4(t,s), \tag{1.5}$$
$$\sigma_1(u,s,x)\leq\sigma_1(v,s,x), \tag{1.6}$$

where

$$\sup_{0\leq t\leq 1}\int_0^1 g_1(t,s)\mathrm{d}s<\infty,\quad \sup_{0\leq t\leq 1}\int_0^1 |g_j(t,s)|^p\mathrm{d}s<\infty,\quad j=2,3,4,\quad \forall p\geq 1,$$

and $g_j(r,s)\leq g_j(t,s), j=2,3,4$ for any $0\leq s\leq r\leq t\leq 1$. $b_1(t,s,x)$ represents the partial derivative of $b(t,s,x)$ with respect to the first variable. $\sigma_1(t,s,x)$ and $\sigma_3(t,s,x)$ represent the partial derivatives of $\sigma(t,s,x)$ with respect to the first variable and the third variable respectively. $\sigma_{ij}(t,s,x)$ means that $\sigma(t,s,x)$ first seeks a partial derivative of the i-th variable, and then seeks a partial derivative of the j-th variable, where $i,j=1,3$, and $i\neq j$.

(H2) For all $t,t',s\in[0,1]$ and $x\in\mathbb{R}^d$,

$$|b(t',s,x)-b(t,s,x)|\leq F_1(t',t,s), \tag{1.7}$$
$$|\sigma(t's,x)-\sigma(t,s,x)|^2\leq F_2(t',t,s), \tag{1.8}$$
$$|\sigma_3(t's,x)-\sigma_3(t,s,x)|^2\leq F_3(t',t,s), \tag{1.9}$$

where $F_i(t',t,s), i=1,2,3$, are nonnegative functions on $[0,1]\times[0,1]\times[0,1]$, and satisfy for some $\gamma>1$,

$$\int_0^{t\vee t'}\big[F_1^2(t',t,s)+F_2(t',t,s)+F_3(t',t,s)\big]\mathrm{d}s\le C|t-t'|^{\gamma}. \tag{1.10}$$

(H3) For all $t,s\in[0,1]$ and $x,y\in\mathbb{R}^d$,

$$|b(t,s,x)-b(t,s,y)|\le h_1(t,s)|x-y|, \tag{1.11}$$
$$|\sigma_3(t,s,x)-\sigma_3(t,s,y)|\le h_2(t,s)|x-y|, \tag{1.12}$$

where $h_i(t,s), i=1,2$, are nonnegative functions on $[0,1]\times[0,1]$, and satisfy

$$\sup_{0\le t\le 1}\int_0^1\big[h_1^{\eta_1}(t,s)+h_2^{\eta_2}(t,s)\big]\mathrm{d}s\le C \tag{1.13}$$

for some $\eta_i\ge 1$, $i=1,2$. Moreover, we assume that $h_j(r,s)\le h_j(t,s)$ for any $0\le s\le r\le t\le 1$, $j=1,2$.

(H4) For all $t\in[0,1]$ and $x\in\mathbb{R}^d$,

$$b(t,t,x)=\sigma(t,t,x)=\sigma_3(t,t,x)=0. \tag{1.14}$$

In what follows, we consider the particular partition

$$\Delta_n: 0<1/n<\cdots<j/n<(j+1)/n<\cdots<1,\quad \Delta_{n,j}=(j/n,(j+1)/n],$$

and the linear interpolation

$$W_t^n=(nt-j)W(\Delta_{n,j})+W_{\frac{j}{n}},\quad j/n\le t\le (j+1)/n,$$

where

$$W(\Delta_{n,j})=W_{\frac{j+1}{n}}-W_{\frac{j}{n}},\quad n\ge 1.$$

The Wong-Zakai approximation $\{X_t^n\}_{0\le t\le 1}$ associated with Eq. (1.1) is defined by

$$X_t^n=\xi+\int_0^t b\big(t,s,X_s^n\big)\mathrm{d}s+\int_0^t\sigma\big(t,s,X_s^n\big)\mathrm{d}W_s^n,\quad t\in[0,1], \tag{1.15}$$

where the second integral is to be understood in the Lebesgue-Stieltjies sense.

In order to prove the convergence of X_t^n to X_t, we need the following additional assumptions:

(H5) For all $s\in[0,t_n]\subseteq[0,1]$ and $x\in\mathbb{R}^d$,

$$\sup_{s\in[0,t_n]}\sup_{x\in\mathbb{R}^d}\Big[n\Big(\int_0^s\sigma(s,r,x)\mathrm{d}r-\int_0^{s_n}\sigma(s_n,r,x)\mathrm{d}r\Big)\\ -(ns-\lfloor ns\rfloor)\int_0^s\sigma_1(s,u,x)\mathrm{d}u\Big]^2\le\frac{C}{n^{\varsigma_1}}, \tag{1.16}$$

where $\varsigma_1>0, s_n:=\lfloor ns\rfloor/n, s_n^+:=(\lfloor ns\rfloor+1)/n$, and $\lfloor a\rfloor$ denotes the integer part of a real number a.

Remark 1.1. Under (H1)-(H3), it is easy to derive that there exists a unique continuous adapted solution X_t to Eq. (1.15) by similarly the proof of Wang [42]. (H2) and (H3) are called regularity conditions, which play an important role in proving the conclusion of this paper.

Remark 1.2. $g_j(t,s), j = 1,4$, in (H1) and $F_i(t',t,s), i = 1,2,3$, in (H2) are not relaxed to $g_j(t,s)(1 + |x|^{\varrho})$ and $F_i(t',t,s)(1 + |x|^{\rho})$ with $\varrho, \rho \in (0,1]$, respectively. The reason is that there are some technical difficulties in the proof of this paper. This point is determined by the characteristics of stochastic Volterra equations.

Remark 1.3. In Sections 4 and 5, we give two examples which satisfy (H1)-(H5).

Remark 1.4. In the paper, we introduce the hypothesis (H5) to deal with the $\mathscr{O}_{n,1}(t)$ (see (3.25) below). At first glance, this hypothesis in this paper is very strange, but through the two examples given in Sections 4 and 5, we show that this hypothesis is reasonable.

The main purposes in this paper is devoted to proving that

Theorem 1.1. *Under (H1)-(H5), we have*

$$\lim_{n\to\infty} \sup_{0\le t\le 1} \mathbb{E}\big|X_t^n - X_t\big|^2 = 0, \tag{1.17}$$

where X_t and X_t^n are the solutions of Eqs. (1.1) *and* (1.15)*, respectively.*

The main proof of this paper is an operation of pure algebra, that does not appeal to Itô's formula. The reason is that the coefficients of stochastic Volterra equations contain the time t. This point is different from most existing results. The proof of Theorem 1.1 requires a few propositions and lemmas which we give below. The proof is involved and delicate. We need to complete the squares in last stage of the proof. Therefore we have to be very careful with the each term in the estimates.

Remark 1.5. Because we assume that $g_i(r,s) \le g_i(t,s), i = 2,3,4$, and $h_j(r,s) \le h_j(t,s)$, $j = 1,2$, hold for any $0 \le s \le r \le t \le 1$ in (H1) and (H3), we can get (1.17). If we remove these, it is easy to obtain the following result:

$$\lim_{n\to\infty} \mathbb{E}\big|X_t^n - X_t\big|^2 = 0, \quad \forall t \in [0,1],$$

which is weaker than (1.17). This point can be seen by using Gronwall's inequality for (3.29).

Although the Wong-Zakai approximation for SDEs has been proposed and studied for more than fifty years, the boundedness of the derivative of diffusion coefficient with respect to its time variable and the boundedness of the derivative of diffusion coefficient with respect to its space variable have been always restrictive assumptions to derive the strong convergence of Wong-Zakai approximations for SDEs. The boundedness assumption in this article is relaxed to two integrable functions and the strong convergence of Wong-Zakai approximations for stochastic Volterra equations without the boundedness of the derivatives of diffusion coefficient with respect to its time and space variables is proposed. In other words, the first contribution of this paper is to prove the strong convergence of Wong-Zakai approximations for stochastic Volterra equations without the boundedness of the derivatives of diffusion coefficient with respect to its time and space variables, and may apply it with a class of SDEs with the kernel of fractional

Brownian motion with Hurst parameter $H \in (1/2, 1)$ and subfractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

On the other hand, if the stochastic integral with respect to Brownian motion in Wong-Zakai approximations theory of the classical SDEs is an Itô integral, then the corresponding corrected term is $\sigma'\sigma/2$ which is well known. However, although the stochastic integral with respect to Brownian motion in the stochastic Volterra equation is an Itô integral, its corrected term is unclear, which is different from the classical case. The correction term in this paper is zero, which is very important and nontrivial to prove the result of this paper. If the correction term in this paper is nonzero, then the nonzero correction term can not be offset by other terms, or treated by classical methods or coped by estimating other terms approaches. Moreover, we can not get the Wong-Zakai approximation of stochastic Volterra equation. This is why the correction term in this article is zero. This point is essentially different from the result of existence and is seen by two examples below. As far as we know, it seems that this is the main reason why the problem of the Wong-Zakai approximation of stochastic Volterra equation has not been solved so far. This is the second contribution of this paper. In Sections 4 and 5 of this paper, two examples are given to show that the hypothesises can be verified.

Remark 1.6. The correction term $\sigma'\sigma/2$ is zero in this paper. It should be pointed out that in the particular case of classical Itô SDEs

$$X_t = \xi + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$$

this is a consequence of (1.14) from (H4).

Note that $L^p\,(p \geq 2)$ convergence rate for Wong-Zakai approximations of other types SDEs (for example, Reflection SDEs, BSDEs) can be presented. The reason is that their proofs rely on the classical stochastic analysis tools, for example, Itô's formula. However, we can not use this in the paper. To get the convergence, we employ a dense result. It is the fact that we only know the convergence of the dense result, but not the convergence rate. This is a very important reason why we can not give the convergence rate in this paper. In addition, since some estimates are not available (e.g. (2.2) for $L^p\,(p > 2)$ case), we can not prove $L^p\,(p > 2)$ convergence in this paper. It seems that this paper can only get an L^2 convergence, but not propose a convergence rate.

Finally we hope to apply our results to obtain the support theorem of stochastic Volterra equations and also to get Wong-Zakai approximations and support theorems of stochastic Volterra equations in Banach space. We like to leave this study in a forthcoming paper given the length 20 of the current article. For the sake of simplicity, we assume $d = m = 1$ in the proof. The multidimensional proof process is similar to this.

This paper is organized as follows. In Section 2, we prepare some necessary propositions and lemmas for later use. In Section 3, we prove our main result. In Sections 4 and 5, we apply our result to SDEs with the kernels of fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, and subfractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, respectively. In Appendixes A and B, (H1)-(H5) are verified for the kernel associated to fractional Brownian motion and subfractional Brownian motion, separately.

2. Preliminaries

In this section, we will present some propositions and lemmas.

Proposition 2.1. *Under (H1)-(H3), Eq.* (1.1) *has a unique strong solution* $\{X_t\}_{0\le t\le 1}$ *such that*

$$\sup_{0\le t\le 1}\mathbb{E}|X_t|^p<\infty, \tag{2.1}$$

where $p\ge 2$.

Proof. Similar to the proof of Wang [42], it is easy to derive that there exists a unique strong solution X_t to Eq. (1.1). Set

$$\tau_{k_1}:=\inf\{t>0,|X_t|>k_1\},\quad k_1\in\mathbb{N}.$$

If this inequality (2.1) holds for the process $X_{\tau_{k_1}\wedge t}$, let $k_1\to\infty$, by Fatou's lemma it follows that this inequality (2.1) also holds for X_t. So we might as well suppose that X_t is bounded. Moreover, by (1.3), (1.4), Hölder's inequality, Young's inequality and moment inequality [28, Theorem 7.1], we have for any $u\in[0,1]$,

$$\begin{aligned}\mathbb{E}|X_u|^p&\le C+C\sup_{0\le t\le u}\mathbb{E}\Big[\int_0^t|b(t,s,X_s)|\mathrm{d}s+\Big|\int_0^t\sigma(t,s,X_s)\mathrm{d}W_s\Big|\Big]^p\\&\le C+C\sup_{0\le t\le u}\mathbb{E}\Big[\int_0^t|b(t,s,X_s)|\mathrm{d}s\Big]^p+C\sup_{0\le t\le u}\mathbb{E}\Big[\Big|\int_0^t\sigma(t,s,X_s)\mathrm{d}W_s\Big|\Big]^p\\&\le C+C\sup_{0\le t\le u}\mathbb{E}\Big[\int_0^t|b(t,s,X_s)|^p\mathrm{d}s\Big]+C\sup_{0\le t\le u}\mathbb{E}\Big[\int_0^t|\sigma(t,s,X_s)|^p\mathrm{d}s\Big]\\&\le C+C\sup_{0\le t\le u}\int_0^t\big[g_2^p(t,s)+g_3^p(t,s)\big]\mathbb{E}\big(1+|X_s|^{p\lambda_1}+|X_s|^{p\lambda_2}\big)\mathrm{d}s\\&\le C+C\int_0^u\big[g_2^p(u,s)+g_3^p(u,s)\big]\big(1+\mathbb{E}|X_s|^p\big)\mathrm{d}s.\end{aligned}$$

Put

$$f(u):=\mathbb{E}|X_u|^p.$$

Then

$$f(u)\le C+C\int_0^u\big[g_2^p(u,s)+g_3^p(u,s)\big]f(s)\mathrm{d}s,$$

which, by Gronwall's inequality (cf. [16, Theorem 16]), means that (2.1) holds. □

The following proposition shows that the Wong-Zakai approximations are well defined.

Proposition 2.2. *Under the assumptions (H1)-(H5), for each* $n\in\mathbb{N}$ *the Eq.* (1.15) *has a unique solution* $\{X_t^n\}_{0\le t\le 1}$ *such that*

$$\sup_{n\in\mathbb{N}}\sup_{0\le t\le 1}\mathbb{E}\big|X_t^n\big|^2<\infty. \tag{2.2}$$

Proof. The proof of existence and uniqueness can be established by the standard Picard approximations (see [42] for the stochastic case). Set

$$\tau_{k_2}:=\inf\big\{t>0,\big|X_t^n\big|>k_2\big\},\quad k_2\in\mathbb{N}.$$

If this inequality (2.2) holds for the process $X^n_{\tau_{k_2}\wedge t}$, let $k_2\to\infty$, by Fatou's lemma it derives from this inequality (2.2) also holds for X_t^n. So we can suppose that X_t^n is bounded.

Note that

$$\int_0^t \sigma\big(t,s,X_s^n\big)\mathrm{d}W_s^n = \int_0^{t_n} \sigma\big(t,s,X_s^n\big)\mathrm{d}W_s^n + \int_{t_n}^t \sigma\big(t,s,X_s^n\big)\mathrm{d}W_s^n =: L_n(t) + M_n(t). \tag{2.3}$$

For $L_n(t)$, by Newton-Leibniz formula, (1.14) and (1.15), we have

$$\begin{aligned} L_n(t) &= n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}} \sigma\big(t,s,X_s^n\big)\mathrm{d}s \\ &= n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}} \bigg\{\sigma\Big(t,s,X_{\frac{j}{n}}^n\Big) + \int_{\frac{j}{n}}^{s} \sigma_3\big(t,s,X_u^n\big) \\ &\qquad \times\bigg[\int_0^u \sigma_1\big(u,r,X_r^n\big)\mathrm{d}W_r^n + \int_0^u b_1\big(u,r,X_r^n\big)\mathrm{d}r\bigg]\mathrm{d}u\bigg\}\mathrm{d}s \\ &=: L_{n,1}(t) + L_{n,2}(t) + L_{n,3}(t), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} L_{n,1}(t) &= n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}} \sigma\Big(t,s,X_{\frac{j}{n}}^n\Big)\mathrm{d}s, \\ L_{n,2}(t) &= n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} \sigma_3\big(t,s,X_u^n\big)\bigg(\int_0^u \sigma_1\big(u,r,X_r^n\big)\mathrm{d}W_r^n\bigg)\mathrm{d}u\mathrm{d}s, \\ L_{n,3}(t) &= n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} \sigma_3\big(t,s,X_u^n\big)\bigg(\int_0^u b_1\big(u,r,X_r^n\big)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s. \end{aligned}$$

For $L_{n,1}(t)$, by the independence of the increments of the Brownian motion, the Cauchy-Schwarz inequality, Fubini's theorem and (1.4), we have

$$\begin{aligned} \mathbb{E}|L_{n,1}(t)|^2 &= n^2\sum_{j=0}^{\lfloor nt\rfloor-1} \mathbb{E}\Bigg[W^2(\Delta_{n,j})\bigg|\int_{\frac{j}{n}}^{\frac{j+1}{n}} \sigma\Big(t,s,X_{\frac{j}{n}}^n\Big)\mathrm{d}s\bigg|^2\Bigg] \\ &= n^2\sum_{j=0}^{\lfloor nt\rfloor-1} \mathbb{E}W^2(\Delta_{n,j})\mathbb{E}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}} \sigma\Big(t,s,X_{\frac{j}{n}}^n\Big)\mathrm{d}s\bigg]^2 \\ &\le n\sum_{j=0}^{\lfloor nt\rfloor-1} \mathbb{E}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}} \Big|\sigma\Big(t,s,X_{\frac{j}{n}}^n\Big)\Big|^2\mathrm{d}s\int_{\frac{j}{n}}^{\frac{j+1}{n}} 1\mathrm{d}s\bigg] \\ &= \sum_{j=0}^{\lfloor nt\rfloor-1} \mathbb{E}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}} \Big|\sigma\Big(t,s,X_{\frac{j}{n}}^n\Big)\Big|^2\mathrm{d}s\bigg] \\ &\le C\int_0^t g_3^2(t,s)\mathbb{E}\big(1+\big|X_{s_n}^n\big|^{2\lambda_2}\big)\mathrm{d}s. \end{aligned} \tag{2.5}$$

Observe that

$$L_{n,2}(t) = L_{n,2}^{(1)}(t) + L_{n,2}^{(2)}(t), \tag{2.6}$$

where

$$L_{n,2}^{(1)}(t) = n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} \sigma_3\big(t,s,X_u^n\big)\left(\int_0^{\frac{j}{n}} \sigma_1\big(u,r,X_r^n\big)\mathrm{d}W_r^n\right)\mathrm{d}u\mathrm{d}s,$$

$$L_{n,2}^{(2)}(t)=n\sum_{j=0}^{\lfloor nt\rfloor-1}W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\big(t,s,X_u^n\big)\left(\int_{\frac{j}{n}}^{u}\sigma_1\big(u,r,X_r^n\big)\mathrm{d}W_r^n\right)\mathrm{d}u\mathrm{d}s.$$

For $L_{n,2}^{(1)}(t)$, by the Cauchy-Schwarz inequality, Fubini's theorem, the property of $g_1(u,r)$, the independence of the increments of the Brownian motion, (1.2) and (1.5), we have

$$\begin{aligned}
&\mathbb{E}\big|L_{n,2}^{(1)}(t)\big|^2\\
&=n^2\mathbb{E}\left[\sum_{j=0}^{\lfloor nt\rfloor-1}W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\big(t,s,X_u^n\big)\left(\int_0^{\frac{j}{n}}\sigma_1\big(u,r,X_r^n\big)\mathrm{d}W_r^n\right)\mathrm{d}u\mathrm{d}s\right]^2\\
&\le n^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\big(t,s,X_u^n\big)\left(\int_0^{\frac{j}{n}}\sigma_1\big(u,r,X_r^n\big)\mathrm{d}W_r^n\right)\mathrm{d}u\mathrm{d}s\right]^2\\
&\le Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[W^2(\Delta_{n,j})\left|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}g_4(t,s)\left|\int_0^{\frac{j}{n}}g_1(u,r)\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|\mathrm{d}r\right|\mathrm{d}u\mathrm{d}s\right|^2\right]\\
&=Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}W^2(\Delta_{n,j})\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}g_4(t,s)\left|\int_0^{\frac{j}{n}}g_1(u,r)\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|\mathrm{d}r\right|\mathrm{d}u\mathrm{d}s\right]^2\\
&\le Cn^2\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4(t,s)\left|\int_0^{\frac{j}{n}}g_1(u,r)\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|\mathrm{d}r\right|\mathrm{d}u\right)^2\mathrm{d}s\int_{\frac{j}{n}}^{\frac{j+1}{n}}1\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\left|\int_0^{\frac{j}{n}}g_1(u,r)\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|\mathrm{d}r\right|^2\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\mathbb{E}\left|\int_0^{\frac{j}{n}}g_1(u,r)\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|\mathrm{d}r\right|^2\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\int_0^{\frac{j}{n}}\int_0^{\frac{j}{n}}\prod_{i=1}^{2}g_1(u,r_i)\mathbb{E}\left(\prod_{i=1}^{2}\left|\frac{\mathrm{d}W_{r_i}^n}{\mathrm{d}r_i}\right|\right)\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\int_0^1\int_0^1\prod_{i=1}^{2}g_1(u,r_i)\left(\mathbb{E}\left|\frac{\mathrm{d}W_{r_i}^n}{\mathrm{d}r_i}\right|^2\right)^{\frac{1}{2}}\mathrm{d}r_i\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\left[\int_0^1g_1(u,r)\left(\mathbb{E}\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|^2\right)^{\frac{1}{2}}\mathrm{d}r\right]^2\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\sup_{0\le r\le1}\mathbb{E}\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|^2\left[\int_0^1g_1(u,r)\mathrm{d}r\right]^2\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}\sup_{0\le r\le1}\mathbb{E}\left|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\right|^2\sup_{0\le u\le1}\left[\int_0^1g_1(u,r)\mathrm{d}r\right]^2\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(\int_{\frac{j}{n}}^{s}g_4^2(t,s)\mathrm{d}u\right)\left(\int_{\frac{j}{n}}^{s}n\mathrm{d}u\right)\mathrm{d}s\right]\\
&\le C\sup_{0\le t\le1}\int_0^1g_4^2(t,s)\mathrm{d}s<\infty.
\end{aligned}\tag{2.7}$$

For $L^{(2)}_{n,2}(t)$, by (1.2), (1.5), the Cauchy-Schwarz inequality, and property of $g_1(u,r)$, it is easy to derive that

$$\begin{aligned}
\mathbb{E}\big|L^{(2)}_{n,2}(t)\big|^2 &= n^4\mathbb{E}\bigg[\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\big(t,s,X^n_u\big)\bigg(\int_{\frac{j}{n}}^{u}\sigma_1\big(u,r,X^n_r\big)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\bigg]^2\\
&\le Cn^4\mathbb{E}\bigg[\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}g_4(t,s)\bigg(\int_{\frac{j}{n}}^{u}g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\bigg]^2\\
&\le Cn^5\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\bigg[W^4(\Delta_{n,j})\bigg|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}g_4(t,s)\bigg(\int_{\frac{j}{n}}^{u}g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\bigg|^2\bigg]\\
&\le Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\bigg|\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}g_1(u,r)\mathrm{d}r\mathrm{d}u\mathrm{d}s\bigg|^2\\
&\le Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\Big(\frac{1}{n}\Big)^2\bigg|\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\bigg|^2\bigg[\sup_{0\le u\le 1}\int_0^1 g_1(u,r)\mathrm{d}r\bigg]^2\\
&\le C\sup_{0\le t\le 1}\int_0^1 g_4^2(t,s)\mathrm{d}s<\infty. \qquad (2.8)
\end{aligned}$$

By (2.6)-(2.8), we have

$$\mathbb{E}|L_{n,2}(t)|^2\le C,\quad 0\le t\le 1. \qquad (2.9)$$

For $L_{n,3}(t)$, by (1.2), (1.5) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}|L_{n,3}(t)|^2 &= n^2\mathbb{E}\bigg[\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\big(t,s,X^n_u\big)\bigg(\int_0^{u}b_1\big(u,r,X^n_r\big)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\bigg]^2\\
&\le Cn^2\mathbb{E}\bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}|W(\Delta_{n,j})|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}g_4(t,s)\bigg(\int_0^{u}g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\bigg]^2\\
&\le Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\bigg[W^2(\Delta_{n,j})\bigg|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}g_4(t,s)\bigg(\int_0^{u}g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\bigg|^2\bigg]\\
&\le Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{n^3}\bigg|\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\bigg|^2\bigg[\sup_{0\le u\le 1}\int_0^1 g_1(u,r)\mathrm{d}r\bigg]^2\\
&\le \frac{C}{n}\sup_{0\le t\le 1}\int_0^1 g_4^2(t,s)\mathrm{d}s\le\frac{C}{n}<\infty. \qquad (2.10)
\end{aligned}$$

By (2.4), (2.5)-(2.10) and Young's inequality, we get

$$\begin{aligned}
\mathbb{E}|L_n(t)|^2 &\le C+C\int_0^t g_3^2(t,s)\mathbb{E}\big(1+\big|X^n_{s_n}\big|^{2\lambda_2}\big)\mathrm{d}s\\
&\le C+\int_0^t g_3^2(t,s)\mathbb{E}\big|X^n_{s_n}\big|^2\mathrm{d}s,\quad 0\le t\le 1. \qquad (2.11)
\end{aligned}$$

For $M_n(t)$, by (1.5), (2.3), Fubini's theorem, Young's inequality and the Cauchy-Schwarz inequality, we have for any $a>1$ and $0<a\lambda_2<1$,

$$\mathbb{E}|M_n(t)|^2=\mathbb{E}\bigg|\int_{t_n}^t\sigma\big(t,s,X^n_s\big)\mathrm{d}W^n_s\bigg|^2$$

$$
\begin{aligned}
&= n^2\mathbb{E}\Big|\int_{t_n}^t \sigma\big(t,s,X_s^n\big)W(\Delta_{n,j})\mathrm{d}s\Big|^2\\
&\le n^2\mathbb{E}\Big[\int_{t_n}^t \big|\sigma\big(t,s,X_s^n\big)W(\Delta_{n,j})\big|^2\mathrm{d}s\int_{t_n}^t 1\mathrm{d}s\Big]\\
&\le n\int_{t_n}^t \mathbb{E}\big|\sigma\big(t,s,X_s^n\big)W(\Delta_{n,j})\big|^2\mathrm{d}s\\
&\le n\int_{t_n}^t \Big[\mathbb{E}\big|\sigma\big(t,s,X_s^n\big)\big|^{2a}\Big]^{\frac{1}{a}}\Big[\mathbb{E}W^{\frac{2a}{a-1}}(\Delta_{n,j})\Big]^{\frac{a-1}{a}}\mathrm{d}s\\
&\le C\int_{t_n}^t g_3^2(t,s)\Big[\mathbb{E}\big(1+\big|X_s^n\big|^{\lambda_2}\big)^{2a}\Big]^{\frac{1}{a}}\mathrm{d}s\\
&\le C\int_{t_n}^t g_3^2(t,s)\Big[\mathbb{E}\big(1+\big|X_s^n\big|^{2a\lambda_2}\big)\Big]^{\frac{1}{a}}\mathrm{d}s\\
&\le C\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big(1+\big|X_s^n\big|^2\big)\mathrm{d}s\\
&\le C\sup_{0\le t\le 1}\int_0^1 g_3^2(t,s)\mathrm{d}s+C\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big|X_s^n\big|^2\mathrm{d}s\\
&\le C+C\int_0^t g_3^2(t,s)\mathbb{E}\big|X_s^n\big|^2\mathrm{d}s.
\end{aligned}
\tag{2.12}
$$

By (1.3), Fubini's theorem, Young's inequality and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\mathbb{E}\Big|\int_0^t b\big(t,s,X_s^n\big)\mathrm{d}s\Big|^2 &\le C\int_0^t g_2^2(t,s)\big(1+\mathbb{E}\big|X_s^n\big|^{2\lambda_1}\big)\mathrm{d}s\\
&\le C+C\int_0^t g_2^2(t,s)\mathbb{E}\big|X_s^n\big|^2\mathrm{d}s.
\end{aligned}
\tag{2.13}
$$

Thus, by (1.15), (2.3) and (2.11)-(2.13), we have for any $r\in[0,1]$,

$$
\sup_{t\le r}\mathbb{E}\big|X_t^n\big|^2\le C+C\int_0^r\big[g_2^2(r,s)+g_3^2(r,s)\big]\sup_{r\le s}\mathbb{E}\big|X_r^n\big|^2\mathrm{d}s. \tag{2.14}
$$

By Gronwall's inequality (see [16, Theorem 16]) it is easy to see that (2.2) holds. Hence, we complete the proof. □

Lemma 2.1. *Under the assumptions (H1)-(H5), there exist $C,\alpha,$ and $\beta>0$ such that*

$$
\mathbb{E}\Big(\sup_{0\le s\le 1}\big|X_s^n-X_{s_n}^n\big|^4\Big)\le\frac{C}{n^\alpha},\quad n\ge 1, \tag{2.15}
$$

$$
\sup_{0\le s\le 1}\mathbb{E}|X_s-X_{s_n}|^4\le\frac{C}{n^\beta},\qquad n\ge 1, \tag{2.16}
$$

where C is independent of n.

Proof. Note that

$$
X_s^n-X_{s_n}^n=\int_0^{s_n}\big[b\big(s,r,X_r^n\big)-b\big(s_n,r,X_r^n\big)\big]\mathrm{d}r
$$

$$
\begin{aligned}
&+\int_0^{s_n}\big[\sigma\big(s,r,X_r^n\big)-\sigma\big(s_n,r,X_r^n\big)\big]\mathrm{d}W_r^n\\
&+\int_{s_n}^{s} b\big(s,r,X_r^n\big)\mathrm{d}r+\int_{s_n}^{s}\sigma\big(s,r,X_r^n\big)\mathrm{d}W_r^n\\
=:\ & I_1(s)+I_2(s)+I_3(s)+I_4(s).
\end{aligned}
\tag{2.17}
$$

For $I_1(s)$, by (1.7), (1.10) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le s\le 1}|I_1(s)|^4\Big)&\le C\mathbb{E}\sup_{0\le s\le 1}\bigg[\int_0^{s_n}\big|b\big(s,r,X_r^n\big)-b\big(s_n,r,X_r^n\big)\big|^2\mathrm{d}r\bigg]^2\\
&\le C\sup_{0\le s\le 1}\bigg[\int_0^{s_n}F_1^2(s,s_n,r)\mathrm{d}r\bigg]^2\le\frac{C}{n^{2\gamma}}.
\end{aligned}
\tag{2.18}
$$

For $I_2(s)$, we have by (1.8), (1.10) and Hölder's inequality

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le s\le 1}|I_2(s)|^4\Big)&\le C\mathbb{E}\bigg\{\sup_{0\le s\le 1}\bigg[\int_0^{s_n}\big|\sigma\big(s,r,X_r^n\big)-\sigma\big(s_n,r,X_r^n\big)\big|^2\mathrm{d}r\bigg]^2\int_0^1\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|^4\mathrm{d}r\bigg\}\\
&\le C\sup_{0\le s\le 1}\bigg[\int_0^{s_n}F_2(s_n,s,r)\mathrm{d}r\bigg]^2\int_0^1\mathbb{E}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|^4\mathrm{d}r\le\frac{C}{n^{2(\gamma-1)}}.
\end{aligned}
\tag{2.19}
$$

For $I_3(s)$, by (1.3), (2.2), Young's inequality and Hölder's inequality, we have for any $q>1$ and $0<2\lambda_1 q<1$,

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le s\le 1}|I_3(s)|^4\Big)&\le\frac{C}{n^3}\mathbb{E}\sup_{0\le s\le 1}\bigg[\int_{s_n}^{s}\big|b\big(s,r,X_r^n\big)\big|^4\mathrm{d}r\bigg]\\
&\le\frac{C}{n^3}\mathbb{E}\bigg[\sup_{0\le s\le 1}\int_0^1 g_2^4(s,r)\big(1+\big|X_r^n\big|^{\lambda_1}\big)^4\mathrm{d}r\bigg]\\
&\le\frac{C}{n^3}\sup_{0\le s\le 1}\int_0^1 g_2^{\frac{4q}{q-1}}(s,r)\mathrm{d}r+\frac{C}{n^3}\mathbb{E}\bigg[\sup_{0\le s\le 1}\int_0^1\big(1+\big|X_r^n\big|^{\lambda_1}\big)^{4q}\mathrm{d}r\bigg]\\
&\le\frac{C}{n^3}+\frac{C}{n^3}\sup_{0\le r\le 1}\mathbb{E}\big|X_r^n\big|^{4\lambda_1 q}\\
&\le\frac{C}{n^3}+\frac{C}{n^3}\sup_{0\le r\le 1}\Big(\mathbb{E}\big|X_r^n\big|^{4\lambda_1 q\cdot\frac{1}{2\lambda_1 q}}\Big)^{2\lambda_1 q}\\
&\le\frac{C}{n^3}\bigg[1+\Big(\sup_{n\in\mathbb{N}}\sup_{0\le r\le 1}\mathbb{E}\big|X_r^n\big|^2\Big)^{2\lambda_1 q}\bigg]\le\frac{C}{n^3}.
\end{aligned}
\tag{2.20}
$$

As for $I_4(s)$, we have by (1.4), (2.2) and Hölder's inequality for any $p_1,p_2>1$ and $0<2\lambda_2p_1p_2<1$,

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{0\le s\le 1}|I_4(s)|^4\Big)&\le Cn\mathbb{E}\sup_{0\le s\le 1}\bigg[\int_{s_n}^{s}\big|\sigma\big(s,r,X_r^n\big)W(\Delta_{n,j})\big|^4\mathrm{d}r\bigg]\\
&\le nC\mathbb{E}\bigg[W^4(\Delta_{n,j})\sup_{0\le s\le 1}\int_0^1 g_3^4(s,r)\big(1+\big|X_r^n\big|^{\lambda_2}\big)^4\mathrm{d}r\bigg]\\
&\le Cn\Big[\mathbb{E}W^{\frac{4p_1}{p_1-1}}(\Delta_{n,j})\Big]^{\frac{p_1-1}{p_1}}\bigg[\mathbb{E}\sup_{0\le s\le 1}\int_0^1 g_3^{4p_1}(s,r)\big(1+\big|X_r^n\big|^{\lambda_2}\big)^{4p_1}\mathrm{d}r\bigg]^{\frac{1}{p_1}}\\
&\le\frac{C}{n}\bigg[\mathbb{E}\sup_{0\le s\le 1}\int_0^1 g_3^{4p_1}(s,r)\big(1+\big|X_r^n\big|^{\lambda_2}\big)^{4p_1}\mathrm{d}r\bigg]^{\frac{1}{p_1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C}{n}\left[\sup_{0\le s\le 1}\int_0^1 g_3^{\frac{4p_1p_2}{p_2-1}}(s,r)\mathrm{d}r+\mathbb{E}\sup_{0\le s\le 1}\int_0^1\left(1+\left|X_r^n\right|^{\lambda_2}\right)^{4p_1p_2}\mathrm{d}r\right]^{\frac{1}{p_1}}\\
&\le \frac{C}{n}\left(1+\int_0^1\mathbb{E}\left|X_r^n\right|^{4\lambda_2p_1p_2}\mathrm{d}r\right)^{\frac{1}{p_1}}\\
&\le \frac{C}{n}\left[1+\int_0^1\left(\mathbb{E}\left|X_r^n\right|^{4\lambda_2p_1p_2\cdot\frac{1}{2\lambda_2p_1p_2}}\right)^{2\lambda_2p_1p_2}\mathrm{d}r\right]^{\frac{1}{p_1}}\\
&\le \frac{C}{n}\left[1+\int_0^1\left(\sup_{n\in\mathbb{N}}\sup_{0\le r\le 1}\mathbb{E}\left|X_r^n\right|^2\right)^{2\lambda_2p_1p_2}\mathrm{d}r\right]^{\frac{1}{p_1}}\le\frac{C}{n}. \qquad (2.21)
\end{aligned}
$$

By (2.17)-(2.21), we can conclude that (2.15) holds, where $\alpha=\min\{2(\gamma-1),1\}$.

Next, we turn to prove (2.16). Note that

$$
X_s-X_{s_n}=J_1(s)+J_2(s)+J_3(s)+J_4(s), \qquad (2.22)
$$

where

$$
\begin{aligned}
J_1(s)&=\int_0^{s_n}[b(s,r,X_r)-b(s_n,r,X_r)]\mathrm{d}r,\\
J_2(s)&=\int_0^{s_n}[\sigma(s,r,X_r)-\sigma(s_n,r,X_r)]\mathrm{d}W_r,\\
J_3(s)&=\int_{s_n}^{s}b(s,r,X_r)\mathrm{d}r,\\
J_4(s)&=\int_{s_n}^{s}\sigma(s,r,X_r)\mathrm{d}W_r.
\end{aligned}
$$

For $J_1(s)$, by (1.7), (1.10) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\sup_{0\le s\le 1}\mathbb{E}|J_1(s)|^4&\le C\sup_{0\le s\le 1}\mathbb{E}\left[\int_0^{s_n}\left|b\big(s,r,X_r^n\big)-b\big(s_n,r,X_r^n\big)\right|^2\mathrm{d}r\right]^2\\
&\le C\sup_{0\le s\le 1}\left[\int_0^{s_n}F_1^2(s,s_n,r)\mathrm{d}r\right]^2\le\frac{C}{n^{2\gamma}}. \qquad (2.23)
\end{aligned}
$$

For $J_2(s)$, we have by (1.8), (1.10) and moment inequality [28, Theorem 7.1]

$$
\begin{aligned}
\sup_{0\le s\le 1}\mathbb{E}|J_2(s)|^4&\le C\sup_{0\le s\le 1}\mathbb{E}\left[\int_0^{s_n}\big(\sigma(s,r,X_r)-\sigma(s_n,r,X_r)\big)^2\mathrm{d}r\right]^2\\
&\le C\sup_{0\le s\le 1}\left[\int_0^{s_n}F_2(s,s_n,r)\mathrm{d}r\right]^2\le\frac{C}{n^{2\gamma}}. \qquad (2.24)
\end{aligned}
$$

For $J_3(s)$, by (1.3), (2.1), Fubini's theorem and Hölder's inequality we get

$$
\begin{aligned}
\sup_{0\le s\le 1}\mathbb{E}|J_3(s)|^4&\le\frac{C}{n^3}\sup_{0\le s\le 1}\int_0^s\left[g_2^4(s,r)\mathbb{E}\big(1+|X_r|^{\lambda_1}\big)^4\right]\mathrm{d}r\\
&\le\frac{C}{n^3}\sup_{0\le s\le 1}\int_0^1\left[g_2^4(s,r)\mathbb{E}\big(1+|X_r|^{4\lambda_1}\big)\right]\mathrm{d}r\\
&\le\frac{C}{n^3}\sup_{0\le s\le 1}\int_0^1\left[g_2^4(s,r)\big(1+\big(\mathbb{E}|X_r|^4\big)^{\lambda_1}\big)\right]\mathrm{d}r\le\frac{C}{n^3}. \qquad (2.25)
\end{aligned}
$$

As for $J_4(s)$, we have by (1.5), (2.1), the Cauchy-Schwarz inequality, Young's inequality, moment inequality [28, Theorem 7.1] and Fubini's theorem

$$\begin{aligned}\sup_{0\le s\le 1}\mathbb{E}|J_4(s)|^4 &\le C\sup_{0\le s\le 1}\mathbb{E}\bigg|\int_{s_n}^{s} g_3^2(s,r)\big(1+|X_r|^{\lambda_2}\big)^2\mathrm{d}r\bigg|^2\\ &\le \frac{C}{n}\sup_{0\le s\le 1}\int_0^1 g_3^4(s,r)\mathbb{E}\big(1+|X_r|^{\lambda_2}\big)^4\mathrm{d}r\\ &\le \frac{C}{n}\sup_{0\le s\le 1}\int_0^1 g_3^4(s,r)\mathbb{E}\big(1+|X_r|^{4\lambda_2}\big)\mathrm{d}r\\ &\le \frac{C}{n}\sup_{0\le s\le 1}\int_0^1 g_3^4(s,r)\Big(1+\mathbb{E}|X_r|^{4\lambda_2\cdot\frac{1}{\lambda_2}}\Big)\mathrm{d}r\\ &\le \frac{C}{n}\sup_{0\le s\le 1}\int_0^1 g_3^4(s,r)\big(1+\mathbb{E}|X_r|^4\big)\mathrm{d}r\le\frac{C}{n}. \end{aligned}\tag{2.26}$$

By (2.22)-(2.26), we obtain (2.16), where $\beta = 1$. Hence, we finish the proof of Lemma 2.1. □

The following density result is also needed in the proof.

Lemma 2.2. *Under the assumption (H1), we have*

$$\lim_{n\to\infty}\sup_{0\le t\le 1}\mathbb{E}\left[\int_0^1\bigg|n\int_{s_n}^{s_n^+}\sigma\big(t,r,X_{s_n}^n\big)\mathrm{d}r-\sigma\big(t,s,X_{s_n}^n\big)\bigg|^2\mathrm{d}s\right]=0.\tag{2.27}$$

Proof. By the similar to the proof of [33, Lemma 6.1.3, Proposition 6.1.2], and [26, Case 3, Page 47], and using Lebesgue dominated convergence theorem it is easy to derive that (2.27) holds. □

3. Wong-Zakai Approximations

Proof of Theorem 1.1. By Eqs. (1.1) and (1.15), we have

$$X_t^n - X_t = \mathscr{I}_n(t)+\mathscr{J}_n(t)+\mathscr{K}_n(t),\tag{3.1}$$

where

$$\begin{aligned}\mathscr{I}_n(t)&=\int_0^t\big[b\big(t,s,X_s^n\big)-b(t,s,X_s)\big]\mathrm{d}s,\\ \mathscr{J}_n(t)&=L_n(t)-\int_0^{t_n}\sigma(t,s,X_s)\mathrm{d}W_s,\\ \mathscr{K}_n(t)&=M_n(t)-\int_{t_n}^t\sigma(t,s,X_s)\mathrm{d}W_s.\end{aligned}$$

For $\mathscr{I}_n(t)$, by the Cauchy-Schwarz inequality and (1.11), it is easy to see that

$$\mathbb{E}|\mathscr{I}_n(t)|^2\le C\int_0^t h_1^2(t,s)\mathbb{E}|X_s^n-X_s|^2\mathrm{d}s.\tag{3.2}$$

For $\mathscr{K}_n(t)$, by the Cauchy-Schwarz inequality, Young's inequality, Fubini's theorem, (1.4), (2.1) and (2.2), we have

$$\mathbb{E}|\mathscr{K}_n(t)|^2\le C\bigg[\mathbb{E}\bigg|\int_{t_n}^t\sigma\big(t,s,X_s^n\big)nW(\Delta_{n,j})\mathrm{d}s\bigg|^2+\mathbb{E}\int_{t_n}^t|\sigma(t,s,X_s)|^2\mathrm{d}s\bigg]$$

$$
\begin{aligned}
&\le C\Big\{n\int_{t_n}^t g_3^2(t,s)\mathbb{E}\Big[W^2(\Delta_{n,j})\big(1+\big|X_s^n\big|^{\lambda_2}\big)^2\Big]\mathrm{d}s+\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big(1+|X_s|^{\lambda_2}\big)^2\mathrm{d}s\Big\}\\
&\le C\Big\{n\int_{t_n}^t g_3^2(t,s)\big[\mathbb{E}W^4(\Delta_{n,j})\big]^{\frac12}\Big[\mathbb{E}\big(1+\big|X_s^n\big|^{\lambda_2}\big)^4\Big]^{\frac12}\mathrm{d}s+\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big(1+|X_s|^{2\lambda_2}\big)\mathrm{d}s\Big\}\\
&\le C\Big\{\int_{t_n}^t g_3^2(t,s)\Big[\mathbb{E}\big(1+\big|X_s^n\big|^{4\lambda_2}\big)\Big]^{\frac12}\mathrm{d}s+\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big(1+|X_s|^{2\lambda_2}\big)\mathrm{d}s\Big\}\\
&\le C\Big\{\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big(1+\big|X_s^n\big|^2\big)\mathrm{d}s+\int_{t_n}^t g_3^2(t,s)\mathbb{E}\big(1+|X_s|^2\big)\mathrm{d}s\Big\}\\
&\le C\Big[\int_{t_n}^t g_3^2(t,s)\mathrm{d}s\Big]\le\frac{C}{\sqrt n}\Big[\int_{t_n}^t g_3^4(t,s)\mathrm{d}s\Big]^{\frac12}\\
&\le\frac{C}{\sqrt n}\Big[\sup_{0\le t\le 1}\int_0^1 g_3^4(t,s)\mathrm{d}s\Big]^{\frac12}\le\frac{C}{\sqrt n}.
\end{aligned}\tag{3.3}
$$

Now let us deal with $\mathscr{J}_n(t)$. Taking into account (2.4) we have

$$
\mathscr{J}_n(t)=\mathscr{J}_{n,1}(t)+\mathscr{J}_{n,2}(t)+L_{n,3}(t),\tag{3.4}
$$

where

$$
\begin{aligned}
&\mathscr{J}_{n,1}(t)=L_{n,1}(t)-\int_0^{t_n}\sigma(t,s,X_s)\mathrm{d}W_s,\\
&\mathscr{J}_{n,2}(t)=L_{n,2}(t).
\end{aligned}
$$

Note that $\mathscr{J}_{n,1}(t)$ can be decomposed as

$$
\begin{aligned}
\mathscr{J}_{n,1}(t)&=\int_0^{t_n}\Big[n\int_{s_n}^{s_n^+}\sigma\big(t,r,X_{s_n}^n\big)\mathrm{d}r-\sigma(t,s,X_s)\Big]\mathrm{d}W_s\\
&=\int_0^{t_n}\Big[n\int_{s_n}^{s_n^+}\sigma\big(t,r,X_{s_n}^n\big)\mathrm{d}r-\sigma\big(t,s,X_{s_n}^n\big)\Big]\mathrm{d}W_s\\
&\quad+\int_0^{t_n}\big[\sigma\big(t,s,X_{s_n}^n\big)-\sigma\big(t,s,X_s^n\big)\big]\mathrm{d}W_s\\
&\quad+\int_0^{t_n}\big[\sigma\big(t,s,X_s^n\big)-\sigma(t,s,X_s)\big]\mathrm{d}W_s.
\end{aligned}
$$

By (1.5) and Itô-isometry [28, Theorem 5.21], we have

$$
\begin{aligned}
\mathbb{E}|\mathscr{J}_{n,1}(t)|^2&\le C\mathbb{E}\Big[\int_0^{t_n}\Big|n\int_{s_n}^{s_n^+}\sigma\big(t,r,X_{s_n}^n\big)\mathrm{d}r-\sigma\big(t,s,X_{s_n}^n\big)\Big|^2\mathrm{d}s\Big]\\
&\quad+C\int_0^{t_n}g_4^2(t,s)\mathbb{E}\big|X_{s_n}^n-X_s^n\big|^2\mathrm{d}s+C\int_0^{t_n}g_4^2(t,s)\mathbb{E}\big|X_s^n-X_s\big|^2\mathrm{d}s.
\end{aligned}\tag{3.5}
$$

Moreover, by (2.15), (2.16) and (2.27), we get

$$
\mathbb{E}|\mathscr{J}_{n,1}(t)|^2\le C\int_0^t g_4^2(t,s)\mathbb{E}\big|X_s^n-X_s\big|^2\mathrm{d}s+\delta_n,\quad \delta_n\ \to\ 0.\tag{3.6}
$$

The term $\mathscr{J}_{n,2}(t)$ can be written as

$$
\mathscr{J}_{n,2}(t)=\sum_{j=1}^7\mathscr{J}_{n,2}^{(j)}(t),\tag{3.7}
$$

where

$$\mathscr{I}_{n,2}^{(1)}(t) = n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\Big(t,s,X^n_{\frac{j}{n}}\Big)\bigg(\int_0^{\frac{j}{n}}\sigma_1\big(u,r,X^n_r\big)\mathrm{d}W^n_r\bigg)\mathrm{d}u\mathrm{d}s,$$

$$\mathscr{I}_{n,2}^{(2)}(t) = n\sum_{j=0}^{\lfloor nt\rfloor-1} W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\Big[\sigma_3\big(t,s,X^n_u\big)-\sigma_3\Big(t,s,X^n_{\frac{j}{n}}\Big)\Big]\bigg(\int_0^{\frac{j}{n}}\sigma_1\big(u,r,X^n_r\big)\mathrm{d}W^n_r\bigg)\mathrm{d}u\mathrm{d}s,$$

$$\mathscr{I}_{n,2}^{(3)}(t) = n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\big(t,s,X^n_u\big)\bigg(\int_{\frac{j}{n}}^{u}\big(\sigma_1\big(u,r,X^n_r\big)-\sigma_1(u,r,X_r)\big)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s,$$

$$\mathscr{I}_{n,2}^{(4)}(t) = n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\big(\sigma_3\big(t,s,X^n_u\big)-\sigma_3(t,s,X_u)\big)\int_{\frac{j}{n}}^{u}\sigma_1(u,r,X_r)\mathrm{d}r\mathrm{d}u\mathrm{d}s,$$

$$\mathscr{I}_{n,2}^{(5)}(t) = n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\Big(t,s,X_{\frac{j}{n}}\Big)\int_{\frac{j}{n}}^{u}\sigma_1\Big(u,r,X_{\frac{j}{n}}\Big)\mathrm{d}r\mathrm{d}u\mathrm{d}s,$$

$$\mathscr{I}_{n,2}^{(6)}(t) = n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\Big(t,s,X_{\frac{j}{n}}\Big)\int_{\frac{j}{n}}^{u}\Big[\sigma_1(u,r,X_r)-\sigma_1\Big(u,r,X_{\frac{j}{n}}\Big)\Big]\mathrm{d}r\mathrm{d}u\mathrm{d}s,$$

$$\mathscr{I}_{n,2}^{(7)}(t) = n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\Big[\sigma_3(t,s,X_u)-\sigma_3\Big(t,s,X_{\frac{j}{n}}\Big)\Big]\int_{\frac{j}{n}}^{u}\sigma_1(u,r,X_r)\mathrm{d}r\mathrm{d}u\mathrm{d}s.$$

For $\mathscr{I}_{n,2}^{(1)}(t)$, by the independence of the increments of the Brownian motion, the Cauchy-Schwarz inequality, Fubini's theorem, (1.2) and (1.5), we have

$$\begin{aligned}
\mathbb{E}|\mathscr{I}_{n,2}^{(1)}(t)|^2 &= n^2\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\bigg|W(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\sigma_3\Big(t,s,X^n_{\frac{j}{n}}\Big)\int_{\frac{j}{n}}^{s}\bigg(\int_0^{\frac{j}{n}}\sigma_1\big(u,r,X^n_r\big)\mathrm{d}W^n_r\bigg)\mathrm{d}u\mathrm{d}s\bigg|^2\\
&\le Cn^2\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}W^2(\Delta_{n,j})\mathbb{E}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\int_{\frac{j}{n}}^{s}\int_0^{\frac{j}{n}}g_1(u,r)\bigg|\frac{\mathrm{d}W^n_r}{\mathrm{d}r}\bigg|\mathrm{d}r\mathrm{d}u\mathrm{d}s\bigg]^2\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4^2(t,s)\mathrm{d}s\bigg]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\Big(s-\frac{j}{n}\Big)\int_{\frac{j}{n}}^{s}\mathbb{E}\bigg(\int_0^1 g_1(u,r)\bigg|\frac{\mathrm{d}W^n_r}{\mathrm{d}r}\bigg|\mathrm{d}r\bigg)^2\mathrm{d}u\mathrm{d}s\\
&= Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4^2(t,s)\mathrm{d}s\bigg]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\Big(s-\frac{j}{n}\Big)\\
&\quad\times\int_{\frac{j}{n}}^{s}\int_0^1\int_0^1 g_1(u,r_1)g_1(u,r_2)\mathbb{E}\bigg(\bigg|\frac{\mathrm{d}W^n_{r_1}}{\mathrm{d}r_1}\bigg|\bigg|\frac{\mathrm{d}W^n_{r_1}}{\mathrm{d}r_1}\bigg|\bigg)\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}u\mathrm{d}s\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4^2(t,s)\mathrm{d}s\bigg]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\Big(s-\frac{j}{n}\Big)\int_{\frac{j}{n}}^{s}\int_0^1\int_0^1 g_1(u,r_1)g_1(u,r_2)\\
&\quad\times\bigg(\mathbb{E}\bigg|\frac{\mathrm{d}W^n_{r_1}}{\mathrm{d}r_1}\bigg|^2\bigg)^{\frac{1}{2}}\bigg(\mathbb{E}\bigg|\frac{\mathrm{d}W^n_{r_1}}{\mathrm{d}r_1}\bigg|^2\bigg)^{\frac{1}{2}}\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}u\mathrm{d}s\\
&\le Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4^2(t,s)\mathrm{d}s\bigg]\sup_{0\le r\le 1}\mathbb{E}\bigg|\frac{\mathrm{d}W^n_r}{\mathrm{d}r}\bigg|^2\int_{\frac{j}{n}}^{\frac{j+1}{n}}\Big(s-\frac{j}{n}\Big)^2\mathrm{d}s
\end{aligned}$$

$$\leq \frac{C}{n}\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}} g_4^2(t,s)\mathrm{d}s \leq \frac{C}{n}\sup_{0\leq u\leq 1}\int_0^1 g_4^2(u,s)\mathrm{d}s \leq \frac{C}{n}. \tag{3.8}$$

For $\mathscr{I}_{n,2}^{(2)}(t)$, we obtain by the Cauchy-Schwarz inequality, Fubini's theorem, (1.2), (1.12), (2.15), [30, Page 203] (or [11, Eq. (12)]),

$$\begin{aligned}
&\mathbb{E}\big|\mathscr{I}_{n,2}^{(2)}(t)\big|^2\\
&\leq Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[|W(\Delta_{n,j})|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} h_2(t,s)\Big|X_u^n - X_{\frac{j}{n}}^n\Big|\int_0^{\frac{j}{n}} g_1(u,r)\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\mathrm{d}r\mathrm{d}u\mathrm{d}s\right]^2\\
&\leq Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} h_2(t,s)\Big|X_u^n - X_{\frac{j}{n}}^n\Big||W(\Delta_{n,j})|\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\int_0^{\frac{j}{n}} g_1(u,r)\mathrm{d}r\mathrm{d}u\mathrm{d}s\right]^2\\
&\leq Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} h_2(t,s)\Big|X_u^n - X_{\frac{j}{n}}^n\Big||W(\Delta_{n,j})|\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\sup_{0\leq u\leq 1}\int_0^1 g_1(u,r)\mathrm{d}r\mathrm{d}u\mathrm{d}s\right]^2\\
&\leq Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} h_2(t,s)\sup_{\frac{j}{n}\leq u\leq s}\Big|X_u^n - X_{\frac{j}{n}}^n\Big||W(\Delta_{n,j})|\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\mathrm{d}u\mathrm{d}s\right]^2\\
&\leq Cn\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} h_2(t,s)\sup_{\frac{j}{n}\leq u\leq s}\Big|X_u^n - X_{\frac{j}{n}}^n\Big||W(\Delta_{n,j})|\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\mathrm{d}s\right]^2\\
&\leq C\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}} h_2^2(t,s)\mathbb{E}\left[\sup_{\frac{j}{n}\leq u\leq s}\Big|X_u^n - X_{\frac{j}{n}}^n\Big||W(\Delta_{n,j})|\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\right]^2\mathrm{d}s\\
&\leq C\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}} h_2^2(t,s)\left[\mathbb{E}\sup_{\frac{j}{n}\leq u\leq s}\Big(X_u^n - X_{\frac{j}{n}}^n\Big)^4\right]^{\frac{1}{2}}\left[\mathbb{E}\Big(|W(\Delta_{n,j})|\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|\Big)^4\right]^{\frac{1}{2}}\mathrm{d}s\\
&\leq \frac{C}{n^{\frac{\alpha}{2}}}\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}} h_2^2(t,s)\left[\mathbb{E}W^8(\Delta_{n,j})\mathbb{E}\Big(\sup_{0\leq r\leq 1}\Big|\frac{\mathrm{d}W_r^n}{\mathrm{d}r}\Big|^8\Big)\right]^{\frac{1}{4}}\mathrm{d}s\\
&\leq \frac{C(1+\log n)}{n^{\frac{\alpha}{2}}}\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}} h_2^2(t,s)\mathrm{d}s \leq \frac{C(1+\log n)}{n^{\frac{\alpha}{2}}}\sup_{0\leq t\leq 1}\int_0^1 h_2^2(t,s)\mathrm{d}s\\
&\leq \frac{C(1+\log n)}{n^{\frac{\alpha}{2}}}.
\end{aligned}\tag{3.9}$$

By (1.2) and (1.5), we have

$$\begin{aligned}
|\mathscr{I}_{n,2}^{(3)}(t)| &\leq Cn^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}} g_4(t,s)\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u} g_1(u,r)\big|X_r^n - X_r\big|\mathrm{d}r\mathrm{d}u\mathrm{d}s\\
&\leq \mathscr{I}_{n,21}^{(3)}(t) + \mathscr{I}_{n,22}^{(3)}(t) + \mathscr{I}_{n,23}^{(3)}(t),
\end{aligned}\tag{3.10}$$

where

$$\mathscr{I}_{n,21}^{(3)}(t) = Cn^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}} g_4(t,s)\mathrm{d}s\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u} g_1(u,r)\Big|X_r^n - X_{\frac{j}{n}}^n\Big|\mathrm{d}r\mathrm{d}u,$$

$$\mathscr{I}_{n,22}^{(3)}(t)=Cn^2\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}g_1(u,r)\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|\mathrm{d}r\mathrm{d}u,$$

$$\mathscr{I}_{n,23}^{(3)}(t)=Cn^2\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}g_1(u,r)\Big|X_{\frac{j}{n}}-X_r\Big|\mathrm{d}r\mathrm{d}u.$$

For $\mathscr{I}_{n,21}^{(3)}(t)$, by (2.15), [3, Example 5.3] and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\big|\mathscr{I}_{n,21}^{(3)}(t)\big|^2&=Cn^4\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}g_1(u,r)\Big|X_r^n-X_{\frac{j}{n}}^n\Big|\mathrm{d}r\mathrm{d}u\Bigg]^2\\
&\le Cn^4\mathbb{E}\Big[\sup_j W^4(\Delta_{n,j})\sup_{0\le r\le 1}\big|X_r^n-X_{r_n}^n\big|^2\Big]\\
&\quad\times\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}g_1(u,r)\mathrm{d}r\mathrm{d}u\mathrm{d}s\Bigg]^2\\
&\le Cn^4\Big[\mathbb{E}\sup_j W^8(\Delta_{n,j})\Big]^{\frac12}\mathbb{E}\Big[\sup_{0\le r\le 1}\big|X_r^n-X_{r_n}^n\big|^4\Big]^{\frac12}\\
&\quad\times\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\int_{\frac{j}{n}}^{s}\bigg(\sup_{0\le u\le 1}\int_0^1 g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\Bigg]^2\\
&\le\frac{C[1+(\log n)^2]}{n^{\frac{\alpha}{2}}}\sup_{0\le t\le 1}\Bigg(\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{2^n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\Bigg)^2\\
&\le\frac{C[1+(\log n)^2]}{n^{\frac{\alpha}{2}}}\bigg(\sup_{0\le t\le 1}\int_0^1 g_4^2(t,s)\mathrm{d}s\bigg)\le\frac{C[1+(\log n)^2]}{n^{\frac{\alpha}{2}}}.
\end{aligned}\tag{3.11}$$

Similarly for $\mathscr{I}_{n,23}^{(3)}(t)$, we have

$$\mathbb{E}\big|\mathscr{I}_{n,23}^{(3)}(t)\big|^2\le\frac{C[1+(\log n)^2]}{n^{\frac{\beta}{2}}}\bigg(\sup_{0\le t\le 1}\int_0^1 g_4^2(t,s)\mathrm{d}s\bigg)\le\frac{C[1+(\log n)^2]}{n^{\frac{\beta}{2}}}.\tag{3.12}$$

For $\mathscr{I}_{n,22}^{(3)}(t)$, by the independence of the increments of the Brownian motion, the Cauchy-Schwarz inequality, Fubini's theorem and Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\big|\mathscr{I}_{n,22}^{(3)}(t)\big|^2&=Cn^4\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}g_4(t,s)\mathrm{d}s\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}g_1(u,r)\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|\mathrm{d}r\mathrm{d}u\Bigg]^2\\
&\le Cn^4\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}W^2(\Delta_{n,j})\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|g_4(t,s)\mathrm{d}s\int_{\frac{j}{n}}^{s}\bigg(\sup_{0\le u\le 1}\int_0^1 g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\Bigg]^2\\
&\le Cn^2\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}W^2(\Delta_{n,j})\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|g_4(t,s)\mathrm{d}s\Bigg]^2\\
&\le Cn^2\mathbb{E}\bigg[\int_0^t W^2(\Delta_{n,\lfloor ns\rfloor})\big|X_{s_n}^n-X_{s_n}\big|g_4(t,s)\mathrm{d}s\bigg]^2\\
&\le Cn^2\mathbb{E}\int_0^t W^4(\Delta_{n,\lfloor ns\rfloor})\big|X_{s_n}^n-X_{s_n}\big|^2g_4^2(t,s)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le Cn^2\int_0^t \mathbb{E}W^4(\Delta_{n,\lfloor ns\rfloor})\mathbb{E}\big|X^n_{s_n}-X_{s_n}\big|^2 g_4^2(t,s)\mathrm{d}s\\
&\le C\int_0^t \mathbb{E}\big|X^n_{s_n}-X_{s_n}\big|^2 g_4^2(t,s)\mathrm{d}s\\
&\le C\int_0^t \mathbb{E}\big|X^n_{s}-X_{s}\big|^2 g_4^2(t,s)\mathrm{d}s + C\int_0^t \mathbb{E}\big|X^n_{s}-X^n_{s_n}\big|^2 g_4^2(t,s)\mathrm{d}s\\
&\quad + C\int_0^t \mathbb{E}|X_s-X_{s_n}|^2 g_4^2(t,s)\mathrm{d}s\\
&\le C\int_0^t \mathbb{E}\big|X^n_{s}-X_{s}\big|^2 g_4^2(t,s)\mathrm{d}s + \frac{C}{n^{\frac{\alpha}{2}}}+\frac{C}{n^{\frac{\beta}{2}}}\\
&\le C\int_0^t \mathbb{E}\big|X^n_{s}-X_{s}\big|^2 g_4^2(t,s)\mathrm{d}s + \delta_n, \quad \delta_n \;\longrightarrow\; 0. \qquad (3.13)
\end{aligned}
$$

By (3.10)-(3.13), we find that

$$
\mathbb{E}\big|\mathscr{I}^{(3)}_{n,2}(t)\big|^2 \le C\int_0^t \mathbb{E}\Big[\big|X^n_s-X_s\big|^2\Big] g_4^2(t,s)\mathrm{d}s+\delta_n, \quad \delta_n \;\longrightarrow\; 0. \qquad (3.14)
$$

For $\mathscr{I}^{(4)}_{n,2}(t)$, we can write

$$
\begin{aligned}
\big|\mathscr{I}^{(4)}_{n,2}(t)\big| &\le n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\big|\sigma_3\big(t,s,X^n_u\big)-\sigma_3(t,s,X_u)\big|\int_{\frac{j}{n}}^{u}|\sigma_1(u,r,X_r)|\mathrm{d}r\mathrm{d}u\mathrm{d}s\\
&\le \mathscr{I}^{(4)}_{n,21}(t)+\mathscr{I}^{(4)}_{n,22}(t)+\mathscr{I}^{(4)}_{n,23}(t), \qquad (3.15)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{I}^{(4)}_{n,21}(t) &= n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\Big|\sigma_3\big(t,s,X^n_u\big)-\sigma_3\Big(t,s,X^n_{\frac{j}{n}}\Big)\Big|\int_{\frac{j}{n}}^{u}|\sigma_1(u,r,X_r)|\mathrm{d}r\mathrm{d}u\mathrm{d}s,\\
\mathscr{I}^{(4)}_{n,22}(t) &= n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\Big|\sigma_3\Big(t,s,X^n_{\frac{j}{n}}\Big)-\sigma_3\Big(t,s,X_{\frac{j}{n}}\Big)\Big|\int_{\frac{j}{n}}^{u}|\sigma_1(u,r,X_r)|\mathrm{d}r\mathrm{d}u\mathrm{d}s,\\
\mathscr{I}^{(4)}_{n,23}(t) &= n^2\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\Big|\sigma_3\Big(t,s,X_{\frac{j}{n}}\Big)-\sigma_3(t,s,X_u)\Big|\int_{\frac{j}{n}}^{u}|\sigma_1(u,r,X_r)|\mathrm{d}r\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

For $\mathscr{I}^{(4)}_{n,21}(t)$, by (1.2), (1.12), (2.15), [3, Example 5.3], Fubini's theorem and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\mathbb{E}\big|\mathscr{I}^{(4)}_{n,21}(t)\big|^2 &\le Cn^4\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} h_2(t,s)\Big|X^n_u-X^n_{\frac{j}{n}}\Big|\int_{\frac{j}{n}}^{u} g_1(u,r)\mathrm{d}r\mathrm{d}u\mathrm{d}s\Bigg]^2\\
&\le Cn^4\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1} W^2(\Delta_{n,j})\sup_{0\le r\le 1}\big|X^n_r-X^n_{r_n}\big|\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} h_2(t,s)\Big(\sup_{0\le u\le 1}\int_0^1 g_1(u,r)\mathrm{d}r\Big)\mathrm{d}u\mathrm{d}s\Bigg]^2\\
&\le Cn^2\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\sup_j W^2(\Delta_{n,j})\sup_{0\le r\le 1}\big|X^n_r-X^n_{r_n}\big|\int_{\frac{j}{n}}^{\frac{j+1}{n}} h_2(t,s)\mathrm{d}s\Bigg]^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq Cn^2\mathbb{E}\bigg[\sup_j W^2(\Delta_{n,j})\sup_{0\leq r\leq 1}\big|X_r^n-X_{r_n}^n\big|\sum_{j=0}^{\lfloor nt\rfloor-1}\bigg(\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2(t,s)\mathrm{d}s\bigg)\bigg]^2\\
&\leq Cn^2\mathbb{E}\Big[\sup_j W^8(\Delta_{n,j})\Big]^{\frac{1}{2}}\mathbb{E}\Big[\sup_{0\leq r\leq 1}\big|X_r^n-X_{r_n}^n\big|^4\Big]^{\frac{1}{2}}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2(t,s)\mathrm{d}s\Bigg]^2\\
&\leq \frac{C[1+(\log n)^2]}{n^{\frac{\alpha}{2}}}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2(t,s)\mathrm{d}s\Bigg]^2\\
&\leq \frac{C[1+(\log n)^2]}{n^{\frac{\alpha}{2}}}\sup_{0\leq t\leq 1}\int_0^1 h_2^2(t,s)\mathrm{d}s\leq \frac{C[1+(\log n)^2]}{n^{\frac{\alpha}{2}}}. \qquad (3.16)
\end{aligned}
$$

Similarly,

$$
\mathscr{I}_{n,23}^{(4)}(t)\leq \frac{C[1+(\log n)^2]}{n^{\frac{\beta}{2}}}. \qquad (3.17)
$$

For $\mathscr{I}_{n,22}^{(4)}(t)$, by Fubini's theorem, the Cauchy-Schwarz inequality, (1.12), (2.15) and (2.16), we have

$$
\begin{aligned}
\mathbb{E}\big|\mathscr{I}_{n,22}^{(4)}(t)\big|^2 &\leq Cn^4\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}h_2(t,s)\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|\bigg(\sup_{0\leq u\leq 1}\int_0^1 g_1(u,r)\mathrm{d}r\bigg)\mathrm{d}u\mathrm{d}s\Bigg]^2\\
&\leq Cn^2\mathbb{E}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2(t,s)\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|\mathrm{d}s\Bigg]^2\\
&\leq Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[W^4(\Delta_{n,j})\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|^2\right]\Bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2(t,s)\mathrm{d}s\Bigg]^2\\
&= Cn^3\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}W^4(\Delta_{n,j})\mathbb{E}\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|^2\Bigg[\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2(t,s)\mathrm{d}s\Bigg]^2\\
&\leq C\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}}h_2^2(t,s)\mathbb{E}\Big|X_{\frac{j}{n}}^n-X_{\frac{j}{n}}\Big|^2\mathrm{d}s\\
&\leq C\int_0^t\mathbb{E}\big|X_{s_n}^n-X_{s_n}\big|^2h_2^2(t,s)\mathrm{d}s\\
&\leq C\int_0^t\mathbb{E}\big|X_s^n-X_s\big|^2h_2^2(t,s)\mathrm{d}s+C\int_0^t\mathbb{E}\big|X_s^n-X_{s_n}^n\big|^2h_2^2(t,s)\mathrm{d}s\\
&\quad +C\int_0^t\mathbb{E}\big|X_s-X_{s_n}\big|^2h_2^2(t,s)\mathrm{d}s\\
&\leq C\int_0^t\mathbb{E}\big|X_s^n-X_s\big|^2h_2^2(t,s)\mathrm{d}s+\delta_n,\quad \delta_n\ \longrightarrow\ 0. \qquad (3.18)
\end{aligned}
$$

By (3.15)-(3.18), we have

$$
\mathbb{E}\big|\mathscr{I}_{n,2}^{(4)}(t)\big|\leq C\int_0^t\mathbb{E}\big|X_s^n-X_s\big|^2h_2^2(t,s)\mathrm{d}s+\delta_n,\quad \delta_n\ \longrightarrow\ 0. \qquad (3.19)
$$

Likewise, we have

$$
\mathbb{E}\big|\mathscr{I}_{n,2}^{(i)}\big|^2\leq C\int_0^t\mathbb{E}\big|X_s^n-X_s\big|^2h_2^2(t,s)\mathrm{d}s+\delta_n,\quad i=6,7. \qquad (3.20)
$$

Before dealing with $\mathscr{I}_{n,2}^{(5)}(t)$, let us give some prior estimates. For all $s\in[0,1]$ and $x\in\mathbb{R}^d$, by (1.14) we have $\sigma_1(s,s,x)=0$. Moreover, by (1.6) we have $\sigma_1(u,s,x)\geq\sigma_1(u,u,x)=0$ for $0\leq s\leq u\leq 1$ and $x\in\mathbb{R}^d$. Thus, $\sigma(u,s,x)$ is a monotonically increasing function with respect to the first variable u for $0\leq s\leq u\leq 1$ and $x\in\mathbb{R}^d$. By (1.14), we have

$$n\int_{s_n}^{s}\sigma(s_n,r,X_{s_n})\mathrm{d}r\leq n\int_{s_n}^{s}\sigma(r,r,X_{s_n})\mathrm{d}r\leq 0, \tag{3.21}$$

for any $s\in[0,1]$.

For any $s\in[0,1]$, by (1.6) we have

$$(ns-\lfloor ns\rfloor)\bigg(\int_0^s\big[\sigma_1\big(s_n+\theta(s-s_n),r,X_{s_n}\big)-\sigma_1(s,r,X_{s_n})\big]\mathrm{d}r\bigg)\leq 0, \tag{3.22}$$

where $\theta\in(0,1)$. Note that

$$\begin{aligned}
&n\bigg(\int_0^s\sigma(s,r,X_{s_n})\mathrm{d}r-\int_0^{s_n}\sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg)-(ns-\lfloor ns\rfloor)\int_0^s\sigma_1(s,r,X_{s_n})\mathrm{d}r\\
&=n\int_{s_n}^s\sigma(s_n,r,X_{s_n})\mathrm{d}r+n\bigg(\int_0^s[\sigma(s,r,X_{s_n})-\sigma(s_n,r,X_{s_n})]\mathrm{d}r\bigg)\\
&\quad-(ns-\lfloor ns\rfloor)\int_0^s\sigma_1(s,r,X_{s_n})\mathrm{d}r\\
&=n\int_{s_n}^s\sigma(s_n,r,X_{s_n})\mathrm{d}r+(ns-\lfloor ns\rfloor)\bigg(\int_0^s\big[\sigma_1\big(s_n+\theta(s-s_n),r,X_{s_n}\big)-\sigma_1(s,r,X_{s_n})\big]\mathrm{d}r\bigg).
\end{aligned} \tag{3.23}$$

Squaring both sides of (3.23) and using (1.16), (3.21), (3.22), we have

$$\begin{aligned}
&\bigg|n\int_{s_n}^s\sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg|^2\\
&\leq\bigg|n\bigg(\int_0^s\sigma(s,r,X_{s_n})\mathrm{d}r-\int_0^{s_n}\sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg)-(ns-\lfloor ns\rfloor)\int_0^s\sigma_1(s,r,X_{s_n})\mathrm{d}r\bigg|^2\leq\frac{C}{n^{\varsigma_1}}.
\end{aligned} \tag{3.24}$$

Next, we write the term $\mathscr{I}_{n,2}^{(5)}(t)$ in the form

$$\begin{aligned}
\mathscr{I}_{n,2}^{(5)}(t)&=n^2\sum_{j=0}^{\lfloor nt\rfloor-1}W^2(\Delta_{n,j})\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s}\sigma_3\Big(t,s,X_{\frac{j}{n}}\Big)\int_{\frac{j}{n}}^{u}\sigma_1\big(u,r,X_{\frac{j}{n}}\big)\mathrm{d}r\mathrm{d}u\mathrm{d}s\\
&=\int_0^{t_n}n^2W^2(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s\sigma_3(t,s,X_{s_n})\int_{s_n}^u\sigma_1(u,r,X_{s_n})\mathrm{d}r\mathrm{d}u\mathrm{d}s\\
&=\int_0^{t_n}n^2W^2(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s\sigma(s,r,X_{s_n})\mathrm{d}r\sigma_3(t,s,X_{s_n})\mathrm{d}s\\
&=:\mathscr{O}_{n,1}+\mathscr{O}_{n,2},
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{O}_{n,1}(t)&=\int_0^{t_n}n^2W^2(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s\sigma(s_n,r,X_{s_n})\mathrm{d}r\sigma_3(t,s,X_{s_n})\mathrm{d}s,\\
\mathscr{O}_{n,2}(t)&=\int_0^{t_n}n^2W^2(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s\sigma(s,r,X_{s_n})-\sigma(s_n,r,X_{s_n})\mathrm{d}r\sigma_3(t,s,X_{s_n})\mathrm{d}s.
\end{aligned}$$

For $\mathscr{O}_{n,1}(t)$, by (1.5), (3.24), the independence of the increments of the Brownian motion, Fubini's theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}|\mathscr{O}_{n,1}(t)|^2 &= \mathbb{E}\bigg[\int_0^{t_n} n^2W^2(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s \sigma(s_n,r,X_{s_n})\mathrm{d}r\sigma_3(t,s,X_{s_n})\mathrm{d}s\bigg]^2\\
&\le \mathbb{E}\bigg[\int_0^{t_n} n^2W^2(\Delta_{n,\lfloor ns\rfloor})\bigg|\int_{s_n}^s \sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg||\sigma_3(t,s,X_{s_n})|\mathrm{d}s\bigg]^2\\
&\le C\mathbb{E}\bigg[\int_0^{t_n} n^2W^2(\Delta_{n,\lfloor ns\rfloor})\bigg|\int_{s_n}^s \sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg|g_4(t,s)\mathrm{d}s\bigg]^2\\
&\le C\int_0^{t_n} n^4\mathbb{E}W^4(\Delta_{n,\lfloor ns\rfloor})\mathbb{E}\bigg|\int_{s_n}^s \sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg|^2 g_4^2(t,s)\mathrm{d}s\\
&= C\int_0^{t_n} \mathbb{E}\bigg|n\int_{s_n}^s \sigma(s_n,r,X_{s_n}g)\mathrm{d}r\bigg|^2 g_4^2(t,s)\mathrm{d}s\\
&\le \frac{C}{n^{\varsigma_1}}\int_0^{t_n} g_4^2(t,s)\mathrm{d}s \le \frac{C}{n^{\varsigma_1}}\sup_{0\le t\le 1}\int_0^1 g_4^2(t,s)\mathrm{d}s\\
&\le \frac{C}{n^{\varsigma_1}} \ \to\ 0 \quad \text{as} \quad n\ \to\ \infty.
\end{aligned} \tag{3.25}$$

For $\mathscr{O}_{n,2}(t)$, by (1.5), (1.10), Fubini's theorem, the Cauchy-Schwarz inequality and dominated convergence theorem, we have

$$\begin{aligned}
\mathbb{E}|\mathscr{O}_{n,2}(t)|^2 &= \mathbb{E}\bigg[\int_0^{t_n} n^2W^2(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s \sigma(s,r,X_{s_n})-\sigma(s_n,r,X_{s_n})\mathrm{d}r\sigma_3(t,s,X_{s_n})\mathrm{d}s\bigg]^2\\
&\le \mathbb{E}\bigg[\int_0^{t_n} n^2W^2(\Delta_{n,\lfloor ns\rfloor})\bigg|\int_{s_n}^s \sigma(s,r,X_{s_n})-\sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg||\sigma_3(t,s,X_{s_n})|\mathrm{d}s\bigg]^2\\
&\le C\mathbb{E}\bigg[\int_0^{t_n} n^2W^2(\Delta_{n,\lfloor ns\rfloor})\bigg|\int_{s_n}^s \sigma(s,r,X_{s_n})-\sigma(s_n,r,X_{s_n})\mathrm{d}r\bigg|g_4(t,s)\mathrm{d}s\bigg]^2\\
&\le C\mathbb{E}\bigg[\int_0^{t_n} n^4(s-s_n)W^4(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s |\sigma(s,r,X_{s_n})-\sigma(s_n,r,X_{s_n})|^2\mathrm{d}r g_4^2(t,s)\mathrm{d}s\bigg]\\
&\le C\mathbb{E}\bigg[\int_0^{t_n} n^3W^4(\Delta_{n,\lfloor ns\rfloor})\int_{s_n}^s F_2(s,s_n,r)\mathrm{d}r g_4^2(t,s)\mathrm{d}s\bigg]\\
&\le C\int_0^{t_n} n^3\mathbb{E}W^4(\Delta_{n,\lfloor ns\rfloor})\int_0^s F_2(s,s_n,r)\mathrm{d}r g_4^2(t,s)\mathrm{d}s\\
&\le \frac{C}{n^{\lambda-1}}\int_0^{t_n} g_4^2(t,s)\mathrm{d}s \le \frac{C}{n^{\lambda-1}}\sup_{0\le t\le 1}\int_0^1 g_4^2(t,s)\mathrm{d}s \ \to\ 0 \quad \text{as} \quad n\ \to\ \infty.
\end{aligned} \tag{3.26}$$

From (3.25)-(3.26) we obtain

$$\mathbb{E}\big|\mathscr{I}_{n,2}^{(5)}(t)\big|^2 \le \delta_n, \quad \delta_n\ \to\ 0. \tag{3.27}$$

Now, utilizing (3.7)-(3.20) and (3.27) we have

$$\mathbb{E}\big|\mathscr{I}_{n,2}(t)\big|^2 \le \delta_n + C\int_0^t \big[h_2^2(t,s)+g_4^2(t,s)\big]\mathbb{E}\big|X_s^n - X_s\big|^2\mathrm{d}s. \tag{3.28}$$

Finally, by (2.10), (3.1)-(3.6) and (3.28), we have for any $u\in[0,1]$,

$$\sup_{t\le u}\mathbb{E}|X_t^n - X_t|^2 \le \delta_n + C\int_0^u \big[h_1^2(u,s)+h_2^2(u,s)+g_4^2(u,s)\big]\sup_{r\le s}\mathbb{E}|X_r^n - X_r|^2\mathrm{d}s, \tag{3.29}$$

where $\delta_n \to 0$. By (3.29) and Gronwall's inequality (see [16, Theorem 16]) it follows that

$$\sup_{0\le t\le 1} \mathbb{E}\big|X_t^n - X_t\big|^2 \;\to\; 0. \tag{3.30}$$

The proof is finished. □

4. Stochastic Volterra Equation with Fractional Brownian Motion Kernel

For any $H \in (1/2, 1)$, we set

$$K_H(t,s) := \begin{cases} c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, & 0 < s \le t \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

where

$$c_H = \left(H - \frac{1}{2}\right)\sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}},$$

and Γ denotes the usual Gamma function.

The fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ may be defined by (cf. [14])

$$B_t := \int_0^t K_H(t,s)\mathrm{d}W_s,$$

which has the covariance function

$$R_H(s,t) = \mathbb{E}(B_t B_s) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big).$$

The fractional Brownian motion has the following properties: Long-range dependence, self-similarity, and Höder continuity. The fractional Brownian motion is neither a semimartingale nor a Markov process.

We consider the following stochastic Volterra equation with the kernel function K_H:

$$X_t = \xi + \int_0^t K_H(t,s) b(X_s)\mathrm{d}s + \int_0^t K_H(t,s)\sigma(X_s)\mathrm{d}W_s \tag{4.2}$$

for any $t \in [0,1]$, where $\xi \in \mathbb{R}^d, b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable functions. Eq. (4.2) has been investigated in many fields, including nonlinear filtering [13] using fractional Brownian motion kernels, pharmacokinetic models [29] (Langevin equation driven by fractional Brownian motion), fluid turbulence [10], and turbulence modelling in atmospheric winds or energy prices [4, 12] using Brownian semistationary processes.

Let X_t^n solve the following equation:

$$X_t^n = \xi + \int_0^t K_H(t,s) b\big(X_s^n\big)\mathrm{d}s + \int_0^t K_H(t,s)\sigma\big(X_s^n\big)\mathrm{d}W_s^n, \quad t \in [0,1]. \tag{4.3}$$

In the section we introduce the following assumption.

(H6) Assume that $b \in C_b^1(\mathbb{R}^d), \sigma \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d\times m})$.

Theorem 4.1. *Let $H \in (1/2, 1)$. Under the assumption (H6) we have*

$$\lim_{n\to\infty} \sup_{0\le t\le 1} \mathbb{E}\big|X_t^n - X_t\big|^2 = 0,$$

where X_t and X_t^n are the solutions of Eqs. (4.2) *and* (4.3)*, respectively.*

Proof. By Lemma A.1, it is easy to verify that (H1)-(H4) hold for all $H \in (1/2, 1)$. By Lemma A.2, we find that (H5) holds for all $H \in (1/2, 1)$. Thus, we derive from Theorem 1.1 that Theorem 4.1 is true. We complete the proof. □

5. Stochastic Volterra Equation with Subfractional Brownian Motion Kernel

For any $H \in (1/2, 1)$, we set

$$\mathcal{K}_H(t,s) := \begin{cases} b_H s^{\frac{3}{2}-H} \displaystyle\int_s^t (u^2 - s^2)^{H-\frac{3}{2}} \mathrm{d}u, & 0 \le s \le t \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

where

$$b_H = \sqrt{\frac{\Gamma(1+2H)\sin(\pi H)}{\pi}}.$$

The subfractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ may be defined by (cf. [8,17])

$$\mathcal{B}_t := \int_0^t \mathcal{K}_H(t,s) \mathrm{d}W_s,$$

which has the covariance function

$$\mathcal{R}_H(s,t) = \mathbb{E}(\mathcal{B}_t \mathcal{B}_s) = s^{2H} + t^{2H} - \frac{1}{2}\big[(s+t)^{2H} + (t-s)^{2H}\big].$$

The subfractional Brownian motion and the fractional Brownian motion have similar properties: long-range dependence, self-similarity, and Hölder continuity. The subfractional Brownian motion is neither a semimartingale nor a Markov process. But, compared with the fractional Brownian motion, the subfractional Brownian motion has non-stationary increments and the increments over non-overlapping intervals are weakly correlated (cf. [8,17]).

We consider the following stochastic Volterra equation with the kernel function $\mathcal{K}_H$:

$$X_t = \xi + \int_0^t \mathcal{K}_H(t,s) b(X_s) \mathrm{d}s + \int_0^t \mathcal{K}_H(t,s) \sigma(X_s) \mathrm{d}W_s \tag{5.2}$$

for any $t \in [0,1]$, where $\xi \in \mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable functions. The subfractional Brownian motion is an extension of the Brownian motion that retains many properties of fractional Brownian motion, but not the stationary increments. This property makes subfractional Brownian motion a possible candidate for models that include long-range dependence, self-similarity, and non-stationary increments which is suitable for the construction of stochastic models in finance and non-stationary queueing systems. Eq. (5.2) is applied in the systems, containing stochastic behavior, long-range dependence, and non-stationary increments (cf. [1,24,32]).

Let X_t^n solve the following equation:

$$X_t^n = \xi + \int_0^t \mathcal{K}_H(t,s) b\big(X_s^n\big)\mathrm{d}s + \int_0^t \mathcal{K}_H(t,s)\sigma\big(X_s^n\big)\mathrm{d}W_s^n, \quad t \in [0,1]. \tag{5.3}$$

(H7) Assume that $b \in C_b^1(\mathbb{R}^d), \sigma \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d\times m})$.

Theorem 5.1. *Let $H \in (1/2, 1)$. Under the assumption (H7) we have*

$$\lim_{n\to\infty} \sup_{0\le t\le 1} \mathbb{E}|X_t^n - X_t|^2 = 0,$$

where X_t and X_t^n are the solutions of Eqs. (5.2) *and* (5.3)*, respectively.*

Proof. By Lemmas B.1-B.2, it is easy to derive that Theorem 5.1 holds. □

Appendix A

The property of K_H is very important in the proof of Theorem 4.1. The following result is taken from [13, 14].

Lemma A.1. (i_1) *The mappings $s \to K_H(t,s)$ is continuous on the set $0 < s \le t$ and there exists a positive constant θ_H such that*

$$K_H(t,s) \le \theta_H s^{\frac{1}{2}-H}, \quad 0 < s \le t \le 1. \tag{A.1}$$

(i_2) *For every $0 < s \le t$*

$$\int_0^t |K_H(t,r) - K_H(s,r)|^2 \mathrm{d}r = (t-s)^{2H}. \tag{A.2}$$

(i_3) *The mappings $t \to K_H(t,s)$ is differentiable on the set $0 < s < t$ and*

$$\frac{\partial}{\partial t} K_H(t,s) = c_H \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \tag{A.3}$$

(i_4) *For each $1 \le p < 2/(2H-1)$,*

$$\sup_{0\le t\le 1} \int_0^1 K_H^p(t,s)\mathrm{d}s < \infty. \tag{A.4}$$

(i_5) *For each $f \in L^\infty([0,1])$ the function*

$$g(t) = \int_0^t K_H(t,s) f(s)\mathrm{d}s, \quad 0 \le t \le 1$$

is derivable and

$$g'(t) = \int_0^t \frac{\partial}{\partial t} K_H(t,s) f(s)\mathrm{d}s. \tag{A.5}$$

The following lemma states that under assumption (H6), the condition (1.16) in the Theorem 4.1 is satisfied.

Lemma A.2. *Under the assumption (H6), we have for any* $s \in [0, t_n] \subseteq [0,1]$,

$$\left| g_n(s) - (ns - \lfloor ns \rfloor) \int_0^s K'_H(s,r)\mathrm{d}r \right|^4 \le \frac{C}{n^{4H-2}}, \tag{A.6}$$

where $g_n : [0,1] \to \mathbb{R}$ *is a function defined by*

$$g_n(s) = n\bigg(\int_0^s K_H(s,r)\mathrm{d}r - \int_0^{s_n} K_H(s_n, r)\mathrm{d}r \bigg).$$

Proof. By

$$\int_0^s K_H(s,r)\mathrm{d}r = a_H s^{H+\frac{1}{2}}$$

and the mean value theorem, we find that for any $s \in [0,1]$ there is a $\theta_1 \in (s_n, s)$ such that

$$\begin{aligned}
&\left| g_n(s) - \left(H + \frac{1}{2}\right) a_H (ns - \lfloor ns \rfloor) s^{H-\frac{1}{2}} \right| \\
=& \left| a_H n \big(s^{H+\frac{1}{2}} - (s_n)^{H+\frac{1}{2}}\big) - \left(H + \frac{1}{2}\right) a_H (ns - \lfloor ns \rfloor) s^{H-\frac{1}{2}} \right| \\
=& \left| \left(H + \frac{1}{2}\right) a_H n (s - s_n) \theta_1^{H-\frac{1}{2}} - \left(H + \frac{1}{2}\right) a_H (ns - \lfloor ns \rfloor) s^{H-\frac{1}{2}} \right| \\
=& \left| \left(H + \frac{1}{2}\right) a_H (ns - \lfloor ns \rfloor) \theta_1^{H-\frac{1}{2}} - \left(H + \frac{1}{2}\right) a_H (ns - \lfloor ns \rfloor) s^{H-\frac{1}{2}} \right| \\
=& \left| \left(H + \frac{1}{2}\right) a_H (ns - \lfloor ns \rfloor) \Big(s^{H-\frac{1}{2}} - \theta_1^{H-\frac{1}{2}} \Big) \right| \\
\le& \, a_H \left(H + \frac{1}{2}\right) \big| s^{H-\frac{1}{2}} - s_n^{H-\frac{1}{2}} \big|.
\end{aligned} \tag{A.7}$$

Moreover, by (A.7) and the mean value theorem, we have

$$\begin{aligned}
&\left| g_n(s) - (ns - \lfloor ns \rfloor) \int_0^s K'_H(s,r)\mathrm{d}r \right|^4 \\
\le& \, C 1_{\{0 \le s < \frac{1}{n}\}} \big| s^{H-\frac{1}{2}} \big|^4 + C 1_{\{\frac{1}{n} \le s \le t_n\}} \big| s^{H-\frac{1}{2}} - s_n^{H-\frac{1}{2}} \big|^4 \\
\le& \, 1_{\{0 \le s < \frac{1}{n}\}} \frac{C}{n^{4H-2}} + 1_{\{\frac{1}{n} \le s \le t_n\}} \frac{C}{n^4} \sup_{\frac{1}{n} \le \theta_2 \le 1} \theta_2^{4H-6} \\
\le& \, \frac{C}{n^{4H-2}} 1_{\{0 \le s \le t_n\}} \le \frac{C}{n^{4H-2}}.
\end{aligned} \tag{A.8}$$

The proof is complete. □

Appendix B

The property of $\mathcal{K}_H$ is very important in the proof of Theorem 5.1. Because the following result can be concluded by simple calculations, we omit the proof.

Lemma B.1. ($\mathbb{T}_1$) *The mappings* $s \to \mathcal{K}_H(t,s)$ *is continuous on the set* $0 < s \le t$ *and there exists a positive constant* d_H *such that*

$$\mathcal{K}_H(t,s) \le d_H s^{H-\frac{1}{2}}, \quad 0 < s \le t \le 1. \tag{B.1}$$

($\mathbb{T}_2$) *For every $0 < s \leq t$ there is a positive constant e_H such that*

$$\int_0^t |\mathcal{K}_H(t,r) - \mathcal{K}_H(s,r)|^2 \mathrm{d}r \leq e_H(t-s)^{2H}. \tag{B.2}$$

($\mathbb{T}_3$) *The mappings $t \to \mathcal{K}_H(t,s)$ is differentiable on the set $0 < s \leq t$ and*

$$\frac{\partial}{\partial t}\mathcal{K}_H(t,s) = b_H\left(\frac{t}{s}+1\right)^{H-\frac{3}{2}}(t-s)^{H-\frac{3}{2}}. \tag{B.3}$$

($\mathbb{T}_4$) *For each $1 \leq p < 2/(2H-1)$,*

$$\sup_{0\leq t\leq 1}\int_0^1 \mathcal{K}_H^p(t,s)\mathrm{d}s < \infty. \tag{B.4}$$

($\mathbb{T}_5$) *For each $f \in L^\infty([0,1])$ the function*

$$g(t) = \int_0^t \mathcal{K}_H(t,s)f(s)\mathrm{d}s, \quad 0 \leq t \leq 1$$

is derivable and

$$g'(t) = \int_0^t \frac{\partial}{\partial t}\mathcal{K}_H(t,s)f(s)\mathrm{d}s. \tag{B.5}$$

The following lemma implies that under assumption (H7), the condition (1.16) in the Theorem 5.1 is satisfied. In addition, this lemma can be proved in the same way we did for Lemma A.2. So we omit the details.

Lemma B.2. *Under the assumption (H7), we have for any $s \in [0,t_n] \subseteq [0,1]$,*

$$\left|\mathfrak{g}_n(s) - (ns - \lfloor ns \rfloor)\int_0^s \mathcal{K}'_H(s,r)\mathrm{d}r\right|^4 \leq \frac{C}{n^{4H-2}}, \tag{B.6}$$

where $\mathfrak{g}_n : [0,1] \to \mathbb{R}_+$ is a function defined by

$$\mathfrak{g}_n(s) = n\bigg(\int_0^s \mathcal{K}_H(s,r)\mathrm{d}r - \int_0^{s_n}\mathcal{K}_H\big(s_n,r\big)\mathrm{d}r\bigg).$$

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