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ANALYSIS OF TWO ANY ORDER SPECTRAL VOLUME METHODS FOR 1-D LINEAR HYPERBOLIC EQUATIONS WITH DEGENERATE VARIABLE COEFFICIENTS*

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Abstract

In this paper, we analyze two classes of spectral volume (SV) methods for one-dimensional hyperbolic equations with degenerate variable coefficients. Two classes of SV methods are constructed by letting a piecewise k-th order ($k \geq 1$ is an integer) polynomial to satisfy the conservation law in each control volume, which is obtained by refining spectral volumes (SV) of the underlying mesh with k Gauss-Legendre points (LSV) or Radaus points (RSV) in each SV. The L^2-norm stability and optimal order convergence properties for both methods are rigorously proved for general non-uniform meshes. Surprisingly, we discover some very interesting superconvergence phenomena: At some special points, the SV flux function approximates the exact flux with $(k+2)$-th order and the SV solution itself approximates the exact solution with $(k+3/2)$-th order, some superconvergence behaviors for element averages errors have been also discovered. Moreover, these superconvergence phenomena are rigorously proved by using the so-called correction function method. Our theoretical findings are verified by several numerical experiments.

Mathematics subject classification: 65N30, 65N25, 65N15.
Key words: Spectral Volume Methods, L^2 stability, Error estimates, Superconvergence.

1. Introduction

Hyperbolic equations have wide applications in chemical reactions, combustion, explosions, and multi-phase flow problems, transmission of electrical signal in the animal nervous system and so on. Numerical simulation becomes more and more important in the study of hyperbolic

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equations. Recently, high-order (or high resolution) numerical schemes attracted a lot of attention in the study of numerical simulation for hyperbolic equations. An incomplete list of high-order schemes include the high-order k-exact FV method [1,9], the essentially nonoscillatory (ENO) method [4,10], the weighted ENO (WENO) method [14,19], and the discontinuous Galerkin (DG) method [6–8,11,17], and the spectral volume (SV) method [26–30]. Among all the above methods, the SV and the DG are two comparable high-order methods which share many common advantages such as: Both are capable of achieving arbitrary high-orders, both can be established on nonuniform and/or unstructured grids, both have compact stencils, both only require the information of the immediate cell neighbours to evaluate the residuals of one target cell and thus are easily parallelizable. Compared with the DG method, the SV method enjoys some advantages such as sub-element-level local conservation property, high-resolution for the discontinuity. We refer to [20,31] for more comparisons between the SV and DG methods.

It is known that the mathematical theory for the DG method, including the stability and convergence properties (see, e.g. [5,12,13,16,18]), has been heavily studied. To our knowledge, the theory for the SV method is far less developed, however. The SV method was first introduced and studied by Wang and his colleagues [15,21,22,26–30]. Van den Abeele *et al.* [23,25] investigated the influence of the partitioning approaches which divide a spectral volume into control volumes and the wave propagation properties of the SV method for 1D and 2D hyperbolic equations. By studying the wave propagation properties, they showed that the third- and fourth-order SV schemes based on the Gauss-Lobatto distributions are weakly unstable. Later, Van den Abeele *et al.* [24] proved that the second-order SV scheme on 3D tetrahedral grids was stable while a two-parameters family of third-order scheme on 3D tetrahedral grids was unstable. Zhang and Shu [31] used Fourier type analysis to show the stability of p-th order ($p \leq 3$) SV schemes derived by uniform subdividing of spectral volumes in 1D uniform grids. All the above analysis are for lower-order SV schemes upon uniform grids. All the above analysis are case-by-case and are based on lower-order SV schemes over uniform meshes. To our best knowledge, there is no stability analysis of any high-order SV schemes on non-uniform meshes in the literature. Moreover, it seems that no theoretic analysis for the convergence order and superconvergence phenomenon has been reported in the literature yet, even for 1D constant-coefficient scalar equations.

In this paper, we will extend the analysis in [3] to the following variable-coefficients problem:

$$\begin{cases} u_t + (\alpha u)_x = g(x,t), & (x,t) \in [0,2\pi] \times [0,T], \\ u(x,0) = u_0(x), & x \in [0,2\pi], \\ u(0,t) = u(2\pi,t), & t \in [0,T], \end{cases} \tag{1.1}$$

where $u_0(x), \alpha(x)$ and $g(x,t)$ are given smooth functions. We emphasize that α might be degenerate in the sense that it has a finite number of zero points in $[0,2\pi]$. Instead of case-by-case studies for low-order SV schemes, we will propose a unified approach to analyze two classes of any order SV schemes, which are constructed by dividing each SV (an interval element) with Gauss-Legendre points (LSV) or Radaus points (RSV) into control volumes (CVs).

Essentially, the SV method is a Petrov-Galerkin method. Its trial space is the standard discontinuous finite element space with respect to SVs, while its test space is the piecewise constant space with respect to CVs. Therefore, standard analysis tools for a Galerkin method can not be applied directly to a SV method, novel tools need to be developed for the unified analysis of the SV method. To overcome this difficulty, we first introduce a special from-trial-to-test-space mapping, and then with the help of this mapping, we represent the SV method as a special Galerkin method, of which the SV bilinear form can be regarded as a perturbation or numerical quadrature of the corresponding DG bilinear form. Based on the analysis approach for the Galerkin method and the estimates of the difference between the SV and DG method, we then establish a uniform analysis framework for the stability and convergence properties of any order SV schemes. Finally, we prove that both the LSV and RSV are stable and can achieve optimal convergence orders (i.e. $(k+1)$-th order) in L^2 norm, even the underlying meshes are non-uniform and the coefficient α is degenerate.

The superconvergence phenomena on the SV methods have been not reported in the literature. Our numerical evidences indicate that the SV method does have some very interesting superconvergence phenomena, even the coefficient α in (1.1) is degenerate. For instances, our numerical experiments show that the RSV solution is superconvergent at Radau points, while the LSV solution is superconvergent at Gauss points. Moreover, we find out that the convergence rates and the location of superconvergence points depend on some specific properties of α (e.g. the multiplicity of zeros of α and the sign of the coefficient α). To figure out a mathematical theory behind, we first develop a correction technique to improve the error order between the SV solution and a special interpolation operator of the exact solution. With this supercloseness, we then establish the superconvergence of the SV solution at some special points. Finally, we prove that, for the solution itself, the superconvergence order can achieve $\mathcal{O}(h^{k+3/2})$ at Radau points for RSV and at Gauss points for LSV, for the flux function approximation, the superconvergence order can be improved to $\mathcal{O}(h^{k+2})$, and for the derivative of the SV solution, the superconvergence order achieves $\mathcal{O}(h^{k+1})$ at some special points which will be specified later.

We would like to point out that the coefficient α here might be degenerate, the wind direction will change according to the sign of the coefficient α. Consequently, suitable numerical fluxes and dividing points should be wisely chosen to match the wind direction change. Moreover, our correction function should also be constructed accordingly, which makes the analysis for degenerate coefficient problems very sophisticated.

The rest of the paper is organized as follows. In Section 2, we present two classes of SV methods for linear conservation laws with degenerate variable coefficients. In Section 3, we study the stability of the RSV and LSV methods, where inequalities in energy norm and flux function norm are established. In Section 4, optimal error estimates in the L^2-norm for both RSV and LSV are proved. Sections 5 is dedicated to the analysis of the superconvergence behavior of the SV solution itself and flux function approximation. We show that the superconvergence phenomenon exists for general variable coefficient hyperbolic equations, and the superconvergence rate may depend upon specific properties of the variable coefficient function. In Section 6, we provide some numerical examples to support our theoretical findings. Finally, some concluding remarks are presented in Section 7.

2. RSV and LSV Methods

Let $\Omega=[0,2\pi]$ and $0=x_0<x_{1/2}<x_{3/2}<\cdots<x_{N+1/2}=2\pi$ be $N+1$ distinct points which split Ω into N elements. For any positive integer l, we denote $\mathbb{Z}_l=\{1,2,\ldots,l\}$ and $\mathbb{Z}_l^0=\{0,1,\ldots,l\}$. For $i\in\mathbb{Z}_N$, we denote $\mathbf{V}_i=[x_{i-1/2},x_{i+1/2}]$, and $h_i=x_{i+1/2}-x_{i-1/2}$. Let $\overline{h}_i=h_i/2$ and $h=\max_{i\in\mathbb{Z}_N}h_i$. We also assume that the mesh is shape-regular, i.e. $h\lesssim h_i, i\in\mathbb{Z}_N$. Here and in the following, the notation $x\lesssim y$ means that x can be bounded by y multiplied by a constant C, which is independent of the mesh size. Let $-1=s_0<s_1<s_2<\cdots<s_{k+1}=1$ be $k+2$ distinct points in the reference element $[-1,1]$. We get a partition

of each element $\mathbf{V}_i$ via the following linear transformation:

$$x_{i,j} = \frac{h_i}{2} s_j + \frac{1}{2}\big(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}\big), \quad i \in \mathbb{Z}_N, \quad j \in \mathbb{Z}^0_{k+1}. \tag{2.1}$$

Denote $\mathbf{V}_{i,j} = [x_{i,j}, x_{i,j+1}], j \in \mathbb{Z}^0_k$, and define the finite element space V_h and the piecewise constant function space $\mathcal{V}_h$ by

$$V_h = \big\{v_h : v_h|_{\mathbf{V}_i} \in \mathcal{P}_k(\mathbf{V}_i), i \in \mathbb{Z}_N\big\},$$
$$\mathcal{V}_h = \big\{v_h^* : v_h^*|_{\mathbf{V}_{i,j}} \in \mathcal{P}_0(\mathbf{V}_{i,j}), i \in \mathbb{Z}_N, j \in \mathbb{Z}^0_k\big\},$$

where $\mathcal{P}_k$ is the space of polynomials of degree at most k. Let

$$\mathcal{H}_h = \big\{v : v|_{\mathbf{V}_i} \in H^1, i \in \mathbb{Z}_N\big\}.$$

Here and throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D, and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$ and $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$ and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$.

The SV scheme for (1.1) read as: Find $u_h \in V_h$ such that

$$\int_{x_{i,j}}^{x_{i,j+1}} (u_h)_t \mathrm{d}x + \alpha\widehat{u}_h|_{i,j+1} - \alpha\widehat{u}_h|_{i,j} = \int_{x_{i,j}}^{x_{i,j+1}} g \mathrm{d}x, \tag{2.2}$$

where $u|_{i,j} = u(x_{i,j})$ and $\widehat{u}_h$ is the numerical flux. In this paper, we choose the upwind flux. That is,

$$\widehat{u}_h|_{i+\frac{1}{2}} = \begin{cases} u_h^-|_{i+\frac{1}{2}}, & \alpha|_{i+\frac{1}{2}} > 0, \\ u_h^+|_{i+\frac{1}{2}}, & \alpha|_{i+\frac{1}{2}} \le 0, \end{cases} \tag{2.3}$$

where $u_h^+|_{i+1/2}$ and $u_h^-|_{i+1/2}$ denote the right and left limits of u_h at the point $x_{i+1/2}$, respectively.

We observe that the choice of the partition points $\{x_{i,j}\}_{j=0}^{k+1}$ of $\mathbf{V}_i$ has a great influence on the SV scheme (2.2). By taking different s_j or $x_{i,j}$, we get different SV schemes. In this paper, we will consider two classes of SV schemes. One is constructed by using Radau points while the other is using Gauss-Legendre points. We call the corresponding schemes as Radau spectral volume (RSV) method and Gauss Legendre spectral volume (LSV) method.

LSV Scheme: $\{s_j\}_{j=1}^k$ are chosen as Gauss-Legendre points, i.e. $s_j, j \in \mathbb{Z}_k$ are k zeros of the Legendre polynomial L_k of degree k.

RSV Scheme: $\{s_j\}_{j=1}^k$ are chosen according to the sign of α on the element boundary, i.e.

- If $\alpha(x_{i-1/2}) > 0, \alpha(x_{i+1/2}) > 0, \{s_j\}_{j=1}^{k+1}$ are chosen as right Radau points, i.e. zeros of the right Radau polynomial $L_{k+1} - L_k$.
- If $\alpha(x_{i-1/2}) < 0, \alpha(x_{i+1/2}) < 0, \{s_j\}_{j=0}^{k}$ are chosen as left Radau points, i.e. zeros of the right Radau polynomial $L_{k+1} + L_k$.
- Otherwise, either of the left Radau points or right Radau points can be used.

We close this section by writing the above two SV schemes into a Petrov-Galerkin method. Define the SV bilinear form on $\mathcal{H}_h \times \mathcal{V}_h$ on each spectral volume $\mathbf{V}_i$ by

$$a_{h,i}(v,w^*) = \sum_{j=0}^{k} w^*_{i,j} \int_{x_{i,j}}^{x_{i,j+1}} v_t \mathrm{d}x + \sum_{j=0}^{k} w^*_{i,j} \big(\alpha\widehat{v}|_{i,j+1} - \alpha\widehat{v}|_{i,j}\big), \quad \forall v \in \mathcal{H}_h, \quad w^* \in \mathcal{V}_h. \tag{2.4}$$

Here $w^*_{i,j}$ represents the value of w^* on $\mathbf{V}_{i,j}, (i,j) \in \mathbb{Z}_N \times \mathbb{Z}^0_k$. Let the global bilinear form

$$a_h(v,w^*) = \sum_{i=1}^{N} a_{h,i}(v,w^*), \quad \forall v \in \mathcal{H}_h, \quad w^* \in \mathcal{V}_h. \tag{2.5}$$

The SV scheme (2.2) can be rewritten as: Find $u_h \in V_h$ such that

$$a_h(u_h, w^*_h) = (g, w^*_h), \quad \forall w^*_h \in \mathcal{V}_h. \tag{2.6}$$

Remark 2.1. The SV scheme (2.6) is consistent in the sense that the exact solution u of (1.1) also satisfies

$$a_h(u, w^*_h) = (g, w^*_h), \quad \forall w^*_h \in \mathcal{V}_h. \tag{2.7}$$

Consequently, there holds the following Galerkin orthogonality:

$$a_h(u - u_h, w^*_h) = 0, \quad \forall w^*_h \in \mathcal{V}_h. \tag{2.8}$$

3. Stability of RSV and LSV

We begin our analysis of stability by relating the SV schemes to some quadratures.

For any $f \in L^1([-1,1])$, we let

$$Q_k[f] = \sum_{j=0}^{k+1} A_j f(s_j), \quad R[f] = \int_{-1}^{1} f(s)\mathrm{d}s - Q_k[f], \tag{3.1}$$

where

$$A_j = \int_{-1}^{1} l_j(s)\mathrm{d}s, \quad j \in \mathbb{Z}^0_{k+1}$$

with l_j the Lagrange basis function at s_j.

The SV method in the paper is related to the following three types of quadratures:

1. For the Gauss-Legendre quadrature, $\{s_j\}_{j=1}^k$ are k Gauss points and $A_0 = A_{k+1} = 0$. This quadrature is exact for polynomials of degree not more than $2k-1$, and its remainder is

$$R[f] = \frac{2^{2k+1}[k!]^4}{(2k+1)[(2k)!]^3} f^{(2k)}(\xi), \quad \xi \in (-1,1). \tag{3.2}$$

2. For the right Radau quadrature, $\{s_j\}_{j=1}^{k+1}$ are $k+1$ right Radau points and $A_0 = 0$. The quadrature is exact for polynomials of degree not more than $2k$ and its remainder is

$$R[f] = \frac{f^{(2k+1)}(\xi)}{(2k+1)!} \int_{-1}^{1} \prod_{j=1}^{j=k} (s-s_j)^2 (s-1)\mathrm{d}s, \quad \xi \in (-1,1). \tag{3.3}$$

3. For the left Radau quadrature, $\{s_j\}_{j=0}^{k}$ are $k+1$ left Radau points and $A_{k+1}=0$. The quadrature is exact for polynomials of degree not more than $2k$ and its remainder is

$$R[f]=\frac{f^{(2k+1)}(\xi)}{(2k+1)!}\int_{-1}^{1}\prod_{j=1}^{j=k}(s-s_j)^2(s+1)\mathrm{d}s,\quad \xi\in(-1,1). \tag{3.4}$$

We introduce a transformation T from V_h onto $\mathcal{V}_h$ as follows:

$$w^*:=Tw=\sum_{i=1}^{N}\sum_{j=0}^{k}w^*_{i,j}\chi_{\mathbf{V}_{i,j}}(x),\quad w\in V_h, \tag{3.5}$$

where $\chi_A, A\subset[0,2\pi]$, is the characteristic function defined as $\chi_A=1$ in A and $\chi_A=0$ otherwise, and $w^*_{i,j}$ can be obtained by the following recurrence formula:

$$w^*_{i,0}=w^+_{i-\frac{1}{2}}+A_{i,0}w_x(x_{i,0}),\quad w^*_{i,j}=w^*_{i,j-1}+A_{i,j}w_x(x_{i,j}),\quad j\in\mathbb{Z}_k, \tag{3.6}$$

where $A_{i,j}=\overline{h}_iA_j, (i,j)\in\mathbb{Z}_N\times\mathbb{Z}_k^0$. By a direct calculation, we have

$$w^*_{i,k}=\sum_{j=0}^{k}\left(w^*_{i,j}-w^*_{i,j-1}\right)+w^*_{i,0}=w^+_{i-\frac{1}{2}}+\sum_{j=0}^{k+1}A_{i,j}w_x(x_{i,j})-A_{i,k+1}w_x\left(x^-_{i+\frac{1}{2}}\right).$$

Noticing that numerical quadrature is exact for w_x, we get

$$w^*_{i,k}=w^-_{i+\frac{1}{2}}-A_{i,k+1}w_x\left(x^-_{i+\frac{1}{2}}\right). \tag{3.7}$$

Remark 3.1. The values of $w^*_{i,0}, w^*_{i,k}$ are closely related to the choice of the points s_j. Specifically:

- If the partition points are chosen as Gauss-Legendre points, then $w^*_{i,0}=w^+_{i-1/2}$ and $w^*_{i,k}=w^-_{i+1/2}$.
- If the partition points are chosen as right-Radau points, then $w^*_{i,0}=w^+_{i-1/2}$.
- If the partition points are chosen as left-Radau points, then $w^*_{i,k}=w^-_{i+1/2}$.

Lemma 3.1. *Suppose that the partition points are given by the quadrature abscissae described in* (3.1)*, then the transformation T defined in* (3.5) *is bijective and bounded, i.e.*

$$\|Tw\|\lesssim\|w\|,\quad \forall w\in V_h. \tag{3.8}$$

Proof. We first show that the transformation T is injective. We assume that $Tw=0$, then we have $w^*_{i,j}=0, j\in\mathbb{Z}_k^0$, and thus $w_x(x_{i,j})=0, j\in\mathbb{Z}_k$ and either $w^+_{i-1/2}$ or $w^-_{i+1/2}=0$. As $w_x\in\mathcal{P}_{k-1}(\mathbf{V}_i)$, we obtain that $w_x\equiv0$. Therefore, we deduce that $w\equiv0$. Suppose $\{\varphi_{i,j}\}_{j=0}^{k}$ is a basis of V_h, it is easy to prove that $\{T\varphi_{i,j}\}_{j=0}^{k}$ is linear independent, which yields that $\{T\varphi_{i,j}\}_{j=0}^{k}$ is a basis of $\mathcal{V}_h$. Then T is bijective. By using the inverse inequality, (3.8) follows, which indicates that T is bounded. □

Remark 3.2. The SV scheme (2.2) is equivalent to the Galerkin scheme: Find $u_h\in V_h$ such that

$$a_h(u_h,w_h^*)=(g,w_h^*),\quad \forall w_h\in V_h, \tag{3.9}$$

where $w_h^*=Tw_h$.

Theorem 3.1. *Suppose $a_{h,i}(\cdot,\cdot)$ be the SV scheme defined by (2.4) with the partition points given by the quadrature abscissae described in (3.1). For any $v,w\in V_h$, let $V=\partial_x^{-1}v_t$ and $w^*=Tw$ be the piecewise constant defined in (3.5). Then*

$$\begin{aligned}a_{h,i}(v,w^*)&=\big(v_t+(\alpha v)_x,w\big)_i+R_i[(V+\alpha v)w_x]\\&\quad+w^*_{i,0}(\alpha v^+-\alpha\widehat{v})|_{i-\frac12}+w^*_{i,k}(\alpha\widehat{v}-\alpha v^-)|_{i+\frac12}.\end{aligned}\tag{3.10}$$

Proof. Both $v\in V_h$ and α are continuous at the interior points implies $\alpha\hat{v}|_{i,j}=\alpha\hat{v}|_{i,j}$ for all $j=1,\dots,k$, then by (2.4),

$$\begin{aligned}a_{h,i}(v,w^*)&=(v_t,w^*)_i+w^*_{i,0}\big(\alpha v|_{i,0}-\alpha\widehat{v}|_{i-\frac12}\big)\\&\quad+w^*_{i,k}\big(\alpha\widehat{v}|_{i+\frac12}-\alpha v|_{i,k+1}\big)+\sum_{j=0}^{k}w^*_{i,j}\int_{x_{i,j}}^{x_{i,j+1}}(\alpha v)_x\\&=\big(v_t+(\alpha v)_x,w^*\big)_i+w^*_{i,0}(\alpha v^+-\alpha\widehat{v})|_{i-\frac12}+w^*_{i,k}(\alpha\widehat{v}-\alpha v^-)|_{i+\frac12}.\end{aligned}\tag{3.11}$$

On the other hand, a direct calculation from (3.6) and (3.7) yields

$$\begin{aligned}(v,w^*)_i&=\sum_{j=0}^{k}\int_{x_{x,j}}^{x_{i,j+1}}vw^*_{i,j}\mathrm{d}x=\sum_{j=0}^{k}w^*_{i,j}\big(\partial_x^{-1}v|_{i,j+1}-\partial_x^{-1}v|_{i,j}\big)\\&=w^-_{i+\frac12}\big(\partial_x^{-1}v\big)|_{i+\frac12}-w^+_{i-\frac12}\big(\partial_x^{-1}v\big)^+|_{i-\frac12}-\sum_{j=0}^{k+1}\partial_x^{-1}v|_{i,j}A_{i,j}w_x(x_{i,j})\\&=\int_{x_{i-\frac12}}^{x_{i+\frac12}}\big(\partial_x^{-1}vw\big)_x\mathrm{d}x-\int_{x_{i-\frac12}}^{x_{i+\frac12}}\partial_x^{-1}vw_x\mathrm{d}x+R_i\big[\partial_x^{-1}vw_x\big]\\&=\int_{x_{i-\frac12}}^{x_{i+\frac12}}vw\mathrm{d}x+R_i\big[\partial_x^{-1}vw_x\big].\end{aligned}\tag{3.12}$$

Substituting (3.12) into (3.11) leads to the desired result (3.10). □

We next compare the SV scheme (2.6) and the discontinuous Galerkin (DG) scheme, which is of great importance in our later stability analysis. The bilinear form of the DG method is defined by

$$a_h^{DG}(v,w)=\sum_{i=1}^{N}a_{h,i}^{DG}(v,w),\quad\forall v,w\in\mathcal{H}_h,$$

where

$$a_{h,i}^{DG}(v,w)=(v_t,w)_i-(\alpha v,w_x)_i+\alpha\widehat{v}w^-|_{i+\frac12}-\alpha\widehat{v}w^+|_{i-\frac12}.\tag{3.13}$$

Here the numerical flux $\hat{v}$ is chosen as upwind flux. Applying integration by part yields

$$a_{h,i}^{DG}(v,w)=\big(v_t+(\alpha v)_x,w\big)_i+w^+_{i-\frac12}(\alpha v^+-\alpha\widehat{v})|_{i-\frac12}+w^-_{i+\frac12}(\alpha\widehat{v}-\alpha v^-)|_{i+\frac12}.\tag{3.14}$$

Consequently,

$$a_{h,i}(v,w^*)=a_{h,i}^{DG}(v,w)+R_i[(V+\alpha v)w_x]+\bar{R}_i[v,w],\quad v\in V_h,\tag{3.15}$$

where $V=\partial_x^{-1}v_t$ and

$$\bar{R}_i[v,w]=\big(w^*_{i,0}-w^+_{i-\frac12}\big)(\alpha v^+-\alpha\widehat{v})|_{i-\frac12}+\big(w^*_{i,k}-w^-_{i+\frac12}\big)(\alpha\widehat{v}-\alpha v^-)|_{i+\frac12}.$$

Now, we are ready to discuss the stability for both the RSV scheme and the LSV scheme. We first estimate the remainder terms appeared in (3.15).

Lemma 3.2. *If $\alpha \in C^1(\mathbf{V}_i)$ for all $i \in \mathbb{Z}_N$, then for both the LSV and RSV methods, there hold*

$$\big|R_i[\alpha v v_x]\big| \lesssim \|v\|_{0,\mathbf{V}_i}^2, \quad \forall v \in V_h, \tag{3.16}$$

$$\big|\bar{R}_i[v,v]\big| \lesssim \|v\|_{0,\mathbf{V}_i}\big(\|v\|_{0,\mathbf{V}_i} + \|v\|_{0,\mathbf{V}_{i+1}} + \|v\|_{0,\mathbf{V}_{i-1}}\big), \quad \forall v \in V_h. \tag{3.17}$$

Proof. We first show (3.16). Denoting by $\bar{\alpha}_i$ the cell average of α in $\mathbf{V}_i$ for all $i \in \mathbb{Z}_N$, we rewrite the residual into

$$R_i[\alpha v v_x] = R_i[\bar{\alpha}_i v v_x] + R_i[(\alpha - \bar{\alpha}_i) v v_x].$$

Since vv_x is a polynomial of degree $2k-1$ in each $\mathbf{V}_i$, the right or left Radau numerical quadrature is exact for polynomial of degree of $2k$, and the Gauss quadrature is exact for polynomial of degree of $2k-1$, we conclude that

$$R_i[\bar{\alpha}_i v v_x] = 0.$$

Therefore,

$$\begin{aligned} R_i[\alpha v v_x] =&\ R_i[(\alpha - \bar{\alpha}_i) v v_x] \\ =&\ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (\alpha - \bar{\alpha}_i) v v_x \mathrm{d}x - \sum_{j=0}^{k+1} A_{i,j}\left[\alpha(x_{i,j}) - \bar{\alpha}_i\right] v(x_{i,j}) v_x(x_{i,j}) \\ =&: R_{i,1} - R_{i,2}. \end{aligned} \tag{3.18}$$

By the Cauchy-Schwartz inequality and the inverse inequality, we obtain that

$$|R_{i,1}| \lesssim h_i \|\alpha\|_{1,\infty,\mathbf{V}_i} \|v\|_{0,\mathbf{V}_i} \|v_x\|_{0,\mathbf{V}_i} \lesssim \|v\|_{0,\mathbf{V}_i}^2. \tag{3.19}$$

Similarly, we use the inverse inequality to obtain

$$|R_{i,2}| \lesssim (h_i)^2 \|\alpha\|_{1,\infty,\mathbf{V}_i} \|v\|_{0,\infty,\mathbf{V}_i} \|v_x\|_{0,\infty,\mathbf{V}_i} \lesssim h\|v\|_{0,\mathbf{V}_i}^2. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18), we obtain the desired result (3.16).

Next we show (3.17). We observe that if $\alpha(x_{i-1/2})\alpha(x_{i+1/2}) \geq 0$, then $\bar{R}_i[v,v] = 0$, if $\alpha(x_{i-1/2})\alpha(x_{i+1/2}) < 0$, then there exists at least one point $\eta_i \in \mathbf{V}_i$ satisfying $\alpha(\eta_i) = 0$ and thus $\|\alpha\|_{0,\infty,V_i} \lesssim h$. Consequently, we have from (3.6) and (3.7) that

$$\begin{aligned} |\bar{R}_i[v,v]| &\lesssim h\|v_x\|_{0,\infty,\mathbf{V}_i} \|\alpha\|_{0,\infty,\mathbf{V}_i} \Big(|[v]_{i-\frac{1}{2}}| + |[v]_{i+\frac{1}{2}}|\Big) \\ &\lesssim \|v\|_{0,\mathbf{V}_i}\big(\|v\|_{0,\mathbf{V}_i} + \|v\|_{0,\mathbf{V}_{i+1}} + \|v\|_{0,\mathbf{V}_{i-1}}\big), \end{aligned}$$

where in the last step, we have used the inverse inequality again. The proof is complete. □

To study the L^2-norm stability, we need the following equivalent norm defined by:

$$\|v\|_E^2 = \sum_{i=1}^{N} \|v\|_{i,E}^2, \quad \|v\|_{i,E}^2 = (v, v^*)_i, \quad \forall v \in V_h, \tag{3.21}$$

where $v^* = Tv$. In light of (3.12), we have that

$$(v, v^*)_i = (v, v)_i + R_i\big[\partial_x^{-1} v \partial_x v\big] \tag{3.22}$$

with the residual

$$R_i[\partial_x^{-1} v \partial_x v] = 0$$

for the right or left Radau quadrature, and

$$\begin{aligned} R_i\big[\partial_x^{-1} v \partial_x v\big] &= \bar{h}_i^{2k+1} \frac{2^{2k+1}(k!)^4}{(2k+1)[(2k)!]^3} \frac{d^{2k}}{x^{2k}} \big(\partial_x^{-1} v v_x\big) \\ &= \bar{h}_i^{2k} \frac{2^{2k+1} k (k!)^2}{(2k+2)(2k+1)[(2k)!]^2} \big(v^{(k)}, v^{(k)}\big)_i \end{aligned}$$

for the Gauss quadrature. Therefore, in both cases $0 \le R_i[\partial_x^{-1} v \partial_x v] \lesssim \|v\|_{0,\mathbf{V}_i}$ and thus

$$\|v\|_0 \le \|v\|_E \lesssim \|v\|_0, \quad \forall v \in V_h.$$

Now we are ready to show the L^2-norm stability of both the LSV and RSV methods.

Theorem 3.2. *Suppose $\alpha \in C^1(\mathbf{V}_i)$ for all $i \in \mathbb{Z}_N$. Let $a_h(\cdot\,,\cdot)$ be the SV bilinear form defined in* (2.5)*, and u_h the solution of* (2.6)*. Then for both RSV and LSV methods,*

$$\frac{1}{2} \frac{d}{dt} \|u_h\|_E^2 \lesssim \|u_h\|_E^2 + \|g\| \cdot \|u_h\|_E. \tag{3.23}$$

Consequently, both RSV and LSV are stable in the sense that

$$\|u_h(\cdot, t)\|_0 \lesssim \|u_h(\cdot, 0)\|_0, \quad t \in (0, T]. \tag{3.24}$$

Proof. First, we use the energy inequality of the DG method in [2] to get

$$(v_t, v)_i \le a_{h,i}^{DG}(v, v) - \frac{1}{2}(\alpha_x v, v)_i, \quad \forall v \in V_h. \tag{3.25}$$

Secondly, choosing $w = v$ in (3.15), we have

$$a_{h,i}(v, v^*) = a_{h,i}^{DG}(v, v) + R_i\big[\big(\partial_x^{-1} v_t\big) v\big] + R_i[\alpha v v_x] + \bar{R}_i[v, v], \quad v \in V_h. \tag{3.26}$$

Therefore,

$$\begin{aligned} (v_t, v)_i + R_i\big[\big(\partial_x^{-1} v_t\big) v\big] &\le a_{h,i}^{DG}(v, v) + R_i\big[\big(\partial_x^{-1} v_t\big) v\big] - \frac{1}{2}(\alpha_x v, v)_i \\ &\le a_{h,i}(v, v^*) - R_i[\alpha v v_x] - \bar{R}_i[v, v] - \frac{1}{2}(\alpha_x v, v)_i. \end{aligned}$$

Since

$$(v_t, v)_i + R_i\big[\big(\partial_x^{-1} v_t\big) v\big] = (v_t, v^*)_i = \frac{1}{2} \frac{d}{dt} \|v\|_{i,E}^2,$$

we actually obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{i,E}^2 \le a_{h,i}(v, v^*) - R_i[\alpha v v_x] - \bar{R}_i[v, v] - \frac{1}{2}(\alpha_x v, v)_i. \tag{3.27}$$

Summarizing the above estimates for all i, and noticing $\alpha \in C^1(\mathbf{V}_i)$ for all $i \in \mathbb{Z}_N$, and the estimates (3.16) and (3.17), we have that

$$\frac{1}{2} \frac{d}{dt} \|v\|_E^2 \le a_h(v, v^*) + \|v\|_0^2. \tag{3.28}$$

Choosing $v=u_h$ in the above inequality and noticing that $a_h(u_h,u_h^*)=(g,u_h^*)$, we obtain (3.23).

Since $\|v\|_0 \le \|v\|_E$ for all $v\in V_h$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_E^2 \lesssim \|u_h\|_E^2 + \|g\|_E\|u_h\|_E.$$

Then by the Gronwall inequality,

$$\|u_h(\cdot,t)\|_E \lesssim \|u_h(\cdot,0)\|_E, \quad t\in(0,T].$$

The stability result (3.24) follows directly from the norm equivalence between $\|\cdot\|_E$ and $\|\cdot\|_0$ in V_h. The proof is complete. □

4. Optimal Order Error Estimates

We begin with an introduction of some interpolation functions. For $\phi\in\mathcal{H}_h$, we denote by $I_h^+\phi, I_h^-\phi, I_h^{\pm}\phi \in V_h$ the standard Lagrange interpolation of ϕ, which respectively satisfy the following $k+1$ conditions on each volume-element $\mathbf{V}_i (i\in Z_N)$:

$$I_h^-\phi(x_{i,j}) = \phi(x_{i,j}), \quad j=1,2,\dots,k+1, \tag{4.1}$$

$$I_h^+\phi(x_{i,j}) = \phi(x_{i,j}), \quad j=0,1,\dots,k, \tag{4.2}$$

$$I_h^{\pm}\phi(x_{i,j}) = \phi(x_{i,j}), \quad j=0,1,\dots,k-1,k+1. \tag{4.3}$$

The standard interpolation theory implies that

$$\left\|I_h^-\phi-\phi\right\|_0 + \left\|I_h^+\phi-\phi\right\|_0 + \left\|I_h^{\pm}\phi-\phi\right\|_0 \lesssim h^{k+1}|\phi|_{k+1}. \tag{4.4}$$

We define a special interpolation $I_h\phi$ of ϕ as follows:

$$I_h\phi = \begin{cases} I_h^-\phi, & \text{if} \quad \alpha|_{i-\frac{1}{2}} \ge 0, \quad \alpha|_{i+\frac{1}{2}} > 0, \\ I_h^+\phi, & \text{if} \quad \alpha|_{i-\frac{1}{2}} \le 0, \quad \alpha|_{i+\frac{1}{2}} < 0, \\ I_h^{\pm}\phi, & \text{otherwise}. \end{cases} \tag{4.5}$$

Lemma 4.1. *The interpolation I_h is flux exact in the sense that for all $\phi\in C^0[0,2\pi]$,*

$$\widehat{I_h\phi}|_{i+\frac{1}{2}} = \phi\big(x_{i+\frac{1}{2}}\big), \quad \forall i\in\mathbb{Z}_N, \tag{4.6}$$

where $\widehat{I_h\phi}$ is the upwind flux function of $I_h\phi$ defined in (2.3).

Proof. By the definition (2.3), if $\alpha(x_{i+1/2})>0, I_h=I_h^-$ or $I_h^{\pm}$ at $x_{i+1/2}$, then

$$\widehat{I_h\phi}|_{i+\frac{1}{2}} = I_h\phi\big(x^-_{i+\frac{1}{2}}\big) = \phi\big(x_{i+\frac{1}{2}}\big),$$

if $\alpha(x_{i+1/2})\le 0, I_h=I_h^+$ or $I_h^{\pm}$ at $x_{i+\frac{1}{2}}$, then

$$\widehat{I_h\phi}|_{i+\frac{1}{2}} = I_h\phi\big(x^+_{i+\frac{1}{2}}\big) = \phi\big(x_{i+\frac{1}{2}}\big).$$

This ends the proof. □

Recalling (2.4), $a_{h,i}(\cdot\,,\cdot)$ can be divided into two parts. That is, for all $v \in \mathcal{H}_h, w^* \in \mathcal{V}_h$, $i \in \mathbb{Z}_N, a_{h,i}(v,w^*) = b_{i,1}(v,w^*) + b_{i,2}(v,w^*)$, where

$$b_{i,1}(v,w^*) = \sum_{j=0}^{k} w^*_{i,j} \int_{x_{i,j}}^{x_{i,j+1}} v_t \mathrm{d}x, \quad b_{i,2}(v,w^*) = \sum_{j=0}^{k} w^*_{i,j}\big(\alpha\widehat{v}|_{i,j+1} - \alpha\widehat{v}|_{i,j}\big).$$

Lemma 4.2. *Let $u(\cdot,t) \in W^{k+1,\infty}(\Omega)$ the solution of* (1.1) *for all $t \in [0,t_0]$. Then*

$$|b_{i,2}(u - I_h u, v^*)| \lesssim h_i^{k+\frac{3}{2}} \|u\|_{k+1,\infty,\mathbf{V}_i} \|v\|_{0,\mathbf{V}_i}, \quad \forall v \in V_h, \quad i \in \mathbb{Z}_N. \tag{4.7}$$

Proof. Rearranging the items of $b_{i,2}(v,w^*)$, we get

$$b_{i,2}(v,w^*) = w^-_{i+\frac{1}{2}}\alpha\widehat{v}|_{i+\frac{1}{2}} - w^+_{i-\frac{1}{2}}\alpha\widehat{v}|_{i-\frac{1}{2}} - \sum_{j=1}^{k} \alpha(x_{i,j}) v(x_{i,j}) A_{i,j} w_x(x_{i,j}). \tag{4.8}$$

Choosing $v = u - I_h u$ in (4.8) and using (4.6) yields

$$b_{i,2}(u - I_h u, v^*) = -\sum_{j=1}^{k} \alpha(x_{i,j})(u - I_h u)(x_{i,j}) A_{i,j} v_x(x_{i,j}). \tag{4.9}$$

If $I_h = I_h^-$ or I_h^+, we get immediately that $(u - I_h u)(x_{i,j}) = 0$ for all $j \in \mathbb{Z}_k$, which implies $b_{i,2}(u - I_h u, v^*) = 0$. If $I_h = I_h^{\pm}$, then $\alpha_{i+1/2}\alpha_{i-1/2} < 0$, which implies that there exists at least a $\eta_i \in \mathbf{V}_i$ satisfying $\alpha(\eta_i) = 0$. Therefore,

$$b_{i,2}(u - I_h u, v^*) = \big(\alpha(x_{i,k}) - \alpha(\eta_i)\big)(u - I_h u)(x_{i,k}) A_{i,k} v_x(x_{i,k}).$$

By the approximation properties of the interpolation and the inverse inequality, we get

$$\begin{aligned} |b_{i,2}(u - I_h u, v^*)| &\lesssim h_i^{k+3} \cdot \|u\|_{k+1,\infty} \cdot \|v\|_{1,\infty,\mathbf{V}_i} \\ &\lesssim h_i^{k+\frac{3}{2}} \|u\|_{k+1,\infty,\mathbf{V}_i} \|v\|_{0,\mathbf{V}_i}. \end{aligned} \tag{4.10}$$

This finishes our proof. □

Now, we are ready to give the optimal error estimate for $\|u_h - I_h u\|_0$.

Theorem 4.1. *Let $u(\cdot,t) \in H^{k+2}(\Omega)$ for any $t \in [0,t_0]$ be the solution of* (1.1)*, $I_h u$ be the interpolation function of u defined in* (4.5)*, and u_h be the solution of the scheme* (3.9) *with the initial solution $u_h(x,0) = I_h u_0$. Then for both the RSV and LSV methods*

$$\|(u - u_h)(\cdot,t)\|_0 \lesssim h^{k+1} \sup_{\tau\in[0,t]} \|u(\cdot,\tau)\|_{k+2}. \tag{4.11}$$

Proof. In each $\mathbf{V}_i$, we have

$$\begin{aligned} a_{h,i}(u - I_h u, v_h^*) &= b_{i,1}(u - I_h u, v_h^*) + b_{i,2}(u - I_h u, v_h^*) \\ &= \big(u_t - (I_h u)_t, v_h^*\big) + b_{i,2}(u - I_h u, v_h^*). \end{aligned} \tag{4.12}$$

By (3.8), (4.7), and the fact that $(I_h u)_t = I_h u_t$, we obtain that for $v_h \in V_h$ and $i \in \mathbb{Z}_N$,

$$\begin{aligned} |a_{h,i}(u - I_h u, v_h^*)| &\lesssim \|u_t - I_h u_t\|_{0,\mathbf{V}_i} \|v_h^*\|_{0,\mathbf{V}_i} + h^{k+\frac{3}{2}} \|u\|_{k+1,\infty,\mathbf{V}_i} \|v\|_{0,\mathbf{V}_i} \\ &\lesssim h_i^{k+1}\Big(\|u_t\|_{k+1,\mathbf{V}_i} + h_i^{\frac{1}{2}} \|u\|_{k+1,\infty,\mathbf{V}_i}\Big) \|v\|_{0,\mathbf{V}_i} \\ &\lesssim h^{k+1} \|u\|_{k+2,\mathbf{V}_i} \|v\|_{0,\mathbf{V}_i}. \end{aligned} \tag{4.13}$$

Here in the last step, we have used the identity $u_t = -u_x$ and the inequality $\|u\|_{k+1,\infty} \lesssim \|u\|_{k+2}$. Choosing $v = u_h - I_h u$ in (3.28) and using the Galerkin orthogonality (2.8), we get

$$\frac{1}{2}\frac{d}{dt}\|u_h - I_h u\|_E^2 \le a\big((u_h - I_h u)_t, (u_h - I_h u)^*\big) + \|u_h - I_h u\|_0^2$$
$$\lesssim h^{k+1}\|u\|_{k+2}\|u_h - I_h u\|_0 + \|u_h - I_h u\|_0^2.$$

By the Gronwall inequality and using the initial value of u_h, we have

$$\|(u_h - I_h u)(\cdot,t)\|_0 \le \|(u_h - I_h u)(\cdot,t)\|_E \lesssim h^{k+1}\|u\|_{k+2}.$$

The desired (4.11) follows from the triangle inequality and the standard interpolation inequality (4.4). The proof is complete. □

5. Superconvergence

In this section, we will investigate the superconvergence of both the numerical flux and the numerical solution of the SV method. The basic idea of our superconvergence analysis is still based on correction function, which is to construct a correction function $w \in V_h$ such that

$$a_h(u - I_h u, v^*) + a_h(w, v^*)$$

is of higher order for all $v^* \in \mathcal{V}_h$. Note that if $w = 0$, which indicates that no correction is done, then we obtain the optimal convergence rate, just as we did in Theorem 4.1. Different from the construction of the correction function for constant coefficients, as the wind direction may change for variable coefficients, then the correction function could be different in different elements. Therefore, we first divide the whole domain into three parts $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\Omega_1 = \Big\{\mathbf{V}_i : \alpha\big(x_{i-\frac{1}{2}}\big) > 0, \alpha\big(x_{i+\frac{1}{2}}\big) > 0\Big\},$$
$$\Omega_2 = \Big\{\mathbf{V}_i : \alpha\big(x_{i-\frac{1}{2}}\big) < 0, \alpha\big(x_{i+\frac{1}{2}}\big) < 0\Big\},$$
$$\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2).$$

In the following, we construct the correction function w_i corresponding to the domain $\Omega_i, i \le 3$, respectively.

5.1. Construction of correction functions

In the element $\mathbf{V}_i \subset \Omega_1$, noticing that $I_h u = I_h^- u$ and $a_{h,i}(u - I_h u, v^*) = (u_t - I_h^- u_t, v^*)$, then we construct the correction function w_1 as follows. Let

$$\overline{\alpha}_i = \max_{x \in \mathbf{V}_i} |\alpha(x)|, \quad \mathcal{P}_-(\mathbf{V}_i) := \mathcal{P}_k(\mathbf{V}_i) \setminus \mathcal{P}_0(\mathbf{V}_i). \tag{5.1}$$

We define the correction function $w_1 \in V_h$ in each $\mathbf{V}_i \subset \Omega_1$ by

$$w_1\big(x^-_{i+\frac{1}{2}}\big) = 0, \quad (\overline{\alpha}_i w_1, v_x)_i = \big(\partial_t(u - I_h^- u), v^*\big)_i, \quad \forall v \in \mathcal{P}_-(\mathbf{V}_i). \tag{5.2}$$

We have the following properties for w_1.

Lemma 5.1. *Let* $\mathbf{V}_i \in \Omega_1, w_1 \in V_h$ *be defined by* (5.2) *with* $\overline{\alpha}_i$ *given by* (5.1)*. Then for both RSV and LSV methods,*

$$\begin{aligned} &a_{h,i}\big(u - I_h^- u + w_1, v^*\big) \\ &= (\partial_t w_1, v^*)_i - \sum_{j=1}^{k} A_{i,j}\big(\alpha(x_{i,j}) - \overline{\alpha}_i\big) v_x(x_{i,j}) w_1(x_{i,j}), \quad \forall v \in \mathcal{P}_-(\mathbf{V}_i). \end{aligned} \tag{5.3}$$

Furthermore, if $u \in W^{k+3,\infty}$*, then*

$$\|w_1\|_{0,\mathbf{V}_i} + \|\partial_t w_1\|_{0,\mathbf{V}_i} \lesssim \frac{h_i^{k+\frac{5}{2}}}{\|\alpha\|_{0,\infty,\mathbf{V}_i}} \|u\|_{k+3,\infty,\mathbf{V}_i}. \tag{5.4}$$

Proof. First, for any $\mathbf{V}_i \in \Omega_1$, noticing that $\hat{w}_1|_{i+1/2} = \hat{w}_1|_{i-1/2} = 0$, and thus we have from (4.8) that

$$a_{h,i}(w_1, v^*) = (\partial_t w_1, v^*)_i - \sum_{j=1}^{k} A_{i,j}\alpha(x_{i,j}) v_x(x_{i,j}) w_1(x_{i,j}).$$

Noticing that the partial points $x_{i,j}$ are chosen as Gauss or right Radau points and $w_1 v_x \in \mathbb{P}_{2k-1}$ for all $v \in V_h$, then we use the property of Gauss or right Radau numerical quadrature to derive that

$$\sum_{j=1}^{k} A_{i,j} v_x(x_{i,j}) w_1(x_{i,j}) = \sum_{j=1}^{k+1} A_{i,j} v_x(x_{i,j}) w_1(x_{i,j}) = (v_x, w_1)_i, \quad v \in V_h,$$

which yields, together with (5.2) that

$$\begin{aligned} a_{h,i}\big(u - I_h^- u + w_1, v^*\big) &= \big(\partial_t(u - I_h^- u), v^*\big)_i + (\partial_t w_1, v^*)_i - \sum_{j=1}^{k} A_{i,j}\alpha(x_{i,j}) v_x(x_{i,j}) w_1(x_{i,j}) \\ &= (\overline{\alpha}_i w_1, v_x) + (\partial_t w_1, v^*)_i - \sum_{j=1}^{k} A_{i,j}\alpha(x_{i,j})_x(x_{i,j}) w_1(x_{i,j}) \\ &= (\partial_t w_1, v^*)_i - \sum_{j=1}^{k} A_{i,j}\big(\alpha(x_{i,j}) - \overline{\alpha}_i\big) v_x(x_{i,j}) w_1(x_{i,j}), \quad v \in \mathcal{P}_-(\mathbf{V}_i). \end{aligned}$$

This finishes the proof of (5.3).

To estimate w_1, we denote $w_0 = u - I_h^- u$ and suppose w_1 has the following Legendre expansion in each element $\mathbf{V}_i$:

$$w_1|_{\mathbf{V}_i} = \sum_{j=0}^{k} c_{i,j}(t) L_{i,j}(x).$$

Here $L_{i,j}$ denotes the Legendre polynomial of degree j in $\mathbf{V}_i$. Thus we have $\|L_{i,j}\|_{0,\infty,\mathbf{V}_i} \lesssim h_i$. Denoting

$$\phi_{i,j+1} = \int_{x_{i-\frac{1}{2}}}^{x} L_{i,j}(s)\mathrm{ds}$$

and choosing $v = \phi_{i,m+1} (m \in \mathbb{Z}_{k-1}^0)$ in (5.2) leads to

$$|c_{i,m}| = \frac{(2m+1)}{\overline{\alpha}_i h_i}\big|\big(\partial_t w_0, \phi_{i,m+1}^*\big)_i\big| \lesssim \frac{1}{\overline{\alpha}_i h_i}\|\partial_t w_0\|_{0,\mathbf{V}_i}\|\phi_{i,m+1}\|_{0,\mathbf{V}_i} \lesssim \frac{h_i}{\overline{\alpha}_i}\|\partial_t w_0\|_{0,\infty,\mathbf{V}_i}.$$

Using $w_1(x^-_{i+1/2}) = 0$, we obtain that

$$|c_{i,k}| = \left|\sum_{j=0}^{k-1} c_{i,j}\right| \lesssim \frac{h_i}{\overline{\alpha}_i}\|\partial_t w_0\|_{0,\infty,\mathbf{V}_i}. \tag{5.5}$$

Similarly, there holds for all $m \in \mathbb{Z}^0_k$,

$$|\partial_t c_{i,m}| \lesssim \frac{h_i}{\overline{\alpha}_i}\|\partial_{tt} w_0\|_{0,\infty,\mathbf{V}_i}.$$

Consequently,

$$\|w_1\|^2_{0,\mathbf{V}_i} \lesssim h_i \sum_{j=0}^{k} c^2_{i,j} \lesssim \frac{h^3_i}{\overline{\alpha}^2_i}\|\partial_t w_0\|^2_{0,\infty,\mathbf{V}_i},$$

$$\|\partial_t w_1\|^2_{0,\mathbf{V}_i} \lesssim h_i \sum_{j=0}^{k} (\partial_t c_{i,j})^2 \lesssim \frac{h^3_i}{\overline{\alpha}^2_i}\|\partial_{tt} w_0\|^2_{0,\infty,\mathbf{V}_i}.$$

Then (5.4) follows from the approximation property of the interpolation function. This finishes our proof. □

Similarly, we can construct the correction function w_2 in each element $\mathbb{V}_i \subset \Omega_2$. Note that $I_h u - I^+_h u$ and $a_{h,i}(u - I^+_h u, v^*) = (\partial_t(u - I^+_h u), v^*)_i$ in $\mathbb{V}_i \subset \Omega_2$. We define $w_2 \in V_h$ in each element $\mathbf{V}_i \in \Omega_2$ by

$$w_2\big(x^+_{i-\frac{1}{2}}\big) = 0, \quad (\overline{\alpha}_i w_2, v_x)_i = \big(\partial_t(u - I^+_h u), v^*\big)_i, \quad \forall v \in \mathcal{P}_-(\mathbf{V}_i). \tag{5.6}$$

Here α_i and $\mathcal{P}_-(\mathbf{V}_i)$ are the same as in (5.1). Following the same argument as that in Lemma 5.1, we have the following results for w_2.

Lemma 5.2. *Let $\mathbf{V}_i \in \Omega_2, w_2 \in V_h$ be defined by* (5.6) *with $\overline{\alpha}_i$ given in* (5.1)*. Then for both RSV and LSV methods*

$$\begin{aligned} &a_{h,i}\big(u - I^+_h u + w_2, v^*\big) \\ = &(\partial_t w_2, v^*)_i - \sum_{j=1}^{k} A_{i,j}\big(\alpha(x_{i,j}) - \overline{\alpha}_i\big)v_x(x_{i,j})w_2(x_{i,j}), \quad \forall v \in \mathcal{P}_-(\mathbf{V}_i). \end{aligned} \tag{5.7}$$

Furthermore, if $u \in W^{k+3,\infty}$, then

$$\|w_2\|_{0,\mathbf{V}_i} + \|\partial_t w_2\|_{0,\mathbf{V}_i} \lesssim \frac{h_i^{k+\frac{5}{2}}}{\|\alpha\|_{0,\infty,\mathbf{V}_i}}\|u\|_{k+3,\infty,\mathbf{V}_i}. \tag{5.8}$$

Remark 5.1. As we may observe from (5.4) and (5.8), in the element where $\alpha(x)$ achieves its zero, we have $\|\alpha\|_{0,\infty} = \mathcal{O}(h)$ and thus the error bounds $1/\|\alpha\|_{0,\infty,\mathbf{V}_i}$ can not be bounded uniformly, which indicates the convergence rate in (5.4) and (5.8) may decrease. This is the essential difference between the constant and degenerate variable coefficients problems, which makes the superconvergence analysis for variable coefficients problems more sophisticated.

We are ready to construct the global correction function $w \in V_h$ over the whole domain as follows:

$$w = \begin{cases} w_1, & \text{if} \quad \alpha|_{i-\frac{1}{2}} \geq 0, \quad \alpha|_{i+\frac{1}{2}} > 0, \\ w_2, & \text{if} \quad \alpha|_{i-\frac{1}{2}} \leq 0, \quad \alpha|_{i+\frac{1}{2}} < 0, \\ 0, & \text{otherwise}. \end{cases} \tag{5.9}$$

Here w_1 and w_2 are defined by (5.2) and (5.6), respectively.

5.2. Superconvergence of the SV flux

In this subsection, we study the superconvergence result for the flux function αu_h. We shall prove that the flux function of the SV method is superconvergent towards the flux $\alpha I_h u$ in the L^2-norm.

Define

$$\tilde{\alpha}_i^2 := \alpha^2\big(x_{i-\frac{1}{2}}\big) + \alpha^2\big(x_{i+\frac{1}{2}}\big), \quad \|v\|_\alpha^2 := \sum_{i=1}^N \tilde{\alpha}_i^2 \|v\|_{0,\mathbf{V}_i}^2, \quad \|v\|_{\alpha,E}^2 := \sum_{i=1}^N \tilde{\alpha}_i^2 (v, v^*)_i.$$

Note that

$$\begin{aligned}\|\alpha(u_h - I_h u)\|_0 &\lesssim \|u_h - I_h u\|_\alpha + h\|u_h - I_h u\|_0 \\ &\lesssim \|u_h - I_h u\|_\alpha + h^{k+2}\|u\|_{k+2}.\end{aligned}$$

In other words, to study the supercloseness result for the flux function approximation, we may turn to analyzing the error $\|u_h - I_h u\|_\alpha$. Towards this end, we first define a new bilinear form

$$A_{h,i}(v, w^*) = \tilde{\alpha}_i^2 a_{h,i}(v, w^*), \quad A_h(v, w^*) = \sum_{i=1}^N A_{h,i}(v, w^*),$$

and then we set up a stability result for the norm $\|\cdot\|_\alpha$. Note that it has been proved in [2] that

$$\frac{1}{2}\frac{d}{dt}\|v\|_\alpha^2 \lesssim \big|A_h^{DG}(v, v)\big| + \|v\|_\alpha^2, \quad v \in V_h,$$

where

$$A_{h,i}^{DG}(v, w) = \tilde{\alpha}_i^2 a_{h,i}^{DG}(v, w), \quad A_h^{DG}(v, w) = \sum_{i=1}^N A_{h,i}^{DG}(v, w).$$

Then by the same argument as what we did in Theorem 3.2, we have the following stability result for the norm $\|\cdot\|_\alpha$.

Lemma 5.3. *Both the RSV and LSV methods are stable in the flux function norm. That is,*

$$\frac{1}{2}\frac{d}{dt}\|v\|_{\alpha,E}^2 \lesssim |A_h(v, v^*)| + \|v\|_\alpha^2, \quad \forall v \in V_h. \tag{5.10}$$

Theorem 5.1. *Let $u(\cdot, t) \in W^{k+3,\infty}(\Omega)$ be the solution of* (1.1), *and w be the correction function defined by* (5.9). *Then*

$$\begin{aligned}&\widehat{w}|_{i+\frac{1}{2}} = 0, \quad i \in \mathbb{Z}_N, \\ &\|w\|_\alpha + \|\partial_t w\|_\alpha \lesssim h^{k+2}\|u\|_{k+3,\infty}.\end{aligned} \tag{5.11}$$

Moreover, there holds for all $v \in V_h$,

$$|A_h(u - I_h u + w, v^*)| \lesssim h^{k+2}\|u\|_{k+3,\infty}\|v\|_\alpha. \tag{5.12}$$

Proof. First, (5.11) follows directly from (5.2), (5.6), (5.4) and (5.8). As for (5.12), we first note that

$$\begin{aligned}A_h(u - I_h u + w, v^*) = &\sum_{\mathbf{V}_i \in \Omega_1 \cup \Omega_2} \tilde{\alpha}_i^2 a_{h,i}(u - I_h u + w, v^*) \\ &+ \sum_{\mathbf{V}_i \in \Omega_3} \tilde{\alpha}_i^2 a_{h,i}\big(u - I_h^{\pm} u, v^*\big).\end{aligned} \tag{5.13}$$

As a direct consequence of Cauchy-Schwarz inequality and (5.3) and (5.4),

$$\begin{aligned}\sum_{\mathbf{V}_i\in\Omega_1}\tilde{\alpha}_i^2 a_{h,i}(u-I_hu+w_1,v^*) &\lesssim \|\partial_t w_1\|_{\alpha,\Omega_1}\|v\|_{\alpha,\Omega_1}\\ &\quad+\sum_{\mathbf{V}_i\in\Omega_1}\tilde{\alpha}_i^2 h_i^2\|v_x\|_{0,\infty,\mathbf{V}_i}\|w_1\|_{0,\infty,\mathbf{V}_i}\\ &\lesssim \|v\|_{\alpha,\Omega_1}\big(\|w_1\|_{\alpha,\Omega_1}+\|\partial_t w_1\|_{\alpha,\Omega_1}\big)\\ &\lesssim h^{k+2}\|u\|_{k+3,\infty}\|v\|_{\alpha,\Omega_1}.\end{aligned}$$

Here in the second step, we have used the inverse inequality. Similarly, there holds

$$\begin{aligned}\sum_{\mathbf{V}_i\in\Omega_2}\tilde{\alpha}_i^2 a_{h,i}(u-I_hu+w_2,v^*) &\lesssim \|v\|_{\alpha,\Omega_2}\big(\|w_2\|_{\alpha,\Omega_2}+\|\partial_t w_2\|_{\alpha,\Omega_2}\big)\\ &\lesssim h^{k+2}\|u\|_{k+3,\infty}\|v\|_{\alpha,\Omega_2}.\end{aligned}$$

Consequently,

$$\sum_{\mathbf{V}_i\in\Omega_1\bigcup\Omega_2}\tilde{\alpha}_i^2 a_{h,i}(u-I_hu+w,v^*)\lesssim h^{k+2}\|u\|_{k+3,\infty}\|v\|_{\alpha}. \tag{5.14}$$

For any $\mathbf{V}_i\in\Omega_3$, since $\alpha(x_{i-1/2})\alpha(x_{i-1/2})\le 0$, then there exists at least one point $\theta_i\in\mathbf{V}_i$ such that $\alpha(\theta_i)=0$, and thus,

$$\|\alpha\|_{0,\infty,\Omega_3}=\|\alpha'(\xi_i)(x-\theta_i)\|_{0,\infty,\Omega_3}\le h\|\alpha\|_{1,\infty,\Omega_3}\lesssim h. \tag{5.15}$$

Then the combination of (4.13), (5.15) and Cauchy-Schwartz inequality leads to

$$\begin{aligned}\Big|\sum_{\mathbf{V}_i\in\Omega_3}\tilde{\alpha}_i^2 a_{h,i}(u-I_hu,v^*)\Big| &\lesssim h^{k+1}\sum_{\mathbf{V}_i\in\Omega_3}\tilde{\alpha}_i^2\big(\|u\|_{k+2,\mathbf{V}_i}+h_i^{\frac{1}{2}}\|u\|_{k+1,\infty,\mathbf{V}_i}\big)\|v\|_{0,\mathbf{V}_i}\\ &\lesssim h^{k+2}\|u\|_{k+2}\|v\|_{\alpha},\quad \forall v\in V_h.\end{aligned} \tag{5.16}$$

Then (5.12) follows for any $v_h\in\mathcal{P}_-(\mathbf{V}_i)$ from (5.13), (5.14) and (5.16).

For all $v\in\mathcal{P}_0(\mathbf{V}_i)$, a direct calculation (3.12) yields that

$$\begin{aligned}A_h(u-I_hu+w,v^*) &= \sum_{\mathbf{V}_i\in\Omega_1\cup\Omega_2}\tilde{\alpha}_i^2[(u_t-I_hu_t,v)_i+(\partial_t w,v)_i]\\ &\quad+\sum_{\mathbf{V}_i\in\Omega_3}\tilde{\alpha}_i^2(u_t-I_hu_t,v)_i.\end{aligned} \tag{5.17}$$

For any $\mathbf{V}_i\in\Omega_1\cup\Omega_2$, the interpolation points are either Gauss points or right/left Radau points. By utilizing the Newton-interpolating-remainder representation, we get

$$u-I_hu=u[x_{i,1},\ldots,x_{i,k+1},x]\prod_{j=1}^{k+1}(x-x_{i,j})=:\widetilde{u}(x)\widetilde{L}_{i,k+1}(x),$$

where

$$\widetilde{u}(x)=u[x_{i,1},\ldots,x_{i,k+1},x],\quad \widetilde{L}_{i,k+1}(x):=\prod_{j=1}^{k+1}(x-x_{i,j}).$$

Using the properties of Legendre polynomials or Radau polynomials, we have

$$\big(\widetilde{L}_{i,k+1},1\big)_i=0,$$

and thus

$$\big|(u_t - I_h u_t, v)_i\big| = \left| \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (\widetilde{u} - \widetilde{u}_0)\widetilde{L}_{i,k+1}(x) v \,\mathrm{d}x \right| \lesssim h_i^{k+2} \|u\|_{k+2,\mathbf{V}_i} \|v\|_{0,\mathbf{V}_i}. \tag{5.18}$$

Here $\widetilde{u}_0$ denotes the cell average of $\widetilde{u}$ in $\mathbf{V}_i$. Combing (5.17), (5.18) and (5.16), we conclude that the (5.12) is also valid for all $v_h \in \mathcal{P}_0(\mathbf{V}_i)$. The proof is complete. □

Corollary 5.1. *Let $u(\cdot,t) \in W^{k+3,\infty}(\Omega)$ be the solution of* (1.1)*, and u_h be the SV solution of* (2.6) *with initial solution $u_h(\cdot,0) = I_h u_0$ with $I_h u$ the interpolation function of u defined in* (4.5)*. Then for both the RSV and LSV methods,*

$$\|(u_h - I_h u)(\cdot,t)\|_\alpha \lesssim h^{k+2} \|u\|_{k+3,\infty}. \tag{5.19}$$

Proof. Let $u_I = I_h u - w$. By choosing $v = u_h - u_I$ in (5.10) and using (5.12), we have

$$\begin{aligned} \frac{d}{dt}\|u_h - u_I\|_{\alpha,E}^2 &\lesssim \big|A\big(u - u_I, (u_h - u_I)^*\big)\big| + \|u_h - u_I\|_\alpha^2 \\ &\lesssim h^{2(k+2)} \|u\|_{k+3,\infty}^2 + \|u_h - u_I\|_\alpha^2. \end{aligned}$$

Due to the special choice of the initial solution, the equivalence between $\|\cdot\|_{\alpha,E}$ and $\|\cdot\|_\alpha$, and the Gronwall inequality, we get

$$\|u_h(\cdot,t) - u_I(\cdot,t)\|_\alpha \lesssim \|w(\cdot,0)\|_\alpha + h^{k+2}\|u\|_{k+3,\infty},$$

which yields, together with the triangle inequality and (5.11), that

$$\|(u_h - I_h u)(\cdot,t)\|_\alpha \lesssim \|w(\cdot,t)\|_\alpha + \|u_h(\cdot,t) - u_I(\cdot,t)\|_\alpha \lesssim h^{k+2}\|u\|_{k+3,\infty}.$$

This ends the proof. □

Now we are ready to present the superconvergence results for the flux function approximation.

Theorem 5.2. *Let $u(\cdot,t) \in W^{k+3,\infty}$ be the solution of* (1.1)*, and u_h be the SV solution obtained by* (2.6) *with initial solution $u_h(\cdot,0) = I_h u_0$. Then for both the RSV and LSV methods, the flux function αu_h has the following superconvergence property:*

$$e_f := \|\alpha u_h - \alpha u_I\|_0 \lesssim h^{k+2}\|u\|_{k+3,\infty}. \tag{5.20}$$

Consequently,

$$e_{f,n} + e_{f,c} + e_{f,r} \lesssim h^{k+2}\|u\|_{k+3,\infty}, \quad e_{f,l} \lesssim h^{k+1}\|u\|_{k+3,\infty}, \tag{5.21}$$

where

$$\begin{aligned} e_{f,n} &:= \left(\frac{1}{N}\sum_{i=1}^{N} (\alpha u - \alpha \widehat{u}_h)^2 (x_{i+\frac{1}{2}}) \right)^{\frac{1}{2}}, \\ e_{f,c} &:= \left(\frac{1}{N}\sum_{i=1}^{N} \left(\frac{1}{h_i} \int_{\mathbf{V}_i} (\alpha u - \alpha u_h) dx \right)^2 \right), \\ e_{f,r} &:= \left(\frac{1}{N}\sum_{i=1}^{N} \sum_{j=1}^{k+1} (\alpha u - \alpha u_h)^2 (y_{i,j}) \right)^{\frac{1}{2}}, \\ e_{f,l} &:= \left(\frac{1}{N}\sum_{i=1}^{N} \sum_{j=1}^{k} (\alpha \partial_x u - \alpha \partial_x u_h)^2 (z_{i,j}) \right)^{\frac{1}{2}}. \end{aligned}$$

Here $\{y_{i,j}\}_{j=1}^{k+1}$ *are the interpolation nodes of the interpolation operator* I_h *defined in* (4.5)*, and* $\{z_{i,j}\}_{j=1}^{k}$ *are the roots of* $\partial_x(\prod_{j=1}^{k+1}(x-y_{i,j}))$.

Proof. Firstly, a direct calculation yields that

$$\begin{aligned}\|\alpha u_h-\alpha u_I\|_{0,\mathbf{V}_i}^2 &\lesssim \int_{\mathbf{V}_i}\big(\alpha-\alpha_{i-\frac12}\big)^2(u_h-u_I)^2\mathrm{d}x\\ &\quad+\int_{\mathbf{V}_i}\big(\alpha-\alpha_{i+\frac12}\big)^2(u_h-u_I)^2\mathrm{d}x+\|u_h-u_I\|_{\alpha,\mathbf{V}_i}^2\\ &\lesssim h_i^2\|u_h-u_I\|_{0,\mathbf{V}_i}^2+\|u_h-u_I\|_{\alpha,\mathbf{V}_i}^2.\end{aligned}$$

The estimate (5.20) follows immediately by using Corollary 5.1 and Theorem 4.1.

Secondly, we show the first inequality of (5.21). On the one hand, by applying (4.6) and the inverse inequality, we have

$$\begin{aligned}|e_{f,n}|^2 &= \frac1N\sum_{i=1}^N\alpha^2\big(x_{i+\frac12}\big)(I_hu-\widehat{u}_h)^2\big(x_{i+\frac12}\big)\\ &\lesssim \sum_{i=1}^N\big(h^2\|I_hu-u_h\|_{0,\mathbf{V}_i}^2+\|I_hu-u_h\|_{\alpha,\mathbf{V}_i}^2\big)\lesssim h^{2k+4}\|u\|_{k+3,\infty}.\end{aligned}$$

On the other hand, by choosing $v=1$ in (5.18), we for all $\mathbf{V}_i\in\Omega_1\cup\Omega_2$ have

$$\begin{aligned}\int_{\mathbf{V}_i}\alpha(u-I_hu)\mathrm{d}x &= \int_{\mathbf{V}_i}\big(\alpha-\alpha_{i+\frac12}\big)(u-I_hu)\mathrm{d}x+\alpha_{i+\frac12}\int_{\mathbf{V}_i}(u-I_hu)\mathrm{d}x\\ &\lesssim h_i^{k+3}\|u\|_{k+3,\mathbf{V}_i,\infty}.\end{aligned}\tag{5.22}$$

As for $\mathbf{V}_i\in\Omega_3$, we have $\alpha(x_{i-1/2})\alpha(x_{i-1/2})\le 0$ in Ω_3. Then there exists at least one point $\theta_i\in\mathbf{V}_i$ such that $\alpha(\theta_i)=0$, indicating that (5.22) still holds in $\mathbf{V}_i\in\Omega_3$. Therefore,

$$\frac1N\sum_{i=1}^N\left(\frac1{h_i}\int_{\mathbf{V}_i}\alpha(u-I_hu)\mathrm{d}x\right)^2\lesssim\frac1N\sum_{i=1}^N h_i^{2k+4}\|u\|_{k+3,\mathbf{V}_i,\infty}^2\lesssim h^{2k+4}\|u\|_{k+3,\infty}^2.$$

By applying (5.20) and Cauchy-Schwarz inequality, we derive that

$$\frac1N\sum_{i=1}^N\left(\frac1{h_i}\int_{\mathbf{V}_i}\alpha(I_hu-u_h)\mathrm{d}x\right)^2\lesssim\|I_hu-u_h\|_\alpha^2\lesssim h^{2k+4}\|u\|_{k+3,\infty}^2.$$

Thus, the application of triangle inequality yields that

$$\begin{aligned}e_{f,c}^2 &\le \frac1N\sum_{i=1}^N\left(\frac1{h_i}\int_{\mathbf{V}_i}\alpha(I_hu-u)\mathrm{d}x\right)^2+\frac1N\sum_{i=1}^N\left(\frac1{h_i}\int_{\mathbf{V}_i}\alpha(I_hu-u_h)\mathrm{d}x\right)^2\\ &\lesssim h^{2k+4}\|u\|_{k+3,\infty}^2.\end{aligned}$$

We next estimate the error $e_{f,r}$. Using the inverse inequality, we get

$$e_{f,r}=\frac1N\sum_{i=1}^N\sum_{j=1}^{k+1}\alpha^2(y_{i,j})(I_hu-u_h)^2(y_{i,j})$$

$$\lesssim \sum_{i=1}^{N}\sum_{j=1}^{k+1}\alpha^2(y_{i,j})\|I_h u-u_h\|^2_{0,\mathbf{V}_i}$$
$$\lesssim \sum_{i=1}^{N}\big(h_i^2\|I_h u-u_h\|^2_{0,\mathbf{V}_i}+\|I_h u-u_h\|^2_{\alpha,\mathbf{V}_i}\big).$$

Therefore, the first inequality of (5.21) follows from the conclusions in Corollary 5.1 and Theorem 4.1.

Finally, by using the Newton-interpolating-remainder representation, we obtain that

$$u(x)-I_h u(x)=u[y_{i,1},\dots,y_{i,k},y_{i,k+1},x]\prod_{j=1}^{k+1}(x-y_{i,j})=:\widetilde{u}(x)\omega(x),$$

which indicates that at the roots of $\omega(x)$, there holds

$$|\partial_x(u-I_h u)(z_{i,j},t)|\lesssim h_i^{k+1}\|\partial_x\widetilde{u}(x)\|_{0,\mathbf{V}_i,\infty}\lesssim h^{k+1}\|u\|_{k+2,\infty}.$$

In addition, by using (5.19) and the inverse inequality, we get

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}\alpha^2(z_{i,j})\partial_x(u_h-I_h u)^2(z_{i,j},t)$$
$$\lesssim \sum_{i=1}^{N}\sum_{j=1}^{k}\alpha^2(z_{i,j})\|\partial_x(u_h-I_h u)\|^2_{0,\mathbf{V}_i}$$
$$\lesssim h^{-2}\|u_h-I_h u\|^2_{\alpha}+\|u_h-I_h u\|^2_0.$$

Therefore,

$$e_{f,l}^2=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}\alpha^2(z_{i,j})\partial_x(u-I_h u+I_h u-u_h)^2(z_{i,j},t)\lesssim h^{2k+2}\|u\|^2_{k+3,\infty}.$$

The proof is complete. □

5.3. Superconvergence of the SV solution

Since the smooth function $\alpha(x)$ has only a finite number of zeros on Ω, for simplicity, we suppose that α only has one zero point $x=0$ and there exists a positive integer m such that

$$\alpha(0)=\alpha'(0)=\cdots=\alpha^{(m-1)}(0)=0,\quad \alpha^{(m)}(0)\neq 0. \tag{5.23}$$

Note that if m=1, then x=0 is the single root of α, while if m>1, $x=0$ is a multiple root of α.

To study the superconvergence of the SV solution approximation, we follow the similar idea of [2] and modify the correction functions w_1 and w_2 defined in (5.2) and (5.6), i.e.

$$\widetilde{w}_i|_{\mathbf{V}_j}=\begin{cases}0, & \mathbf{V}_j\subset\Lambda=\big[0,x_{i_0+\frac12}\big],\\ w_i, & \mathbf{V}_j\subset\Lambda^+=\Omega\setminus\Lambda,\end{cases} \tag{5.24}$$

where $i\in\mathbb{Z}_2$, the positive integer i_0 satisfies that $x_{i_0-1/2}\le h^{1/m'}\le x_{i_0+/1/2}$, and $m'=\min\{m,k+3\}$. Then we define the global correction function $\widetilde{w}$ by

$$\widetilde{w}|_{\mathbf{V}_i}=\begin{cases}\widetilde{w}_1, & \text{if}\quad \alpha|_{i-\frac12}\ge 0,\quad \alpha|_{i+\frac12}>0,\\ \widetilde{w}_2, & \text{if}\quad \alpha|_{i-\frac12}\le 0,\quad \alpha|_{i+\frac12}<0,\\ 0, & \text{otherwise}.\end{cases} \tag{5.25}$$

Lemma 5.4. *Let $u \in W^{k+3,\infty}$ be the solution of* (1.1)*, $\alpha(x)$ be a sufficiently smooth function satisfying* (5.23)*, and $\widetilde{w}_i (i \in \mathbb{Z}_2)$ be the modified correction functions defined by* (5.24)*. Then*

$$\sum_{i=1}^{2} \big(\|\widetilde{w}_i\|_0 + \|\partial_t \widetilde{w}_i\|_0 \big) \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty}. \tag{5.26}$$

Furthermore, there holds that

$$|a_h(u - I_h u + \widetilde{w}, v^*)| \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v\|_0, \quad v \in V_h. \tag{5.27}$$

Proof. Here we omit the proof of (5.26) and refer to [2] for more detailed information and discussions. We focus our attention to prove (5.27). Replacing w_1 in (5.3) by $\widetilde{w}_1$, and following the same argument as what we did in (5.3) yields

$$\begin{aligned}
&\Big| \sum_{\mathbf{V}_i \in \Omega_1} a_{h,i}\big(u - I_h^- u + \widetilde{w}_1, v^*\big) \Big| \\
\le &\Big| \sum_{\mathbf{V}_i \in \Lambda} \big(\partial_t (u - I_h^- u), v^*\big)_i + \sum_{\mathbf{V}_i \in \Lambda^+} (\partial_t w_1, v^*)_i \\
&\quad - \sum_{\mathbf{V}_i \in \Lambda^+} \sum_{j=1}^{k} \big(\alpha(x_{i,j}) - \overline{\alpha}_i\big) v_x(x_{i,j}) w_1(x_{i,j}) \Big| \\
\lesssim &\, h^{k+1} \|u\|_{k+2,\infty} x_{i_0+\frac{1}{2}}^{\frac{1}{2}} \|v\|_{0,\Lambda} + \big(\|\widetilde{w}_1\|_0 + \|\partial_t \widetilde{w}_1\|_0\big) \|v\|_0 \\
\lesssim &\, h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v\|_0, \quad \forall v \in \mathcal{P}_-(\mathbf{V}_i).
\end{aligned}$$

Here in the last step, we have used $x_{i_0+1/2} \lesssim h^{1/m'}$ and (5.26). As for $v \in \mathcal{P}_0(\mathbf{V}_i)$, we have from (5.18) and (5.26) that

$$\begin{aligned}
&\Big| \sum_{\mathbf{V}_i \in \Omega_1} a_{h,i}\big(u - I_h^- u + \widetilde{w}_1, v^*\big) \Big| \\
\le &\Big| \sum_{\mathbf{V}_i \in \Lambda} \big(\partial_t (u - I_h^- u), v\big)_i + \sum_{\mathbf{V}_i \in \Lambda^+} (\partial_t w_1, v)_i \Big| \\
\lesssim &\, h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v\|_0.
\end{aligned}$$

Therefore, for all $v \in V_h$, $v = v_0 + v_1$, where $v_0 \in \mathcal{P}_0(\mathbf{V}_i)$ and $v_1 \in \mathcal{P}_-(\mathbf{V}_i)$, we have

$$\Big| \sum_{\mathbf{V}_i \in \Omega_1} a_{h,i}\big(u - I_h^- u + \widetilde{w}_1, v^*\big) \Big| \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v\|_0. \tag{5.28}$$

Likewise, we get

$$\Big| \sum_{\mathbf{V}_i \in \Omega_2} a_{h,i}\big(u - I_h^+ u + \widetilde{w}_2, v^*\big) \Big| \lesssim h^{k+1+\frac{1}{2m'}} \|u\|_{k+3,\infty} \|v\|_0, \quad \forall v \in V_h. \tag{5.29}$$

Noticing that

$$\begin{aligned}
a_h\big(u - I_h u + \widetilde{w}, v^*\big) = &\sum_{\mathbf{V}_i \in \Omega_1} a_{h,i}\big(u - I_h^- u + \widetilde{w}_1, v^*\big) \\
&+ \sum_{\mathbf{V}_i \in \Omega_2} a_{h,i}\big(u - I_h^+ u + \widetilde{w}_2, v^*\big) \\
&+ \sum_{\mathbf{V}_i \in \Omega_3} a_{h,i}\big(u - I_h^\pm u, v^*\big).
\end{aligned} \tag{5.30}$$

By using the fact that $|\Omega_3| \lesssim h$, the inequalities (5.15) and (4.10), we obtain

$$
\begin{aligned}
&\Big|\sum_{\mathbf{V}_i\in\Omega_3} a_{h,i}\big(u - I_h^{\pm}u, v^*\big)\Big| \\
=&\Big|\sum_{\mathbf{V}_i\in\Omega_3}\big(\partial_t(u-I_h^{\pm}u), v^*\big)_i - \sum_{\mathbf{V}_i\in\Omega_3}\sum_{j=1}^{k} A_{i,j}\alpha(x_{i,j})\big(u-I_h^{\pm}u\big)_x(x_{i,j})v(x_{i,j})\Big| \\
\lesssim& h^{k+1+\frac{1}{2m'}}\|u\|_{k+3,\infty}\|v\|_0, \quad \forall v\in V_h.
\end{aligned}
\tag{5.31}
$$

Then the inequality (5.27) follows from (5.28)-(5.31). □

Theorem 5.3. *Let $u \in W^{k+3,\infty}$ be the solution of* (1.1)*, and u_h be the solution of* (2.6) *with the initial solution $u_h^0 = I_h u_0$, where I_h is defined by* (4.5)*. Suppose $\alpha(x)$ is a sufficiently smooth function satisfying* (5.23)*. Then for $m' = \min\{m, k+3\}$,*

$$
e_u := \|u_h - I_h u\|_0 \lesssim h^{k+1+\frac{1}{2m'}}\|u\|_{k+3,\infty}. \tag{5.32}
$$

Moreover, there hold

$$
\begin{aligned}
&e_{u,n} + e_{u,c} + e_{u,r} \lesssim h^{k+1+\frac{1}{2m'}}\|u\|_{k+3,\infty},\\
&e_{u,l} \lesssim h^{k+\frac{1}{2m'}}\|u\|_{k+3,\infty},
\end{aligned}
\tag{5.33}
$$

where

$$
\begin{aligned}
e_{u,n} &:= \left(\frac{1}{N}\sum_{i=1}^{N}(u-\widehat{u}_h)^2(x_{i+\frac12})\right)^{\frac12},\\
e_{u,c} &:= \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{h_i}\int_{\mathbf{V}_i}(u-u_h)dx\right)^2\right)^{\frac12},\\
e_{u,r} &:= \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k+1}(u-u_h)^2(y_{i,j})\right)^{\frac12},\\
e_{u,l} &:= \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}(u-u_h)^2(z_{i,j})\right)^{\frac12},
\end{aligned}
$$

where the nodes $\{y_{i,j}\}_{j=1}^{k+1}$ and $\{z_{i,j}\}_{j=1}^{k}$ are the same as in Theorem 5.2.

Proof. By the definition of $\widetilde{w}$ in (5.25) and the property (5.11), it is easy to verify that the numerical flux of $\widetilde{w}$ satisfies $\widehat{\widetilde{w}}|_{i+1/2} = 0$ for all $i \in \mathbb{Z}_N$. Consequently, choosing $v = u_h - I_h u + \widetilde{w}$ in (3.28) and applying the property (4.6) and (5.27), we obtain that

$$
\begin{aligned}
\frac{d}{dt}\|u_h - I_h u + \widetilde{w}\|_0^2 &\lesssim \|u_h - I_h u + \widetilde{w}\|_0^2 + \big|a_h\big(u - I_h u + \widetilde{w}, (u_h - I_h u + \widetilde{w})^*\big)\big|\\
&\lesssim \|u_h - u_I + \widetilde{w}\|_0^2 + h^{k+1+\frac{1}{2m'}}\|u\|_{k+3,\infty}\|u_h - u_I + \widetilde{w}\|_0.
\end{aligned}
$$

Then (5.32) follows from the Gronwall inequality and (5.26). The proof of (5.33) is similar to that of Theorem 5.2 and we omit it here. □

Remark 5.2. For hyperbolic equations with fixed wind direction (i.e. the variable coefficient α does not change its sign), the highest superconvergence rate of the RSV method can be improved to $2k+1$ and that of the LSV method can achieve $2k$, which have been reported numerically in [3]. We refer to [3] for more detailed analysis and information.

6. Numerical Results

In this section, we present some numerical experiments to verify our theoretical findings. Uniform meshes of N elements are used in our numerical experiments. We use the fourth-order Runge-Kutta method with time step $\Delta t = 0.01/N$ to reduce the time discretization. Denoting $e = u - u_h$. In our numerical experiment, we will test various errors including $\|e\|_0, e_f, e_{f,c}, e_{f,r}, e_{f,n}, e_{f,l}, e_u, e_{u,c}, e_{u,r}, e_{u,n}$ and $e_{u,l}$, which are defined in Theorems 5.2 and 5.3. We use "r" to denote the convergence order. To test the difference between the SV methods and DG methods, we denote by $\bar{u}_h$ the numerical solution calculated from the upwind DG method and define $\bar{e} = u_h - \bar{u}_h$,

$$\bar{e}_{f,c} := \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{h_i}\int_{\mathbf{V}_i}(\alpha u_h - \alpha\bar{u}_h)\mathrm{d}x\right)^2\right),$$

$$\bar{e}_{u,c} := \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{h_i}\int_{\mathbf{V}_i}(\bar{u}_h - u_h)\mathrm{d}x\right)^2\right)^{\frac{1}{2}}.$$

Since the numerical results of the two examples obtained by DG methods have been reported in [2], we do not provide the errors and convergence rates calculated from the DG method and refer to [2] for more detailed information.

Example 6.1. We consider the following equation with the periodic boundary condition:

$$u_t + \big(\sin(x)u\big)_x = g(x,t), \quad (x,t) \in [0,2\pi]\times(0,\pi/2], \quad u(x,0) = \sin(x),$$

where $g = g(x,t)$ is chosen such that the solution $u(x,t) = \mathrm{e}^{\sin(x-t)}$. Note that $\alpha(x) = \sin(x)$ has three zeros $x = 0, \pi, 2\pi$, and at these zeros,

$$\partial_x\alpha(x) = \cos(x) \neq 0,$$

which implies that Theorem 5.3 holds with $m = m' = 1$.

We solve the problem using LSV and RSV schemes with $k = 1, 2, 3$. Listed in Table 6.1 are L^2- and L^∞-norms of the error e and their corresponding convergence order. We observe that both LSV and RSV methods have optimal convergence orders of h^{k+1}. This fact confirms our theoretical results in Theorem 4.1.

We report the superconvergence results of SV methods in Tables 6.2-6.5. As for the numerical flux approximation, we observe a convergence rate of $k+2$ for the errors $e_f, e_{f,c}, e_{f,r}, e_{f,n}$ and a rate of $k+1$ for $e_{f,l}$ for both RSV and LSV, which are consistent with the theoretical findings in Theorem 5.2. As for the numerical solution itself, we observe a convergence rate of $(k+3/2)$ for $e_u, e_{u,c}, e_{u,r}$ and $e_{u,n}$, and a rate of $(k+1/2)$ for $e_{u,l}$, which are consistent with our theory in Theorem 5.3. Note that for $k = 1$, no superconvergence phenomenon has been found for the LSV method.

To demonstrate the difference between our SV methods and the upwind DG method, we list in Table 6.6 the L^2 error $\|\bar{e}\|_0$ and the cell average errors $\bar{e}_{f,c}, e_{u,c}$. We observe that the error $\|\bar{e}\|_0$ converges with the order of $k+3/2$ for the RSV while $k+1$ for LSV, indicating that the RSV solution tends to the DG solution more closer than the LSV solution. Furthermore, a convergence rates of $k+2$ is also observe for the errors $\bar{e}_{f,c}$ and $\bar{e}_{u,c}$, indicating the superconvergence phenomenon for the cell average errors.

Table 6.1: Errors and convergence orders of RSV and LSV for Example 6.1.

k	N	RSV				LSV			
		$\|e\|_0$	r	$\|e\|_{0,\infty}$	r	$\|e\|_0$	r	$\|e\|_{0,\infty}$	r
1	128	4.24E-4		1.10E-3		6.31E-4		1.60E-3	
	256	1.06E-4	2.00	2.73E-4	2.00	1.58E-4	2.00	4.09E-4	2.00
	512	2.64E-5	2.00	6.82E-5	2.00	3.95E-5	2.00	1.02E-4	2.00
	1024	6.59E-6	2.00	1.71E-5	2.00	9.87E-6	2.00	2.56E-5	2.00
2	128	2.81E-6		8.02E-6		4.42E-6		1.34E-5	
	256	3.51E-7	3.00	1.00E-6	3.00	5.52E-7	3.00	1.67E-6	3.00
	512	4.39E-8	3.00	1.25E-7	3.00	6.90E-8	3.00	2.09E-7	3.00
	1024	5.49E-9	3.00	1.57E-8	3.00	8.63E-9	3.00	2.61E-8	3.00
3	32	4.82E-6		1.84E-5		7.66E-6		3.27E-5	
	64	2.97E-7	4.02	1.19E-6	3.95	4.75E-7	4.01	2.09E-6	3.97
	128	1.84E-8	4.01	7.50E-8	3.99	2.96E-8	4.00	1.31E-7	3.99
	256	1.15E-9	4.01	4.70E-9	4.00	1.85E-9	4.00	8.22E-9	4.00

Table 6.2: Errors and convergence orders for the RSV flux function approximation for Example 6.1.

k	N	e_f	r	$e_{f,c}$	r	$e_{f,r}$	r	$e_{f,n}$	r	$e_{f,l}$	r
$k=1$	128	2.07E-7		1.60E-6		3.11E-6		3.13E-6		2.35E-4	
	256	3.02E-8	2.78	1.94E-7	3.04	3.97E-7	2.97	3.98E-7	2.98	5.94E-5	1.98
	512	4.05E-9	2.90	2.38E-8	3.02	4.97E-8	3.00	4.98E-8	3.00	1.49E-5	2.00
	1024	5.23E-10	2.95	2.95E-9	3.01	6.22E-9	3.00	6.22E-9	3.00	3.71E-6	2.00
$k=2$	128	8.96E-10		6.68E-10		2.02E-8		2.02E-8		2.46E-6	
	256	3.55E-11	4.66	2.26E-11	4.89	1.25E-9	4.02	1.25E-9	4.02	3.00E-7	3.03
	512	1.48E-12	4.59	8.24E-13	4.78	7.73E-11	4.01	7.73E-11	4.01	3.71E-8	3.01
	1024	6.36E-14	4.54	3.49E-14	4.56	4.82E-12	4.00	4.82E-12	4.00	4.63E-9	3.00
$k=3$	32	9.45E-8		2.78E-8		2.11E-7		1.47E-7		4.92E-6	
	64	2.17E-9	5.44	5.90E-10	5.56	5.74E-9	5.20	4.29E-9	5.09	3.27E-7	3.91
	128	4.92E-11	5.46	1.38E-11	5.42	1.51E-10	5.25	1.22E-10	5.14	2.04E-8	4.00
	256	1.12E-12	5.45	3.25E-13	5.43	4.08E-12	5.21	3.66E-12	5.06	1.26E-9	4.01

Table 6.3: Errors and convergence rates for the RSV solution itself approximation for Example 6.1.

k	N	e_u	r	$e_{u,c}$	r	$e_{u,r}$	r	$e_{u,n}$	r	$e_{u,l}$	r
$k=1$	128	2.60E-7		2.51E-5		2.51E-5		2.41E-5		5.30E-4	
	256	3.82E-8	2.77	4.43E-6	2.50	4.39E-6	2.59	4.24E-6	2.51	1.66E-5	1.66
	512	5.36E-9	2.83	7.84E-7	2.50	7.71E-7	2.55	7.45E-7	2.51	5.42E-5	1.61
	1024	7.58E-10	2.82	1.39E-7	2.50	1.36E-7	2.53	1.31E-7	2.50	1.83E-6	1.56
$k=2$	128	1.56E-8		7.55E-9		5.54E-8		5.45E-8		6.20E-6	
	256	1.59E-9	3.30	7.10E-10	3.41	4.63E-9	3.58	4.55E-9	3.58	1.01E-6	2.62
	512	1.50E-10	3.41	6.67E-11	3.42	3.97E-10	3.54	3.90E-10	3.54	1.71E-7	2.56
	1024	1.36E-11	3.46	6.12E-12	3.45	3.46E-11	3.52	3.40E-11	3.52	2.96E-8	2.53
$k=3$	32	6.05E-7		1.47E-7		9.91E-7		6.56E-7		2.11E-5	
	64	2.75E-8	4.46	6.55E-9	4.49	4.70E-8	4.40	2.89E-8	4.51	1.78E-6	3.57
	128	1.23E-9	4.54	2.95E-10	4.48	2.15E-9	4.45	1.28E-9	4.50	1.54E-7	3.53
	256	5.48E-11	4.52	1.32E-11	4.48	9.66E-11	4.48	5.66E-11	4.50	1.34E-8	3.52

Table 6.4: Errors and convergence orders for the LSV flux function approximation for Example 6.1.

k	N	e_f	r	$e_{f,c}$	r	$e_{f,r}$	r	$e_{f,n}$	r	$e_{f,l}$	r
$k=1$	128	7.36E-6		9.80E-5		7.77E-6		5.32E-6		3.10E-4	
	256	1.85E-6	1.99	2.45E-5	2.00	1.99E-6	1.96	1.33E-6	2.00	8.16E-5	2.00
	512	4.64E-7	2.00	6.14E-6	2.00	4.98E-7	2.00	3.33E-7	2.00	2.05E-5	2.00
	1024	1.16E-7	2.00	1.54E-6	2.00	1.25E-8	2.00	8.33E-8	2.00	5.14E-6	2.00
$k=2$	128	1.41E-9		8.49E-9		3.08E-8		3.08E-8		2.58E-6	
	256	1.14E-10	3.63	5.36E-10	3.98	1.95E-9	3.98	1.94E-9	3.98	3.17E-7	3.03
	512	8.05E-12	3.82	3.36E-11	3.99	1.21E-10	4.00	1.21E-10	4.00	3.92E-8	3.02
	1024	5.35E-13	3.91	2.09E-12	4.01	7.61E-12	4.00	7.61E-12	4.00	4.88E-9	3.01
$k=3$	32	5.24E-8		6.28E-8		2.71E-7		2.42E-7		6.36E-6	
	64	1.13E-9	5.54	1.36E-9	5.53	7.72E-9	5.13	6.94E-9	5.12	4.33E-7	3.88
	128	2.51E-11	5.49	3.03E-11	5.49	2.13E-10	5.18	1.97E-10	5.14	2.63E-8	4.04
	256	5.62E-13	5.48	6.79E-13	5.48	6.24E-12	5.10	5.95E-12	5.05	1.61E-9	4.03

Table 6.5: Errors and convergence rates for the LSV solution itself approximation for Example 6.1.

k	N	e_u	r	$e_{u,c}$	r	$e_{u,r}$	r	$e_{u,n}$	r	$e_{u,l}$	r
$k=1$	128	9.48E-6		1.17E-4		3.63E-5		4.36E-5		7.71E-4	
	256	2.38E-6	1.99	2.90E-5	2.02	9.28E-6	1.97	1.02E-5	2.09	2.43E-4	1.66
	512	5.97E-7	2.00	7.23E-6	2.01	2.36E-6	1.98	2.48E-6	2.05	7.99E-5	1.60
	1024	1.50E-7	2.00	1.80E-6	2.00	5.95E-7	1.99	6.10E-7	2.02	2.70E-5	1.56
$k=2$	128	1.70E-8		3.03E-9		5.65E-8		5.64E-8		6.32E-6	
	256	1.52E-9	3.49	2.17E-9	3.80	4.74E-9	3.57	4.71E-9	3.58	1.03E-6	2.62
	512	1.34E-10	3.50	1.65E-10	3.72	3.96E-10	3.58	3.94E-10	3.58	1.74E-7	2.57
	1024	1.19E-11	3.50	1.33E-11	3.64	3.39E-11	3.55	3.37E-11	3.55	3.00E-8	2.53
$k=3$	32	2.25E-7		3.21E-7		1.09E-6		9.36E-7		2.33E-5	
	64	9.46E-9	4.57	1.45E-8	4.47	5.01E-8	4.44	3.81E-8	4.62	2.08E-6	3.49
	128	4.06E-10	4.54	6.50E-10	4.48	2.27E-9	4.46	1.61E-9	4.56	1.85E-7	3.49
	256	1.77E-11	4.52	2.90E-11	4.49	1.02E-10	4.48	7.00E-11	4.53	1.64E-8	3.49

Table 6.6: Errors between the SV and the upwind DG for Example 6.1.

k	N	RSV						LSV					
		$\|\bar{e}\|_0$	r	$\bar{e}_{f,c}$	r	$\bar{e}_{u,c}$	r	$\|\bar{e}\|_0$	r	$\bar{e}_{f,c}$	r	$\bar{e}_{u,c}$	r
1	128	5.1E-05		5.8E-06		5.8E-06		6.3E-04		5.6E-05		5.4E-05	
	256	9.0E-06	2.5	7.4E-07	3.0	7.4E-07	3.0	1.6E-04	2.0	1.4E-05	2.0	1.4E-05	2.0
	512	1.6E-06	2.5	9.3E-08	3.0	9.3E-08	3.0	4.0E-05	2.0	3.6E-06	2.0	3.4E-06	2.0
	1024	2.8E-07	2.5	1.2E-08	3.0	1.2E-08	3.0	9.9E-06	2.0	9.0E-06	2.0	8.4E-06	2.0
2	128	4.5E-08		1.9E-08		1.9E-08		4.4E-06		2.5E-08		3.4E-08	
	256	3.6E-09	3.6	1.1E-09	4.1	1.1E-09	4.1	5.5E-07	3.0	1.5E-09	4.0	1.5E-09	4.0
	512	3.1E-10	3.5	6.8E-11	4.0	6.8E-11	4.0	6.9E-08	3.0	9.6E-11	4.0	9.4E-11	4.0
	1024	2.7E-11	3.5	4.3E-12	4.0	4.3E-12	4.0	8.6E-09	3.0	6.0E-11	4.0	5.9E-12	4.0
3	32	1.6E-06		4.1E-07		4.1E-07		7.7E-06		1.3E-07		1.2E-07	
	64	6.8E-08	4.6	9.7E-09	5.4	9.7E-09	5.4	4.8E-07	4.0	4.2E-09	4.9	4.0E-09	4.9
	128	3.0E-09	4.5	2.4E-10	5.3	2.4E-10	5.3	3.0E-08	4.0	1.4E-11	5.0	1.3E-11	5.0
	256	1.3E-10	4.5	6.4E-12	5.3	6.4E-12	5.3	1.9E-09	4.0	4.2E-12	5.0	3.9E-12	5.0

Example 6.2. We consider the following equation with the periodic boundary condition:

$$u_t + \left(\sin^2(x)u\right)_x = g(x,t), \quad (x,t) \in [0,2\pi] \times (0,\pi/2], \quad u(x,0) = \mathrm{e}^{\sin(x)},$$

where g is chosen such that the solution $u(x,t) = \mathrm{e}^{\sin(x-t)}$. Note that at the zeros $x = 0, \pi, 2\pi$ of $\alpha = \sin^2 x$, there holds

$$\alpha(x) = \partial_x \alpha(x) = 0, \quad \partial_x^2 \alpha(x) \neq 0,$$

which imply $m = m' = 2$.

Listed in Table 6.7 are L^2 and L^∞ errors for both the LSV and RSV. Similar to Example 6.1, both methods achieve optimal convergence orders. Listed in Tables 6.8-6.11 are superconvergence properties of two SV methods. Again, we observe that $e_f, e_{f,n}, e_{f,r}, e_{f,c}$ converge with the order of at least $k+2$, and $e_{f,l}$ converges with order $k+1$. This confirms the error estimates in Theorems 4.1 and 5.2. We also find that for the cell average error e_c, the convergence order

Table 6.7: Errors and convergence orders of RSV and LSV for Example 6.2.

k	N	RSV				LSV			
		$\|e\|_0$	r	$\|e\|_{0,\infty}$	r	$\|e\|_0$	r	$\|e\|_{0,\infty}$	r
1	128	3.09E-4		9.86E-4		8.59E-4		1.52E-3	
	256	9.94E-5	1.97	2.58E-4	1.94	2.15E-4	2.00	3.72E-4	2.03
	512	2.53E-5	1.98	6.60E-5	1.97	5.41E-5	1.99	9.24E-5	2.01
	1024	6.39E-6	1.98	1.67E-5	1.98	1.36E-5	1.99	2.29E-5	2.01
2	128	3.06E-6		8.50E-6		4.61E-6		1.33E-5	
	256	3.76E-7	3.03	1.08E-6	2.97	5.73E-7	3.01	1.67E-6	3.00
	512	4.62E-8	3.02	1.32E-7	3.03	7.11E-8	3.01	2.09E-7	3.00
	1024	5.70E-9	3.02	1.64E-8	3.01	8.83E-9	3.01	2.61E-8	3.00
3	32	5.18E-6		1.82E-5		6.97E-6		2.92E-5	
	64	3.03E-7	4.09	1.11E-6	4.04	4.32E-7	4.00	2.11E-6	3.79
	128	1.86E-8	4.03	7.15E-7	3.95	2.73E-8	3.98	1.32E-7	4.00
	256	1.15E-9	4.01	4.60E-8	3.96	1.74E-9	3.97	8.51E-9	3.96

Table 6.8: Errors and convergence orders for the RSV flux function approximation for Example 6.2.

k	N	e_f	r	$e_{f,c}$	r	$e_{f,r}$	r	$e_{f,n}$	r	$e_{f,l}$	r
$k=1$	128	4.97E-6		4.81E-6		8.74E-6		7.21E-6		2.18E-4	
	256	6.24E-7	3.00	6.03E-7	3.00	1.10E-6	3.00	9.06E-7	3.00	5.36E-5	2.02
	512	7.80E-8	3.00	7.55E-8	3.00	1.38E-7	3.00	1.13E-7	3.00	1.34E-5	2.00
	1024	9.75E-9	3.00	9.44E-9	3.00	1.71E-8	3.00	1.42E-8	3.00	3.35E-6	2.00
$k=2$	128	2.85E-9		2.86E-9		3.31E-8		1.29E-8		2.97E-6	
	256	8.91E-11	5.00	8.52E-11	5.07	2.07E-9	4.00	6.61E-10	4.30	3.74E-7	3.00
	512	2.78E-12	5.00	2.66E-12	5.00	1.29E-10	4.00	3.33E-11	4.31	4.68E-8	3.00
	1024	8.70E-14	5.00	8.31E-14	5.00	8.06E-11	4.00	1.69E-12	4.30	5.85E-9	3.00
$k=3$	32	4.71E-8		4.70E-8		2.77E-7		9.46E-8		9.03E-6	
	64	8.39E-10	5.81	8.40E-10	5.81	8.67E-9	5.00	2.64E-9	5.16	5.71E-7	3.98
	128	2.25E-11	5.22	2.25E-11	5.22	2.55E-10	5.08	6.60E-11	5.32	3.44E-8	4.05
	256	5.91E-13	5.25	6.01E-13	5.25	7.97E-12	5.01	1.68E-12	5.30	2.15E-9	4.00

Table 6.9: Errors and convergence orders for the RSV solution approximation Example 6.2.

k	N	e_u	r	$e_{u,c}$	r	$e_{u,r}$	r	$e_{u,n}$	r	$e_{u,l}$	r
$k=1$	128	1.58E-6		1.62E-5		5.25E-5		1.51E-4		6.17E-3	
	256	2.16E-7	2.87	2.20E-6	2.88	1.08E-5	2.28	3.19E-5	2.25	2.61E-3	1.24
	512	2.98E-8	2.86	3.03E-7	2.86	2.26E-6	2.26	6.72E-6	2.25	1.10E-3	1.24
	1024	4.13E-9	2.85	4.22E-8	2.84	4.73E-7	2.26	1.41E-6	2.25	4.62E-4	1.25
$k=2$	128	7.97E-8		7.96E-8		6.20E-7		8.37E-7		8.46E-5	
	256	6.26E-9	3.67	6.18E-9	3.67	6.73E-8	3.20	8.90E-8	3.23	1.83E-5	2.21
	512	4.75E-10	3.72	4.71E-10	3.71	7.22E-9	3.22	9.41E-9	3.24	3.91E-6	2.23
	1024	3.58E-11	3.73	3.56E-11	3.72	7.68E-10	3.23	9.92E-10	3.25	8.28E-7	2.24
$k=3$	32	2.50E-7		2.50E-7		4.81E-6		3.87E-6		1.75E-4	
	64	1.27E-8	4.31	1.27E-8	4.31	2.33E-7	4.37	1.78E-7	4.44	1.69E-5	3.36
	128	5.42E-10	4.54	5.43E-10	4.54	1.26E-9	4.21	9.83E-9	4.18	1.80E-6	3.23
	256	2.37E-11	4.52	2.37E-11	4.52	6.68E-10	4.23	5.27E-10	4.22	1.93E-7	3.22

Table 6.10: Errors and convergence orders for the LSV flux function approximation Example 6.2.

k	N	e_f	r	$e_{f,c}$	r	$e_{f,r}$	r	$e_{f,n}$	r	$e_{f,l}$	r
$k=1$	128	1.34E-4		1.24E-4		1.87E-4		1.31E-4		7.74E-4	
	256	3.30E-5	2.01	3.08E-5	2.01	4.63E-5	2.02	3.26E-5	2.01	1.93E-4	2.00
	512	8.25E-6	2.00	7.70E-6	2.01	1.15E-5	2.01	8.14E-6	2.01	4.78E-3	2.00
	1024	2.06E-6	2.00	1.92E-6	2.01	2.87E-6	2.01	2.03E-6	2.01	1.19E-3	2.00
$k=2$	128	3.36E-8		3.46E-8		8.22E-8		5.03E-8		3.21E-6	
	256	2.27E-9	4.00	2.16E-9	4.00	5.18E-9	3.99	3.15E-9	4.00	3.97E-7	3.01
	512	1.42E-10	4.00	1.35E-10	4.00	3.27E-10	3.99	1.98E-10	4.00	4.94E-8	3.01
	1024	8.86E-12	4.00	8.43E-12	4.00	2.04E-11	4.00	1.24E-11	4.00	6.17E-9	3.00
$k=3$	32	1.29E-7		1.29E-7		5.07E-7		2.02E-7		1.68E-5	
	64	5.17E-9	4.64	5.17E-9	4.64	1.84E-8	4.79	7.32E-9	4.79	7.54E-7	3.96
	128	1.95E-10	4.73	1.95E-10	4.73	6.00E-10	4.94	2.31E-10	4.98	4.66E-8	4.01
	256	5.10E-12	5.25	5.10E-12	5.25	1.88E-11	5.00	7.22E-12	5.00	2.91E-9	4.00

Table 6.11: Errors and convergence rates for the LSV solution approximation for Example 6.2.

k	N	e_u	r	$e_{u,c}$	r	$e_{u,r}$	r	$e_{u,n}$	r	$e_{u,l}$	r
$k=1$	128	3.26E-4		2.80E-4		3.25E-4		3.41E-4		7.51E-3	
	256	8.05E-5	2.01	6.92E-5	2.02	8.05E-5	2.02	8.31E-5	2.04	3.07E-3	1.29
	512	2.01E-5	2.00	1.72E-5	2.01	2.00E-5	2.01	2.04E-5	2.02	1.27E-3	1.27
	1024	5.03E-6	2.00	4.28E-6	2.01	4.98E-6	2.01	5.05E-6	2.02	5.34E-4	1.25
$k=2$	128	1.19E-7		1.21E-7		6.16E-7		6.06E-7		8.38E-5	
	256	8.17E-9	3.86	8.28E-9	3.87	6.66E-8	3.21	6.28E-8	3.27	1.80E-5	2.22
	512	5.63E-10	3.86	5.77E-10	3.84	7.16E-9	3.22	6.58E-9	3.26	3.84E-6	2.23
	1024	3.90E-11	3.85	4.09E-11	3.82	7.65E-10	3.23	6.91E-10	3.25	8.13E-7	2.24
$k=3$	32	3.01E-7		3.00E-7		3.26E-6		2.14E-6		1.68E-5	
	64	1.77E-8	4.08	1.77E-8	4.08	1.85E-7	4.14	9.17E-8	4.54	7.54E-7	3.31
	128	7.31E-10	4.59	7.31E-10	4.60	9.81E-9	4.24	5.25E-9	4.13	4.66E-8	3.22
	256	3.03E-11	4.60	3.04E-11	4.60	5.15E-10	4.25	2.80E-10	4.23	4.97E-9	3.23

Table 6.12: Errors between the SV and the upwind DG for Example 6.2.

k	N	RSV						LSV					
		$\|\bar{e}\|_0$	r	$\bar{e}_{f,c}$	r	$\bar{e}_{u,c}$	r	$\|\bar{e}\|_0$	r	$\bar{e}_{f,c}$	r	$\bar{e}_{u,c}$	r
1	128	5.2E-05		9.6E-07		9.4E-07		6.3E-04		5.1E-05		5.0E-05	
	256	9.1E-06	2.5	1.2E-07	3.0	1.2E-07	3.0	1.6E-04	2.0	1.3E-05	2.0	1.2E-05	2.0
	512	1.6E-06	2.5	1.5E-08	3.0	1.4E-08	3.0	4.0E-05	2.0	3.3E-06	2.0	3.2E-06	2.0
	1024	2.9E-07	2.5	1.9E-09	3.0	1.9E-09	3.0	9.9E-06	2.0	8.2E-07	2.0	8.1E-07	2.0
2	128	6.5E-08		4.5E-10		4.2E-10		4.4E-06		2.2E-08		2.0E-08	
	256	4.9E-09	3.7	1.5E-11	4.9	1.4E-11	4.9	5.5E-07	3.0	1.4E-09	4.0	1.3E-09	4.0
	512	4.9E-10	3.3	4.9E-13	4.9	4.4E-13	5.0	6.9E-08	3.0	8.6E-11	4.0	8.5E-11	4.0
	1024	4.3E-11	3.5	1.5E-14	5.0	1.4E-14	5.0	8.6E-09	3.0	5.4E-12	4.0	5.3E-12	4.0
3	32	1.3E-06		1.3E-08		1.2E-08		7.7E-06		7.3E-08		7.0E-08	
	64	5.3E-08	4.6	3.0E-10	5.4	2.9E-10	5.4	4.8E-07	4.0	3.2E-09	4.5	3.0E-09	4.5
	128	2.2E-09	4.6	7.1E-12	5.4	6.9E-12	5.4	3.0E-08	4.0	1.0E-10	5.0	9.8E-11	5.0
	256	9.8E-11	4.5	1.5E-13	5.5	1.3E-11	5.5	1.9E-09	4.0	3.1E-12	5.0	2.9E-12	5.0

in case $k = 2$ for RSV is 5, one order higher than the theoretical result. Again, we have not observed superconvergence phenomenon in case $k = 1$ for LSV.

Listed in Table 6.12 are the differences between our SV methods and the upwind DG methods. Again we observe that $\|\bar{e}\|_0$ of RSV and LSV converges with the rates of $k+3/2$ and $k+1$ respectively, while the convergence rates of $\bar{e}_{f,c}$ and $\bar{e}_{u,c}$ are at least $k + 2$ for both LSV and RSV schemes.

7. Concluding Remarks

In this work, we study the L^2-norm stability, convergence and superconvergence behaviors of LSV and RSV for 1-D linear hyperbolic equations with degenerate variable coefficients. We prove that both two SV schemes are stable and have optimal convergence orders in the L^2-norm. Furthermore, we establish the superconvergence properties for the SV method including: the flux function approximation αu_h are of $(k + 2)$-th order superconvergent towards the flux function $\alpha I_h u$, of $(k+2)$-th order superconvergent at the interpolation points and at downwind points, of $(k + 2)$-th order superconvergent for the cell average, and the derivative of the flux function approximation $(\alpha u_h)_x$ is of $(k + 1)$-th order superconvergent to the derivative of flux function $(\alpha I_h u)_x$, the convergence rate of the SV approximation solution itself u_h depends upon the specific property of α, and the highest superconvergence rate that can be achieved is $k+3/2$, which is half order higher than the optimal convergence rate. These superconvergent results are similar to those of the upwind DG method.

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