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CONVERGENCE AND STABILITY OF THE SPLIT-STEP THETA METHOD FOR A CLASS OF STOCHASTIC VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY NOISE*

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Abstract

In this paper, we investigate the theoretical and numerical analysis of the stochastic Volterra integro-differential equations (SVIDEs) driven by Lévy noise. The existence, uniqueness, boundedness and mean square exponential stability of the analytic solutions for SVIDEs driven by Lévy noise are considered. The split-step theta method of SVIDEs driven by Lévy noise is proposed. The boundedness of the numerical solution and strong convergence are proved. Moreover, its mean square exponential stability is obtained. Some numerical examples are given to support the theoretical results.

Mathematics subject classification: 65C30.
Key words: Stochastic Volterra integro-differential equations, Existence and uniqueness, Stability, Split-step theta method, Convergence.

1. Introduction

In this paper, we concern the following SVIDE driven by Lévy noise:

$$\begin{aligned} Y(t) = \varphi(t) &+ \int_0^t f\left(Y(z), \int_0^z \kappa(z-s)Y(s)\,ds\right)dz \\ &+ \int_0^t g\left(Y(z), \int_0^z \kappa(z-s)Y(s)\,ds\right)dw(z) \\ &+ \int_0^t \int_{\mathbf{Z}} \gamma\left(Y(z), \int_0^z \kappa(z-s)Y(s)\,ds, \xi\right)\tilde{N}(dz, d\xi) \end{aligned} \tag{1.1}$$

for $t \in [0,\infty)$, where $\varphi : [0,\infty) \to \mathbb{R}$ and $\|\varphi\|_\infty^2 = \max_{t\in[0,\infty)} |\varphi(t)|^2 < \infty$. Here $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}\times\mathbb{R}\times\mathbf{Z} \to \mathbb{R}$ are measurable functions. The kernel κ: $[0,\infty) \to \mathbb{R}$ is continuous.

As we known, SVIDE (1.1) can be regarded as an extension of stochastic differential equations (SDEs) (see [14] and the references cited therein) and special types of stochastic Volterra integral equations (SVIEs) (see, e.g. [2, 7] and the references therein). SVIDEs and SDEs are used to mathematically formulate many problems in different kinds of fields. The theoretical analysis of SVIDEs has gained abundant attention in recent decades (see [13, 16] and the references therein).

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In general, explicit solutions of SDEs and SVIDEs are rarely available and we have to resort to numerical methods to gain their approximate solutions. A large number of studies on numerical methods for SDEs have emerged (see, e.g. [5, 6, 15, 17, 24–26]), however, there are only a few numerical results about SVIDEs and SVIEs (see, e.g. [10, 12, 19, 22] and the references therein). Although numerical solutions of SVIEs have attracted more and more attention recently, but the research in this area is still limited. In particular, Liang *et al.* [11] obtained super-convergence of the Euler-Maruyama method for SVIEs. In 2020, we studied theoretical and numerical analysis of the Euler-Maruyama method for the generalized SVIDEs under global Lipschitz condition [23] and a class of SVIDEs with non-globally Lipschitz continuous coefficients [21].

It is necessary to incorporate event driven uncertainty such as market crashes, central bank announcements, changes in credit ratings, defaults, etc. which can have sudden and significant effects on the movements of stock price into a model, and this can be expressed by jumps. The evolution of economics, finance and many other random quantities are often modeled by SVIDEs driven by Lévy noise, which offer the most flexible, numerically accessible mathematical frameworks ([4] and the references therein). Some progress has been made in the recent decades [1, 3, 8, 18, 20].

To the best of our knowledge, due to some new difficulties caused by the stochastic integral (see [9]) and Lévy noise, these are the first results in the literature for such generalized SVIEs driven by Lévy noise.

This paper is organized as follows: We will consider the existence, uniqueness, boundedness and mean square exponential stability of the analytic solution of SVIDE (1.1) in Section 2. The split-step theta (SST) method of SVIDE (1.1) is presented and its boundedness, convergence and mean square exponential stability are established in Section 3. Finally, we will give some numerical examples in Section 4 to illustrate the theoretical results of SVIDE (1.1).

2. Theoretical Analysis of SVIDE Driven by Lévy Noise

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ be the expectation corresponding to $\mathbb{P}$. Let $\mathbf{Z} \subseteq \mathbf{R}_+ - \{\mathbf{0}\}$ be the range space of the impulsive jumps. A one-dimensional Brownian motion defined on the probability space is denoted by $w(t)$ and $N(dt, d\xi)$ is a Poisson random measure defined on σ-finite measure space $(\mathbf{Z}, \mathcal{L}, \nu)$ with intensity measure $\nu \not\equiv 0$ for the case when $\nu \equiv 0$. Set

$$\tilde{N}(dt, d\xi) := N(dt, d\xi) - \nu(d\xi)dt,$$

where

$$\nu(d\xi) = \lambda\phi(\xi)d\xi, \quad \phi(\xi) = \frac{1}{\sqrt{2\pi\xi}} \exp\left(-\frac{(\ln \xi)^2}{2}\right), \quad 0 < \xi < \infty.$$

Moreover, we assume that $w(t)$ is independent of $\tilde{N}(t, \cdot)$. The family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in[0,T]}$ such that $\mathbb{E}|x(t)|^p < \infty$ $(p \geq 1)$ is denoted by $\mathcal{L}^p([0, T]; \mathbb{R})$. We denoted by $\mathcal{M}^2([0, T]; \mathbb{R})$ the family of processes $\{x(t)\}_{t\in[0,T]}$ in $\mathcal{L}^2([0, T]; \mathbb{R})$ such that

$$\mathbb{E}\left(\int_0^T |x(t)|^2\, dt\right) < \infty.$$

For $a, b \in \mathbb{R}$, we use $a \vee b$ and $a \wedge b$ for $\max\{a, b\}$ and $\min\{a, b\}$, respectively. If G is a subset of Ω, its indicator function is denote by $\mathbf{1}_G$.

The assumptions are listed below.

(**A1**) For any $R \geq 1$, there is a constant $K_R > 0$ such that

$$|f(x,y) - f(\bar{x},\bar{y})| \vee |g(x,y) - g(\bar{x},\bar{y})| \leq K_R\big(|x-\bar{x}| + |y-\bar{y}|\big) \tag{2.1}$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq R$.

Remark 2.1. By the elementary inequality and (2.1), we can see that for all $x, y \in \mathbb{R}$ such that

$$|f(x,y)| \vee |g(x,y)| \leq \bar{K}_R(1 + |x| + |y|), \tag{2.2}$$

where $\bar{K}_R = K_R \vee |f(0,0)| \vee |g(0,0)|$.

(**A2**) There are positive constants α_1 and α_2 such that

$$2x^T f(x,y) + |g(x,y)|^2 + \int_{\mathbf{Z}} |\gamma(x,y,\xi)|^2 \, \nu(d\xi) \leq -\alpha_1|x|^2 + \alpha_2|y|^2 \tag{2.3}$$

for all $x, y \in \mathbb{R}$.

(**A3**) There exist positive constants ζ and η such that

$$|\kappa(t)| \leq \zeta e^{-\eta t} \tag{2.4}$$

for any $t \in [0, \infty)$.

(**A4**) There is a positive constant K_1 such that

$$\int_{\mathbf{Z}} |\gamma(x,y,\xi) - \gamma(\bar{x},\bar{y},\xi)|^2 \nu(d\xi) \leq K_1\big(|x-\bar{x}|^2 + |y-\bar{y}|^2\big) \tag{2.5}$$

for $x, y, \bar{x}, \bar{y} \in \mathbb{R}$.

Remark 2.2. Using the elementary inequality and (2.5), we can see that for all $x, y \in \mathbb{R}$ such that

$$\int_{\mathbf{Z}} |\gamma(x,y,\xi)|^2 \, \nu(d\xi) \leq \bar{K}_1\big(1 + |x|^2 + |y|^2\big), \tag{2.6}$$

where $\bar{K}_1 = K_1 \vee |\gamma(0,0,\xi)|$ and $\xi \in \mathbf{Z}$.

(**A5**) There exists a positive constant K_2 such that

$$|\kappa(t) - \kappa(\bar{t})|^2 \leq K_2|t - \bar{t}\,|^2,$$

for $t, \bar{t} \in [0, \infty)$.

(**A6**) There is a constant $K_3 > 0$ such that

$$\begin{aligned} &2(x-\bar{x})^T\big[f(x,y) - f(\bar{x},\bar{y})\big] + |g(x,y) - g(\bar{x},\bar{y})|^2 \\ &\quad + \int_{\mathbf{Z}} |\gamma(x,y,\xi) - \gamma(\bar{x},\bar{y},\xi)|^2 \, \nu(d\xi) \\ &\leq K_3\big(|x-\bar{x}|^2 + |y-\bar{y}|^2\big) \end{aligned} \tag{2.7}$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}$.

2.1. Existence and uniqueness of the analytic solutions

For every integer $R \geq 1$, define the stopping time

$$\tau_R = \inf\{t \geq 0 : |Y(t)| \geq R\}.$$

Lemma 2.1. *Assume that (A2) and (A3) hold. If $Y(t)$ is a solution of* SVIDE (1.1), *then*

$$E|Y(t)|^2 \leq K_0, \quad t \in [0,T], \tag{2.8}$$

where K_0 depends on $\kappa, T, \alpha_1, \alpha_2$ and φ. Moreover, we also have

$$\mathbb{P}\{\tau_R \leq T\} \leq \frac{K_0}{R^2}. \tag{2.9}$$

In particular, $Y(t)$ belongs to $\mathcal{M}^2([0,T];\mathbb{R})$.

Proof. Define $Y_R(t) := Y(t \wedge \tau_R)$ for $t \in [0,T]$. It is easy to see that $Y_R(t)$ satisfies

$$\begin{aligned} Y_R(t) = \varphi(t\wedge\tau_R) &+ \int_0^t f\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\mathbf{1}_{[0,\tau_R]}(z)dz \\ &+ \int_0^t g\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\mathbf{1}_{[0,\tau_R]}(z)dw(z) \\ &+ \int_0^t\int_{\mathbf{Z}} \gamma\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds, \xi\right)\mathbf{1}_{[0,\tau_R]}(z)\tilde{N}(dz,d\xi). \end{aligned}$$

Using the Itô formula, for any $t \in [0,T]$ we obtain

$$\begin{aligned} |Y_R(t)|^2 \leq &|\varphi(t\wedge\tau_R)|^2 \\ &+ \int_0^t 2Y_R^T(z) f\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\mathbf{1}_{[0,\tau_R]}(z)dz \\ &+ \int_0^t \left|g\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\right|^2\mathbf{1}_{[0,\tau_R]}(z)dz \\ &+ \int_0^t 2Y_R^T(z) g\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\mathbf{1}_{[0,\tau_R]}(z)dw(z) \\ &+ \int_0^t\int_{\mathbf{Z}} 2Y^T(s)\gamma\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds, \xi\right)\mathbf{1}_{[0,\tau_R]}(z)\tilde{N}(dz,d\xi) \\ &+ \int_0^t\int_{\mathbf{Z}} \left[\left|Y_R(z) + \gamma\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds, \xi\right)\right|^2 - |Y_R(z)|^2\right. \\ &\qquad \left. -2Y^T(z)\gamma\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds, \xi\right)\right]\mathbf{1}_{[0,\tau_R]}(z)N(dz,d\xi). \end{aligned} \tag{2.10}$$

Taking the expectation of (2.10), we get

$$\begin{aligned} \mathbb{E}|Y_R(t)|^2 \leq &|\varphi(t\wedge\tau_R)|^2 \\ &+ \mathbb{E}\int_0^t 2Y_R^T(z) f\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\mathbf{1}_{[0,\tau_R]}(z)dz \\ &+ \mathbb{E}\int_0^t \left|g\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\right|^2\mathbf{1}_{[0,\tau_R]}(z)dz \\ &+ \mathbb{E}\int_0^t\int_{\mathbf{Z}} \left|\gamma\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)\mathbf{1}_{[0,\tau_R]}(s)ds\right)\right|^2 \nu(d(\xi))\mathbf{1}_{[0,\tau_R]}(z)dz. \end{aligned}$$

By (A2), one can get that

$$\mathbb{E}|Y_R(t)|^2 \le |\varphi(t\wedge\tau_R)|^2 + \mathbb{E}\int_0^t \left(-\alpha_1|Y_R(z)|^2 + \alpha_2\left|\int_0^z \kappa(z-s)Y_R(s)ds\right|^2\right)dz.$$

Applying the Hölder inequality and (A3), we also obtain

$$\begin{aligned}\mathbb{E}\left|\int_0^z \kappa(z-s)Y_R(s)ds\right|^2 &= \mathbb{E}\int_0^z |\kappa(z-s)|dr\int_0^z |\kappa(z-s)|\,|Y_R(s)|^2 ds\\ &\le \mathbb{E}\int_0^z \zeta e^{-\eta(z-s)}dr\int_0^z \zeta e^{-\eta(z-s)}|Y_R(s)|^2 ds\\ &\le \frac{\zeta^2}{\eta}\mathbb{E}\int_0^s e^{-\eta(s-r)}|Y_R(s)|^2 dr\\ &\le \frac{\zeta^2}{\eta}\int_0^t \mathbb{E}|Y_R(s)|^2 ds.\end{aligned}$$

Consequently, we have

$$\mathbb{E}|Y_R(t)|^2 \le |\varphi(t\wedge\tau_R)|^2 + \left(-\alpha_1+\alpha_2\frac{\zeta^2}{\eta}\right)\int_0^t \mathbb{E}|Y_R(s)|^2 ds.$$

The Gronwall inequality implies that

$$\mathbb{E}|Y_R(t)|^2 \le |\varphi(t\wedge\tau_R)|^2 \exp\left[\left(-\alpha_1+\alpha_2\frac{\zeta^2}{\eta}\right)T\right] := K_0.$$

Thus

$$\mathbb{E}|Y(t\wedge\tau_R)|^2 \le K_0.$$

Hence, (2.8) follows by letting $R\to\infty$.

Finally, one gets

$$\mathbb{P}\{\tau_R\le T\} = \mathbb{E}\{\mathbf{1}_{\tau_R\le T}\} \le \frac{1}{R^2}\mathbb{E}|Y(t\wedge\tau_R)|^2 \le \frac{K_0}{R^2}.$$

The proof is complete. □

Theorem 2.1. *Assume (A1)-(A4) hold. Then there exists a unique solution $Y(t)$ to* SVIDE *(1.1).*

Proof. For each $R\ge 1$, define the truncated functions

$$f_R(x,y) = \begin{cases} f(x,y), & \text{if } |x|\vee|y|\le R,\\ f\left((R\wedge|x|)\frac{x}{|x|},(R\wedge|y|)\frac{y}{|y|}\right), & \text{if } |x|>R \text{ or } |y|>R,\end{cases}$$

$$g_R(x,y) = \begin{cases} g(x,y), & \text{if } |x|\vee|y|\le R,\\ g\left((R\wedge|x|)\frac{x}{|x|},(R\wedge|y|)\frac{y}{|y|}\right), & \text{if } |x|>R \text{ or } |y|>R,\end{cases}$$

$$\gamma_R(x,y,\xi) = \begin{cases} \gamma(x,y,\xi), & \text{if } |x|\vee|y|\le R,\\ \gamma\left((R\wedge|x|)\frac{x}{|x|},(R\wedge|y|)\frac{y}{|y|},\xi\right), & \text{if } |x|>R \text{ or } |y|>R.\end{cases}$$

Obviously, f_R, g_R and γ_R satisfy the Lipschitz condition and the linear growth condition. Hence, similarly to the proof of [14, Theorem 3.1], there exists a unique solution $Y_R(\cdot)$ in $\mathcal{M}^2([0,T];\mathbb{R})$ to

$$\begin{aligned}
Y_R(t) = \varphi(t\wedge\tau_R) &+ \int_0^t f_R\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)ds\right)dz \\
&+ \int_0^t g_R\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)ds\right)dw(z) \\
&+ \int_0^t \int_{\mathbf{Z}} \gamma_R\left(Y_R(z), \int_0^z \kappa(z-s)Y_R(s)ds, \xi\right)\tilde{N}(dz,d\xi), \quad t\in[0,T], \qquad (2.11)
\end{aligned}$$

where $\tau_R = T\wedge \inf\{t\in[0,T] : |X_R(t)| \geq R\}$ is defined as that in the proof of Lemma 2.1.

It is easy to see that

$$Y_R(t) = Y_{R+1}(t), \quad t\in[0,\tau_R]. \qquad (2.12)$$

This implies that τ_R is increasing. Applying the linear growth condition, there exists an $R_0 = R_0(\omega)$ such that $\tau_R = T$ whenever $R \geq R_0$.

Define $Y(t) = Y_{R_0}(t)$ for $t\in[0,T]$. Using (2.12), we have $Y(t\wedge\tau_R) = Y_R(t\wedge\tau_R)$. Hence, we have

$$\begin{aligned}
Y(t\wedge\tau_R) &= \varphi(t\wedge\tau_R) + \int_0^{t\wedge\tau_R} f_R\left(Y(z), \int_0^z \kappa(z-s)Y(s)ds\right)dz \\
&\quad + \int_0^{t\wedge\tau_R} g_R\left(Y(z), \int_0^z \kappa(z-s)Y(s)ds\right)dw(z) \\
&\quad + \int_0^{t\wedge\tau_R} \int_{\mathbf{Z}} \gamma_R\left(Y(z), \int_0^z \kappa(z-s)Y(s)ds, \xi\right)\tilde{N}(dz,d\xi) \\
&= \varphi(t\wedge\tau_R) + \int_0^{t\wedge\tau_R} f\left(Y(z), \int_0^z \kappa(z-s)Y(s)ds\right)dz \\
&\quad + \int_0^{t\wedge\tau_R} g\left(Y(z), \int_0^z \kappa(z-s)Y(s)ds\right)dw(z) \\
&\quad + \int_0^{t\wedge\tau_R} \int_{\mathbf{Z}} \gamma\left(Y(z), \int_0^z \kappa(z-s)Y(s)ds, \xi\right)\tilde{N}(dz,d\xi).
\end{aligned}$$

Letting $R\to\infty, Y(t)$ is a solution of SVIDE (1.1). Applying Lemma 2.1, $Y(t)$ belongs to $\mathcal{M}([0,T];\mathbb{R})$. We then obtain the uniqueness by a stopping procedure. □

2.2. Mean square exponential stability of the analytic solution

In this section, suppose $f(0,0) = g(0,0) = \gamma(0,0,\xi) = 0$, $\xi\in\mathbf{Z}$, we get the mean square exponential stability of the SVIDE (1.1) by using the similar way of [13].

Theorem 2.2. *Under (A1)-(A4), for any initial data $x_0\in\mathbb{R}$, if $\alpha_1 > \alpha_2\zeta^2/\eta^2$, there exist positive constants ι and c_0 such that the unique global solution of* SVIDE (1.1) *satisfies*

$$\mathbb{E}|Y(t)|^2 \leq c_0|x_0|^2 e^{-\iota t}, \quad \forall\, t\geq 0.$$

Proof. By the Itô formula, one has

$$
\begin{aligned}
e^{\alpha_1 t}\mathbb{E}|Y(t)|^2 = |\varphi(0)|^2 + \mathbb{E}\int_0^t e^{\alpha_1 s}\Big[2X^T\big(f(X(s),U(s)\big)ds + \big|g\big(Y(s),U(s)\big)\big|^2\Big]\, ds \\
+ \mathbb{E}\int_0^t 2e^{\alpha_1 s}Y^T(s)g\big(Y(s),U(s)\big)dw(s) \\
+ \mathbb{E}\int_0^t\int_{\mathbf{Z}} 2e^{\alpha_1 s}Y^T(s)\gamma\big(Y(s),U(s),\xi\big)\tilde{N}(ds,d\xi) \\
+ \mathbb{E}\int_0^t\int_{\mathbf{Z}} e^{\alpha_1 s}\Big[\big|Y(s)+\gamma\big(Y(s),U(s),\xi\big)\big|^2 - |Y(s)|^2 \\
- 2Y^T(s)\gamma\big(Y(s),U(s),\xi\big)\Big]N(ds,d\xi),
\end{aligned}
$$

where

$$
U(s) = \int_0^s \kappa(s-r)Y(r)dr. \tag{2.13}
$$

Using (A2), we get

$$
e^{\alpha_1 t}\mathbb{E}|Y(t)|^2 \le |\varphi(0)|^2 - \alpha_1\int_0^t e^{\alpha_1 s}\mathbb{E}|Y(s)|^2 ds + \alpha_2\mathbb{E}\int_0^t e^{\alpha_1 s}\mathbb{E}|U(s)|^2 ds. \tag{2.14}
$$

Applying the Hölder inequality, (A3) and (2.13), we can show

$$
\begin{aligned}
|U(s)|^2 &= \left|\int_0^s \kappa(s-r)Y(r)dr\right|^2 \\
&= \int_0^s |\kappa(s-r)|dr\int_0^s |\kappa(s-r)|\,|Y(r)|^2 dr \\
&\le \int_0^s \zeta e^{-\eta(s-r)}dr\int_0^s \zeta e^{-\eta(s-r)}|Y(r)|^2 dr \\
&\le \frac{\zeta^2}{\eta}\int_0^s e^{-\eta(s-r)}|Y(r)|^2 dr.
\end{aligned}
$$

Substituting this into (2.14) yields

$$
\begin{aligned}
e^{\alpha_1 t}\mathbb{E}|Y(t)|^2 \le |\varphi(0)|^2 - \alpha_1\int_0^t e^{\alpha_1 s}\mathbb{E}|Y(s)|^2 ds \\
+ \alpha_2\int_0^t e^{\alpha_1 s}\frac{\zeta^2}{\eta}\int_0^s e^{-\eta(s-r)}\mathbb{E}|Y(r)|^2 dr ds.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\mathbb{E}|Y(t)|^2 &\le |\varphi(0)|^2 e^{-\alpha_1 t} - \alpha_1\int_0^t e^{-\alpha_1(t-s)}\mathbb{E}|Y(s)|^2 ds \\
&\quad + \alpha_2\int_0^t e^{-\alpha_1(t-s)}\frac{\zeta^2}{\eta}\int_0^s e^{-\eta(s-r)}\mathbb{E}|Y(r)|^2 dr ds \\
&\le |\varphi(0)|^2 e^{-\alpha_1 t} + \alpha_2\int_0^t e^{-\alpha_1(t-s)}\frac{\zeta^2}{\eta}\int_0^s e^{-\eta(s-r)}\mathbb{E}|Y(r)|^2 dr ds.
\end{aligned}
$$

Define

$$
L_1(t) := |\varphi(0)|^2 e^{-\alpha_1 t} + \alpha_2\int_0^t e^{-\alpha_1(t-s)}\frac{\zeta^2}{\eta}\int_0^s e^{-\eta(s-r)}L_1(r)dr ds.
$$

Consequently, one obtains

$$\mathbb{E}|Y(t)|^2 - L_1(t) \leq \frac{\alpha_2}{\alpha_1}\frac{\zeta^2}{\eta^2}\int_0^t \left[\mathbb{E}|Y(r)|^2 - L_1(r)\right] dr.$$

By the Gronwall inequality, one gets

$$\mathbb{E}|Y(t)|^2 \leq L_1(t).$$

Set

$$L_2(t) := \int_0^t e^{-\eta(t-r)} L_1(r) dr,$$

we have

$$\begin{aligned} \dot{L}_1(t) &= \ \alpha_2 \frac{\zeta^2}{\eta} L_2(t) - \alpha_1 L_1(t), \\ \dot{L}_2(t) &= -\eta L_2(t) + L_1(t) \end{aligned} \tag{2.15}$$

with initial data $L_1(0) = |\varphi(0)|^2$ and $L_2(0) = 0$. It is known that (2.15) is exponentially stable if and only if $\alpha_1 > \alpha_2\zeta^2/\eta^2$. □

3. The SST Method

Define $t_n := nh$ for $n = 0, 1, \dots$. Hence, the SST method of SVIDE (1.1) can be defined as follows:

$$X_n = x_n + \theta h f\left(X_n, h\sum_{l=0}^{n-1} k(t_n - t_l)X_l\right), \tag{3.1}$$

and

$$\begin{aligned} x_{n+1} = x_n &+ hf\left(X_n, h\sum_{l=0}^{n-1} k(t_n - t_l)X_l\right) + g\left(X_n, h\sum_{l=0}^{n-1} k(t_n - t_l)X_l\right)\Delta w_n \\ &+ \int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma\left(X_n, h\sum_{l=0}^{n} k(t_n - t_l)X_l, \xi\right)\tilde{N}(dz, d\xi), \end{aligned} \tag{3.2}$$

where $\Delta w_n = w(t_{n+1})w(t_n)$.

By induction, (3.2) can be rewritten as the following form:

$$\begin{aligned} x_{n+1} = \varphi(0) &+ \sum_{r=0}^{n} hf\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} \kappa(t_r - t_l)X_l\, ds\right) \\ &+ \sum_{r=0}^{n} g\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} \kappa(t_r - t_l)X_l\, ds\right)\Delta w_r \\ &+ \sum_{r=0}^{n}\int_{t_r}^{t_{r+1}}\int_{\mathbf{Z}} \gamma\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} \kappa(t_r - t_l)X_l\, ds, \xi\right)\tilde{N}(dz, d\xi). \end{aligned} \tag{3.3}$$

3.1. Boundedness of the numerical solution

Having overcome the difficulties caused by random integrals and Lévy noise, we obtain the boundedness of the numerical solution to generalized SVIDE (1.1).

Theorem 3.1. *Assume that (A2) and (A3) hold. For* $\theta \in [1/2, 1]$, $0 < h < h^* < \eta^2/(2\theta\alpha_2\zeta^2)$ *and* $0 \le n \le S$ $(t_S = T)$*, then there exist positive constants* C_0 *and* C_0' *such that*

$$\mathbb{E}|x_n|^2 \le C_0, \quad \mathbb{E}|X_n|^2 \le C_0', \tag{3.4}$$

where C_0 *and* C_0' *depend on* $\kappa, T, \varphi, \alpha_1$ *and* α_2*, but not on* h*.*

Proof. Define

$$U_n := h\sum_{l=0}^{n} \kappa(t_n - t_l)X_l,$$

then by (A3), one has

$$\begin{aligned}
\mathbb{E}|U_n|^2 &= \mathbb{E}\left|h\sum_{l=0}^{n}\kappa(t_n - t_l)X_l\right|^2 \\
&\le \mathbb{E}\zeta^2\left|h\sum_{l=0}^{n}e^{-\eta(t_n-t_l)}\right|\left|h\sum_{l=0}^{n}e^{-\eta(t_n-t_l)}|X_l|^2\right| \\
&\le \frac{\zeta^2}{\eta^2}\sup_{l\in[0,n]}\mathbb{E}|X_l|^2.
\end{aligned} \tag{3.5}$$

Taking the expectation of (3.1) and using the elementary inequality, for all $0 \le t_{n+1} \le T$, we get

$$\begin{aligned}
&\mathbb{E}|X_n|^2 = \mathbb{E}|x_n|^2 + \theta^2h^2\mathbb{E}|f(X_n,U_n)|^2 + 2\theta h\mathbb{E}\langle x_n, f(X_n,U_n)\rangle, \\
&\langle x_n, f(X_n,U_n)\rangle = \langle X_n, f(X_n,U_n)\rangle - \theta h\langle f(X_n,U_n), f(X_n,U_n)\rangle.
\end{aligned}$$

Hence, applying (A2), we have

$$\begin{aligned}
\mathbb{E}|X_n|^2 &= \mathbb{E}|x_n|^2 - \theta^2h^2|f(X_n,U_n)|^2 + 2\theta h\langle X_n, f(X_n,U_n)\rangle \\
&\le \mathbb{E}|x_n|^2 + 2\theta h\langle X_n, f(X_n,U_n)\rangle \\
&\le \mathbb{E}|x_n|^2 + 2\theta h\big(-\alpha_1\mathbb{E}|X_n|^2 + \alpha_2\mathbb{E}|U_n|^2\big) \\
&\le \mathbb{E}|x_n|^2 + 2\theta h\alpha_2\mathbb{E}|U_n|^2.
\end{aligned} \tag{3.6}$$

By (3.2), one gets

$$\begin{aligned}
|x_{n+1}|^2 = |x_n|^2 &+ |hf(X_n,U_n)|^2 + h|g(X_n,U_n)|^2 + 2\langle x_n, hf(X_n,U_n)\rangle \\
&+ h\int_{\mathbf{Z}}|\gamma(X_n,U_n,\xi)|^2 v(d\xi) + M_n,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
M_n = 2\langle x_n + hf(X_n,U_n), g(X_n,U_n)\Delta w_l\rangle &+ \left|\sum_{r=0}^{n} g(X_n,U_n)\Delta w_l\right|^2 - h\left|\sum_{r=0}^{n} g(X_n,U_n)\right|^2 \\
+ 2\left\langle\int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}}\gamma(X_n,U_n,\xi)\tilde{N}(dz,d\xi), g(X_n,U_n)\Delta w_l\right\rangle &
\end{aligned}$$

$$+ 2\left\langle x_n + hf(X_n,U_n), \int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma(X_n,U_n,\xi)\tilde{N}(dz,d\xi)\right\rangle$$
$$+ \left|\int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma(X_n,U_n,\xi)\tilde{N}(dz,d\xi)\right|^2 - h\mathbb{E}\int_{\mathbf{Z}} |\gamma(X_n,U_n,\xi)|^2 \upsilon(d\xi).$$

Since Δw_n is independent of $\mathcal{F}_{t_n}$ and $g(X_n,U_n)$ is $\mathcal{F}_{t_n}$-measurable, we have $g(X_n,U_n)$ is independent of Δw_n. It is obvious that $\mathbb{E}(\Delta w_n) = 0$ and $\mathbb{E}(\Delta w_n)^2 = h$.

Notice that $w(t)$ is independent of $\tilde{N}(t,\cdot)$, we have

$$\mathbb{E}\left\langle \int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma(X_n,U_n,\xi)\tilde{N}(dz,d\xi), g(X_n,U_n)\Delta w_l\right\rangle = 0,$$
$$\mathbb{E}\left(\int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma(X_n,U_n,\xi)\tilde{N}(dz,d\xi)\right) = 0,$$
$$\mathbb{E}\left|\int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma(X_n,U_n,\xi)\tilde{N}(dz,d\xi)\right|^2 = h\mathbb{E}\int_{\mathbf{Z}} |\gamma(X_n,U_n,\xi)|^2\nu(d\xi).$$

Consequently, we have $\mathbb{E}(M_n) = 0$.

Using the elementary inequality, $\theta \in [1/2, 1]$ and (A2), we find that

$$\begin{aligned}
\mathbb{E}|x_{n+1}|^2 &\le \mathbb{E}|x_n|^2 + \mathbb{E}|hf(X_n,U_n)|^2 + h\mathbb{E}|g(X_n,U_n)|^2 \\
&\quad + 2\mathbb{E}\langle x_n, hf(X_n,U_n)\rangle + h\mathbb{E}\int_{\mathbf{Z}} |\gamma(X_n,U_n,\xi)|^2\upsilon(d\xi) \\
&\le \mathbb{E}|x_n|^2 + (1-2\theta)h^2\mathbb{E}|f(X_n,U_n)|^2 + h\mathbb{E}|g(X_n,U_n)|^2 \\
&\quad + 2h\mathbb{E}\langle X_n, f(X_n,U_n)\rangle + h\mathbb{E}\int_{\mathbf{Z}} |\gamma(X_n,U_n,\xi)|^2\upsilon(d\xi) \\
&\le \mathbb{E}|x_n|^2 - \alpha_1 h\mathbb{E}|X_n|^2 + \alpha_2 h\mathbb{E}|U_n|^2 \\
&\le \mathbb{E}|x_n|^2 + \alpha_2 h\mathbb{E}|U_n|^2 \\
&\le \mathbb{E}|x_{n-1}|^2 + \alpha_2 h\mathbb{E}|U_{n-1}|^2 + \alpha_2 h\mathbb{E}|U_n|^2 \\
&\le \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\le |\varphi(0)|^2 + \alpha_2 h\sum_{i=0}^{n}\mathbb{E}|U_i|^2.
\end{aligned}$$

Hence, we show

$$\begin{aligned}
\mathbb{E}|X_n|^2 &\le \mathbb{E}|x_n|^2 + 2\theta h\alpha_2\mathbb{E}|U_n|^2 \\
&\le |\varphi(0)|^2 + \alpha_2 h\sum_{i=0}^{n-1}\mathbb{E}|U_i|^2 + 2\theta h\alpha_2\mathbb{E}|U_n|^2.
\end{aligned} \tag{3.8}$$

Substituting (3.5) into (3.8), we have

$$\mathbb{E}|X_n|^2 \le |\varphi(0)|^2 + \alpha_2 h\sum_{i=0}^{n-1}\frac{\zeta^2}{\eta^2}\sup_{0\le l\le i}\mathbb{E}|X_l|^2 + 2\theta h\alpha_2\frac{\zeta^2}{\eta^2}\sup_{0\le l\le n}\mathbb{E}|X_l|^2.$$

Therefore, we get

$$\sup_{0\le l\le n}\mathbb{E}|X_l|^2 \le \frac{1}{1-2\theta h\alpha_2\zeta^2/\eta^2}|\varphi(0)|^2 + \frac{\alpha_2\zeta^2/\eta^2}{1-2\theta h\alpha_2\zeta^2/\eta^2}h\sum_{i=0}^{n-1}\sup_{0\le l\le i}\mathbb{E}|X_l|^2.$$

The discrete Gronwall inequality implies that

$$\sup_{0\le l\le S}\mathbb{E}|X_l|^2\le C_0',$$

where

$$C_0':=\frac{\eta^2}{\eta^2-2\theta h\alpha_2\zeta^2}|\varphi(0)|^2\exp\left[\frac{\alpha_2\zeta^2}{\eta^2-2\theta h\alpha_2\zeta^2}T\right].$$

Applying (3.5) and (3.6), one gets

$$\mathbb{E}|x_{n+1}|^2\le|\varphi(0)|^2+\alpha_2 h\sum_{i=0}^{n}\frac{\zeta^2}{\eta^2}\sup_{l\in[0,i]}\mathbb{E}|X_l|^2\le C_0,$$

where $C_0:=|\varphi(0)|^2+\alpha_2T\zeta^2C_0'/\eta^2$. The proof is complete. □

3.2. Convergence of the SST method

Now we will propose the approximate time continuous interpolation of discrete numerical approximation, and then get the convergence results of SST method (3.1).

Define

$$s_h:=t_n\quad\text{for}\quad s\in[t_n,t_{n+1}).$$

Let $X(t)$ be the continuous form of X_n with $X(t_n)=X_n$, i.e.

$$X_h(t)=\sum_{n=0}^{\infty}\mathbf{1}_{\{t_n<t\le t_{n+1}\}}X_n \tag{3.9}$$

and

$$\begin{aligned}\bar{x}(t)&=X_h(t)+\int_{t_n}^{t}f\left(X_h(z),\int_0^{z_h}\kappa(z_h-s_h)X_h(s)\,ds\right)dz\\&\quad+\int_{t_n}^{t}g\left(X_h(z),\int_0^{z_h}\kappa(z_h-s_h)X_h(s)\,ds\right)dw(z)\\&\quad+\int_{t_n}^{t}\int_{\mathbf{Z}}\gamma\left(X_h(z),\int_0^{z_h}\kappa(z_h-s_h)X_h(s)\,ds,\xi\right)\tilde{N}(dz,d\xi)\\&=\varphi(0)+\int_0^t f\big(X_h(z),U_h(z_h)\big)\,dz+\int_0^t g\big(X_h(s),U_h(z_h)\big)\,dw(z)\\&\quad+\int_0^t\int_{\mathbf{Z}}\gamma\big(X_h(s),U_h(z_h),\xi\big)\tilde{N}(dz,d\xi),\end{aligned}$$

where $t\in[t_n,t_{n+1})$ with $n=0,1,\dots$, and

$$U_h(z):=\int_0^z\kappa(z-s_h)X_h(s)\,ds.$$

Define the stopping time

$$\rho_R:=\inf\{t\ge 0:|\bar{x}(t)|\vee|X_h(t)|\ge R\}.$$

Lemma 3.1. *Under (A1)-(A4), for $\theta \in [1/2, 1]$ and $0 < h < h^* < \eta^2/(2\theta\alpha_2\zeta^2)$, we have*

$$\mathbb{E}\big[|\bar{x}(t) - X_h(t)|^2 \mathbf{1}_{[0,\rho_R)}(t)\big] \leq C_1(R)h, \quad t \in [0, T], \tag{3.10}$$

where

$$C_1(R) := 3\big(T\bar{K}_R + \bar{K}_R + \bar{K}_1\big)\left[1 + C_0'\left(1 + \frac{\zeta^2}{\eta^2}\right)\right].$$

Proof. For $t \in [t_n, t_{n+1})$, using (A3) and Theorem 3.1, we have

$$\begin{aligned}
\mathbb{E}\,|U_h(z_h)|^2 &= \mathbb{E}\left|\int_0^{z_h} \kappa(z_h - s_h)X_h(s)\,ds\right|^2 \\
&\leq \int_0^{z_h} \zeta e^{-\eta(z_h - s_h)}ds \int_0^{z_h} \zeta e^{-\eta(z_h - s_h)}\mathbb{E}|X_h(s)|^2 ds \\
&\leq \frac{\zeta^2}{\eta}\int_0^{z_h} e^{-\eta(z_h - s_h)}\mathbb{E}|X_h(s)|^2 ds \\
&\leq \frac{\zeta^2}{\eta^2}C_0'.
\end{aligned}$$

By Remark 2.1, (A3), Remark 2.2 and Theorem 3.1, we obtain

$$\begin{aligned}
\mathbb{E}\big[|\bar{x}(t) - X_h(t)|^2 \mathbf{1}_{[0,\rho_R)}(t)\big] &\leq 3(t - t_n)\left[\int_{t_n}^{t} \mathbb{E}\big|f\big(X_h(z), U_h(z_h)\big)\big|^2 dz \mathbf{1}_{[0,\rho_R)}(t)\right] \\
&\quad + 3\left[\int_{t_n}^{t} \mathbb{E}\big|g\big(X_h(z), U_h(z_h)\big)\big|^2 dz \mathbf{1}_{[0,\rho_R)}(t)\right] \\
&\quad + 3\left[\int_{t_n}^{t} \mathbb{E}\left|\int_Z \gamma\big(X_h(z), U_h(z_h), \xi\big)\upsilon(d\xi)\right|^2 dz \mathbf{1}_{[0,\rho_R)}(t)\right] \\
&\leq 3h\int_{t_n}^{t}\Big[\bar{K}_R\left(1 + \mathbb{E}|X_h(z)|^2 + \mathbb{E}|U_h(z_h)|^2\right)dz \mathbf{1}_{[0,\rho_R)}(t)\Big] \\
&\quad + 3\int_{t_n}^{t}\Big[\big(\bar{K}_R + \bar{K}_1\big)\big(1 + \mathbb{E}|X_h(z)|^2 + \mathbb{E}|U_h(z_h)|^2\big)dz \mathbf{1}_{[0,\rho_R)}(t)\Big] \\
&\leq C_1(R)h,
\end{aligned}$$

where

$$C_1(R) := 3\big(T\bar{K}_R + \bar{K}_R + \bar{K}_1\big)\left[1 + C_0'\left(1 + \frac{\zeta^2}{\eta^2}\right)\right].$$

The proof is complete. □

Lemma 3.2. *Under (A1)-(A3), for any $\varepsilon \in (0, 1)$ and $\theta \in [1/2, 1]$, then there exists a constant R_0 satisfies $R > R_0$, let $0 < h < h^* < \eta^2/(2\theta\alpha_2\zeta^2) \wedge h_0(R)$, where $h_0(R) > 0$ such that*

$$\mathbb{P}\{\rho_R \leq T\} \leq \varepsilon. \tag{3.11}$$

Proof. For any $t \in [0, T]$, using the Itô formula, we have

$$\begin{aligned}
|\bar{x}(t \wedge \rho_R)|^2 \leq |\varphi(0)|^2 &+ \int_0^{t\wedge\rho_R} 2\bar{x}^T(z) f\big(X_h(z), U_h(z_h)\big)dz \\
&+ \int_0^{t\wedge\rho_R} \big|g\big(X_h(z), U_h(z_h)\big)\big|^2 dz
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t\wedge\rho_R} 2\bar{x}^T(z)g\big(X_h(z),U_h(z_h)\big)dw(z)\\
&+\int_0^{t\wedge\rho_R}\int_{\mathbf{Z}} 2\bar{x}^T(s)\gamma\big(X_h(z),U_h(z_h),\xi\big)\tilde{N}(dz,d\xi)\\
&+\int_0^{t\wedge\rho_R}\int_{\mathbf{Z}} \Big[\big|\bar{x}(z)+\gamma\big(X_h(z),U_h(z_h),\xi\big)\big|^2-|\bar{x}(s)|^2\\
&\qquad\qquad -2\bar{x}^T(s)\gamma\big(X_h(z),U_h(z_h),\xi\big)\Big]N(dz,d\xi).
\end{aligned}
$$

Applying (A2), one obtains that

$$
\begin{aligned}
\mathbb{E}|\bar{x}(t\wedge\rho_R)|^2 \le{}& |\varphi(0)|^2+\mathbb{E}\int_0^{t\wedge\rho_R}\Big[2X_h^T(z)f\big(X_h(z),U_h(z_h)\big)+\big|g\big(X_h(z),U_h(z_h)\big)\big|^2\Big]dz\\
&+\mathbb{E}\int_0^{t\wedge\rho_R}\int_{\mathbf{Z}}\big|\gamma\big(X_h(z),U_h(z_h),\xi\big)\big|^2\nu(d\xi)dz\\
&+\mathbb{E}\int_0^{t\wedge\rho_R}2\big(\bar{x}(z)-X_h(z)\big)^Tf\big(X_h(z),U_h(z_h)\big)dz\\
\le{}& |\varphi(0)|^2+\int_0^{t\wedge\rho_R}\big(-\alpha_1\mathbb{E}|X_h(s)|^2+\alpha_2\mathbb{E}|U_h(z_h)|^2\big)dz\\
&+\mathbb{E}\int_0^{t}2\big(\bar{x}(z)-X_h(z)\big)^Tf\big(X_h(z),U_h(z_h)\big)\mathbf{1}_{[0,\rho_R]}(z)dz.
\end{aligned}
$$

By Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}\int_0^{t}2\big(\bar{x}(z)-X_h(z)\big)^Tf\big(X_h(z),U_h(z_h)\big)\mathbf{1}_{[0,\rho_R]}(z)dz\\
\le{}& 2\int_0^t\left(\mathbb{E}|\bar{x}(z)-X_h(z)|^2\mathbf{1}_{[0,\rho_R]}(z)\right)^{\frac{1}{2}}\left(\big|f\big(X_h(z),U_h(z_h)\big)\big|^2\mathbf{1}_{[0,\rho_R]}(z)\right)^{\frac{1}{2}}dz\\
\le{}& 2\int_0^t\left(\mathbb{E}|\bar{x}(z)-X_h(z)|^2\mathbf{1}_{[0,\rho_R]}(z)\right)^{\frac{1}{2}}\left(\big|f\big(X_h(z\wedge\rho_R),U_h(z_h\wedge\rho_R)\big)\big|^2\right)^{\frac{1}{2}}dz.
\end{aligned}
$$

Using Remark 2.1, Lemma 3.1 and Theorem 3.1, we get

$$
\begin{aligned}
\mathbb{E}|\bar{x}(t\wedge\rho_R)|^2 \le{}& |\varphi(0)|^2+\int_0^{t\wedge\rho_R}\big(-\alpha_1\mathbb{E}|X_h(s)|^2+\alpha_2\mathbb{E}|U_h(z_h)|^2\big)dz\\
&+2T\sqrt{C_1(R)\bar{K}_R\left[1+C_0'\left(1+\frac{\zeta^2}{\eta^2}\right)\right]}h^{\frac{1}{2}}\\
\le{}& |\varphi(0)|^2+\alpha_2\frac{\zeta^2}{\eta^2}TC_0'+2T\sqrt{C_1(R)\bar{K}_R\left[1+C_0'\left(1+\frac{\zeta^2}{\eta^2}\right)\right]}h^{\frac{1}{2}}.
\end{aligned}
$$

Hence, we get

$$
\mathbb{P}\{\rho_R\le T\}\le\frac{C_2}{R^2}+\frac{C_3(R)h^{\frac{1}{2}}}{R^2},
$$

where

$$
\begin{aligned}
C_2 &:= |\varphi(0)|^2+\alpha_2\frac{\zeta^2}{\eta^2}TC_0',\\
C_3(R) &:= 2T\sqrt{C_1(R)\bar{K}_R\left[1+C_0'\left(1+\frac{\zeta^2}{\eta^2}\right)\right]}.
\end{aligned}
$$

For any $\varepsilon \in (0,1)$, there exist a constant R_0, for $R > R_0$, such that

$$\frac{C_2}{R^2} < \frac{\varepsilon}{2}.$$

Then, choose $h_0(R)$, when $0 < h < h^* < \eta^2/(2\theta\alpha_2\zeta^2) \wedge h_0(R)$, we have

$$\frac{C_3(R)h^{\frac{1}{2}}}{R^2} < \frac{\varepsilon}{2}.$$

Therefore,

$$\mathbb{P}\{\rho_R \leq T\} \leq \varepsilon.$$

The proof is complete. □

Lemma 3.3. *Under (A1)-(A6), define $e(t) := Y(t) - \bar{x}(t)$ and $\vartheta_R = \tau_R \wedge \rho_R$. For any $t \in [0,T]$, $\theta \in [1/2, 1]$ and $0 < h < h^* < \eta^2/(2\theta\alpha_2\zeta^2)$, then*

$$\sup_{r \in [0, t_n \wedge \vartheta_R]} \mathbb{E}|e(r)|^2 \leq C_5(R)h,$$

where

$$\begin{aligned} C_4(R) &:= C_1(R)T + 3(K_R + K_3)K_2 h C_0' T^3 + 3(K_R + K_3)\frac{\zeta^2}{\eta^2} h C_0' T \\ &\quad + 2(K_R + K_3)TC_1(R) + 6(K_R + K_3)\frac{\zeta^2}{\eta^2} T^3 C_1(R), \\ C_5(R) &:= C_4(R) \exp\left[2(K_R + K_3)\left(1 + 3\frac{\zeta^2}{\eta^2}\right)T\right]. \end{aligned}$$

Proof. Define $\bar{e}(t) := Y(t) - X_h(t)$. By Itô's formula, we obtain

$$\begin{aligned} \mathbb{E}|e(t_n \wedge \vartheta_R)|^2 \leq \mathbb{E}\int_0^{t_n \wedge \vartheta_R} &\Big(2e^T(z)\big[f\big(Y(z), U(z)\big) - f\big(X_h(z), U_h(z)\big)\big] \\ &\quad + \big|g\big(Y(z), U(z)\big) - g\big(X_h(z), U_h(z)\big)\big|^2\Big)dz \\ + \mathbb{E}\int_0^{t_n \wedge \vartheta_R} \int_Z &\Big(\big|e(z) + \gamma\big(Y(z), U(z), \xi\big) - \gamma\big(X_h(z), U_h(z), \xi\big)\big|^2 \\ &\quad - |e(z)|^2\Big)\nu(d\xi)dz \\ - \mathbb{E}\int_0^{t_n \wedge \vartheta_R} \int_Z &\Big(2e^T(z)\big[\gamma\big(Y(z), U(z), \xi\big) - \gamma\big(X_h(z), U_h(z), \xi\big)\big]\Big)\nu(d\xi)dz \\ \leq \mathbb{E}\int_0^{t_n \wedge \vartheta_R} &2[X_h(z) - \bar{x}(z)]^T\big[f\big(Y(z), U(z)\big) - f\big(X_h(z), U_h(z)\big)\big]dz \\ + \mathbb{E}\int_0^{t_n \wedge \vartheta_R} &\Big(2\bar{e}^T(z)\big[f\big(Y(z), U(z)\big) - f\big(X_h(z), U_h(z)\big)\big] \\ &\quad + \big|g\big(Y(z), U(z)\big) - g\big(X_h(z), U_h(z)\big)\big|^2\Big)dz \\ + \mathbb{E}\int_0^{t_n \wedge \vartheta_R} \int_Z &\big|\gamma\big(Y(z), U(z), \xi\big) - \gamma\big(X_h(z), U_h(z), \xi\big)\big|^2 \nu(d\xi)dz. \end{aligned}$$

Applying (A6), one has

$$\begin{aligned}
\mathbb{E}|e(t_n\wedge\vartheta_R)|^2 \le& \int_0^{t_n\wedge\vartheta_R}\mathbb{E}|X_h(z)-\bar{x}(z)|^2dz\\
&+\int_0^{t_n\wedge\vartheta_R}\mathbb{E}\big|f\big(Y(z),U(z)\big)-f\big(X_h(z),U_h(z)\big)\big|^2dz\\
&+K_3\mathbb{E}\int_0^{t_n\wedge\vartheta_R}\big[|\bar{e}(z)|^2+|U(z)-U_h(z)|^2\big]dz\\
\le& \int_0^{t_n\wedge\vartheta_R}\mathbb{E}|X_h(z)-\bar{x}(z)|^2dz+K_R\int_0^{t_n\wedge\vartheta_R}\mathbb{E}\big[|\bar{e}(z)|^2+|U(z)-U_h(z)|^2\big]dz\\
&+K_3\mathbb{E}\int_0^{t_n\wedge\vartheta_R}\big[|\bar{e}(z)|^2+|U(z)-U_h(z)|^2\big]dz\\
\le& \int_0^{t_n\wedge\vartheta_R}\mathbb{E}|X_h(z)-\bar{x}(z)|^2dz+(K_R+K_3)\int_0^{t_n\wedge\vartheta_R}\mathbb{E}|\bar{e}(z)|^2dz+(K_R+K_3)H_1,
\end{aligned}$$

where

$$\begin{aligned}
H_1 :=& \int_0^{t_n\wedge\vartheta_R}\mathbb{E}|U(z)-U_h(z)|^2dz\\
=& \int_0^{t_n\wedge\vartheta_R}\mathbb{E}\left|\int_0^{z}\kappa(z-s)Y(s)\,ds-\int_0^{z_h}\kappa(z_h-s_h)X_h(s)\,ds\right|^2dz.
\end{aligned}$$

By (A1), the Cauchy inequality, the Hölder's inequality and the Itô isometry, one obtains

$$\begin{aligned}
H_1 \le& 3\int_0^{t_n\wedge\vartheta_R}\mathbb{E}\left|\int_0^{z}\kappa(z-s)[Y(s)-X_h(s)]\,ds\right|^2dz\\
&+3\int_0^{t_n\wedge\vartheta_R}\mathbb{E}\left|\int_0^{z}[\kappa(z-s)-\kappa(z_h-s_h)]X_h(s)\,ds\right|^2dz\\
&+3\int_0^{t_n\wedge\vartheta_R}\left|\int_{z_h}^{z}\kappa(z_h-s_h)X_h(s)\,ds\right|^2dz.
\end{aligned}$$

Using (A3) and the elementary inequality, we have

$$\begin{aligned}
&\int_0^{t_n\wedge\vartheta_R}\mathbb{E}\left|\int_0^{z}\kappa(z-s)[Y(s)-X_h(s)]\,ds\right|^2dz\\
\le& \int_0^{t_n\wedge\vartheta_R}\left|\int_0^{z}e^{-\eta(z-s)}ds\right|\left|\int_0^{z}e^{-\eta(z-s)}\mathbb{E}|Y(s)-X_h(s)|^2\,ds\right|dz\\
\le& \frac{\zeta^2}{\eta^2}\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|Y(r)-X_h(r)|^2ds.
\end{aligned}$$

Applying (A5), Theorem 3.1 and the elementary inequality, we get

$$\begin{aligned}
&\int_0^{t_n\wedge\vartheta_R}\mathbb{E}\left|\int_0^{z}[\kappa(z-s)-\kappa(z_h-s_h)]X_h(s)\,ds\right|^2dz\\
\le& \int_0^{t_n\wedge\vartheta_R}\left|\int_0^{z}|\kappa(z-s)-\kappa(z_h-s_h)|^2ds\right|\left|\int_0^{z}\mathbb{E}|X_h(s)|^2\,ds\right|dz\\
\le& K_2h^2C_0'T^3.
\end{aligned}$$

By (A3), Theorem 3.1 and the elementary inequality, we have

$$\begin{aligned}
&\int_0^{t_n\wedge\vartheta_R}\left|\int_{z_h}^{z}\kappa(z_h-s_h)X_h(s)\,ds\right|^2 dz\\
\leq\ &\int_0^{t_n\wedge\vartheta_R}\left|\int_{z_h}^{z}e^{-\eta(z_h-s_h)}ds\right|\left|\int_{z_h}^{z}e^{-\eta(z_h-s_h)}\mathbb{E}|X_h(s)|^2\,ds\right|dz\\
\leq\ &\frac{\zeta^2}{\eta^2}h^2C_0'T,
\end{aligned}$$

we get

$$H_1\leq 3\frac{\zeta^2}{\eta^2}\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|Y(r)-X_h(r)|^2ds+3K_2h^2C_0'T^3+3\frac{\zeta^2}{\eta^2}h^2C_0'T.$$

Applying Lemma 3.1, one obtains that

$$\begin{aligned}
\mathbb{E}|e(t_n\wedge\vartheta_R)|^2&\leq C_1(R)Th+3(K_R+K_3)K_2h^2C_0'T^3+3(K_R+K_3)\frac{\zeta^2}{\eta^2}h^2C_0'T\\
&\quad+3(K_R+K_3)\frac{\zeta^2}{\eta^2}T^2\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|Y(r)-X_h(r)|^2ds\\
&\quad+(K_R+K_3)\int_0^{t_n\wedge\vartheta_R}\mathbb{E}|\bar{e}(z)|^2dz\\
&\leq C_1(R)Th+3(K_R+K_3)K_2h^2C_0'T^3+3(K_R+K_3)\frac{\zeta^2}{\eta^2}h^2C_0'T\\
&\quad+(K_R+K_3)\int_0^{t_n\wedge\vartheta_R}\mathbb{E}|\bar{e}(z)|^2dz+6(K_R+K_3)\frac{\zeta^2}{\eta^2}T^2\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|e(r)|^2ds\\
&\quad+6(K_R+K_3)\frac{\zeta^2}{\eta^2}T^2\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|\bar{x}(r)-X_h(r)|^2ds\\
&\leq C_1(R)Th+3(K_R+K_3)K_2h^2C_0'T^3+3(K_R+K_3)\frac{\zeta^2}{\eta^2}h^2C_0'T\\
&\quad+2(K_R+K_3)\int_0^{t_n\wedge\vartheta_R}\mathbb{E}|\bar{x}(z)-X_h(z)|^2dz\\
&\quad+2(K_R+K_3)\int_0^{t_n\wedge\vartheta_R}\mathbb{E}|e(z)|^2dz+6(K_R+K_3)\frac{\zeta^2}{\eta^2}T^3C_1(R)\\
&\quad+6(K_R+K_3)\frac{\zeta^2}{\eta^2}T^2\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|e(r)|^2ds\\
&\leq C_1(R)Th+3(K_R+K_3)K_2h^2C_0'T^3+3(K_R+K_3)\frac{\zeta^2}{\eta^2}h^2C_0'T\\
&\quad+2(K_R+K_3)TC_1(R)h+6(K_R+K_3)\frac{\zeta^2}{\eta^2}T^3C_1(R)h\\
&\quad+2(K_R+K_3)\left(1+3\frac{\zeta^2}{\eta^2}\right)\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|e(r)|^2ds\\
&\leq C_4(R)h+2(K_R+K_3)\left(1+3\frac{\zeta^2}{\eta^2}T^2\right)\int_0^{t_n\wedge\vartheta_R}\sup_{r\in[0,s]}\mathbb{E}|e(r)|^2ds,
\end{aligned}$$

where

$$C_4(R) := C_1(R)T + 3(K_R + K_3)K_2 h C_0' T^3 + 3(K_R + K_3)\frac{\zeta^2}{\eta^2} h C_0' T$$
$$+ 2(K_R + K_3)TC_1(R) + 6(K_R + K_3)\frac{\zeta^2}{\eta^2} T^3 C_1(R).$$

Hence, by Gronwall's equality, we have

$$\sup_{r\in[0,t_n\wedge\vartheta_R]} \mathbb{E}|e(r)|^2 \le C_5(R)h,$$

where

$$C_5(R) := C_4(R)\exp\left[2(K_R + K_3)\left(1 + 3\frac{\zeta^2}{\eta^2}\right)T\right].$$

The proof is complete. □

Theorem 3.2. *Under (A1)-(A6), for $q \in [1, 2)$, and $\theta \in [1/2, 1]$, there exists a constant R_0 satisfies $R > R_0$, let $0 < h < h^* < \eta^2/(2\theta\alpha_2\zeta^2) \wedge h_0(R)$, where $h_0(R) > 0$ such that*

$$\lim_{h\to 0} \sup_{0\le t_n\le T} \mathbb{E}|Y(t_n) - X_n|^q = 0. \tag{3.12}$$

Proof. By the Young inequality, for any $\delta > 0$, one gets that

$$\begin{aligned}
\mathbb{E}\,|Y(t_n) - X_n|^q &= \mathbb{E}\left[|Y(t_n) - X_n|^q \mathbf{1}_{\{\tau_R>T,\,\rho_R>T\}}(\omega)\right] \\
&\quad + \mathbb{E}\left[|Y(t_n) - X_n|^q \mathbf{1}_{\{\tau_R\le T \text{ or } \rho_R\le T\}}(\omega)\right] \\
&\le \mathbb{E}\left[|Y(t_n) - X_n|^q \mathbf{1}_{\{\tau_R>T,\,\rho_R>T\}}(\omega)\right] \\
&\quad + \frac{\delta q}{2}\mathbb{E}|Y(t_n) - X_n|^2 + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\mathbb{P}\{\tau_R > T,\, \rho_R > T\} \\
&\le \mathbb{E}\left[|Y(t_n) - X_n|^q \mathbf{1}_{\{\tau_R>T,\,\rho_R>T\}}(\omega)\right] + \frac{\delta q}{2}\mathbb{E}|Y(t_n) - X_n|^2 \\
&\quad + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\mathbb{P}\{\tau_R > T\} + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\mathbb{P}\{\rho_R > T\}.
\end{aligned} \tag{3.13}$$

Hence, we have

$$\begin{aligned}
\mathbb{E}\left[|Y(t_n) - X_n|^2 \mathbf{1}_{\{\tau_R>T,\,\rho_R>T\}}(\omega)\right] &= \mathbb{E}\left[|Y(t_n) - X_n|^2 \mathbf{1}_{\{\vartheta_R>T\}}(\omega)\right] \\
&\le \mathbb{E}|e(t_n \wedge \vartheta_R)|^2.
\end{aligned}$$

Applying Lemma 2.1 and Theorem 3.1, one has

$$\frac{\delta q}{2}\mathbb{E}|Y(t_n) - X_n|^2 \le \delta q\left(\mathbb{E}|Y(t_n)|^2 + \mathbb{E}|X_n|^2\right) \le \delta q(K_0 + C_0').$$

Together with Lemma 3.2 and (3.13), we get

$$\begin{aligned}
\mathbb{E}|Y(t_n) - X_n|^q &\le C_5^{\frac{q}{2}}(R)h^{\frac{q}{2}} + \delta q\,(K_0 + C_0') + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\frac{K_0}{R^2} \\
&\quad + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\frac{C_2}{R^2} + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\frac{C_3(R)}{R^2}h^{\frac{1}{2}}.
\end{aligned} \tag{3.14}$$

Therefore, for any $\varepsilon \in (0,1)$, choose sufficiently small δ such that

$$\delta q\,(K_0 + C_0') < \frac{\varepsilon}{3},$$

then, choose sufficiently large R_1, for $R > R_1$, such that

$$\frac{2-q}{2\delta^{\frac{q}{2-q}}}\frac{K_0}{R^2} + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\frac{C_2}{R^2} < \frac{\varepsilon}{3},$$

finally, choose $h_1(R)$, for all $h < h_1(R)$, we show that

$$C_5^{\frac{q}{2}}(R)h^{\frac{q}{2}} + \frac{2-q}{2\delta^{\frac{q}{2-q}}}\frac{C_3(R)}{R^2}h^{\frac{1}{2}} < \frac{\varepsilon}{3}.$$

Hence, we get

$$\mathbb{E}|Y(t_n) - X_n|^q \le \varepsilon.$$

Consequently, one obtains

$$\lim_{h\to 0}\sup_{0\le t_n\le T}\mathbb{E}|Y(t_n) - X_n|^q = 0.$$

The proof is complete. □

3.3. Mean-square exponential stability of the SST method

In this section, suppose $f(0,0) = g(0,0) = \gamma(0,0,\xi) = 0, \xi \in \mathbf{Z}$. The following theorem shows that the SST method is mean-square exponentially stable.

Theorem 3.3. *Let (A1)-(A4) hold, $\alpha_1 > (\zeta^2/\eta^2)\alpha_2$ and $\theta \in (1/2,1]$. Then for sufficiently small h, the solution of the SST method is mean-square exponential stable.*

Proof. Using (A2), (3.1) and (3.7), we have

$$\begin{aligned}
|x_{n+1}|^2 &\le |x_n|^2 + \frac{1-2\theta}{\theta^2}|X_n - x_n|^2 + h|g(X_n,U_n)|^2 + 2h\langle X_n, f(X_n,U_n)\rangle \\
&\quad + h\int_{\mathbf{Z}}|\gamma(X_n,U_n,\xi)|^2\,\upsilon(d\xi) + M_n \\
&\le |x_n|^2 + \frac{1-2\theta}{\theta^2}|X_n - x_n|^2 - \alpha_1 h|X_n|^2 + \alpha_2 h|U_n|^2 + M_n \\
&\le \left(1 + \frac{1-2\theta}{\theta^2}\right)|x_n|^2 + 2\frac{2\theta-1}{\theta^2}\langle X_n, x_n\rangle \\
&\quad - \left(\alpha_1 h + \frac{2\theta-1}{\theta^2}\right)|X_n|^2 + \alpha_2 h|U_n|^2 + M_n.
\end{aligned} \tag{3.15}$$

For $\theta \in (1/2,1]$, applying the elementary inequality, we obtain

$$2\frac{2\theta-1}{\theta^2}\langle X_n, x_n\rangle \le \left(\alpha_1 h + \frac{2\theta-1}{\theta^2}\right)|X_n|^2 + \frac{(2\theta-1)^2}{\theta^2(\alpha_1 h\theta^2 + 2\theta - 1)}|x_n|^2.$$

Substituting this into (3.15), one gets

$$\begin{aligned}
|x_{n+1}|^2 &\le \left(1 + \frac{1-2\theta}{\theta^2}\right)|x_n|^2 + \frac{(2\theta-1)^2}{\theta^2\,(\alpha_1 h\theta^2 + 2\theta - 1)}|x_n|^2 + \alpha_2 h|U_n|^2 + M_n \\
&\le \left(1 - \frac{\alpha_1 h(2\theta-1)}{\alpha_1 h\theta^2 + 2\theta - 1}\right)|x_n|^2 + \alpha_2 h|U_n|^2 + M_n \\
&= (1 - \alpha_{1,h}h)|x_n|^2 + \alpha_2 h|U_n|^2 + M_n,
\end{aligned}$$

where

$$\alpha_{1,h} := \frac{\alpha_1(2\theta - 1)}{\alpha_1 h\theta^2 + 2\theta - 1}.$$

It is easy to see that $\alpha_{1,h} \to \alpha_1$ as $h \to 0$.

Therefore, for any fix $\varepsilon > 0$, we can choose h small enough to satisfy

$$\alpha_{1,h} \geq \alpha_1 - \varepsilon =: \alpha_1'.$$

In particularly, $\alpha_1 > (\zeta^2/\eta^2)\alpha_2$, we can choose ε small enough to satisfy

$$\alpha_1' = \alpha_1 - \varepsilon > \frac{\zeta^2}{\eta^2}\alpha_2.$$

Consequently, we obtain

$$|x_{n+1}|^2 \leq (1 - \alpha_1' h)|x_n|^2 + \alpha_2 h|U_n|^2 + M_n. \tag{3.16}$$

For any $A > 1$, we have

$$\begin{aligned} &A^{(i+1)h}|x_{i+1}|^2 - A^{ih}|x_i|^2 \\ &\leq A^{(i+1)h}(1 - \alpha_1' h - A^{-h})|x_i|^2 + A^{(i+1)h}\alpha_2 h|U_i|^2 + A^{(i+1)h}M_i. \end{aligned} \tag{3.17}$$

Summing the both sides of (3.17) from $i = 0$ to $i = n - 1$, we get

$$\begin{aligned} A^{nh}|x_n|^2 \leq |\varphi(0)|^2 + D(h)\sum_{i=0}^{n-1} A^{(i+1)h}|x_i|^2 \\ + \alpha_2 h \sum_{i=0}^{n-1} A^{(i+1)h}|U_i|^2 + \sum_{i=0}^{n-1} A^{(i+1)h}M_i, \end{aligned} \tag{3.18}$$

where

$$D(h) := 1 - \alpha_1' h - A^{-h}.$$

It is obvious that $D(0) = 0$ and

$$D'(h) = -\alpha_1' + A^{-h}\log A \leq 0$$

for $1 < A < e^{\alpha_1'}$. Hence, for any $h > 0$ and $1 < A < e^{\alpha_1'}$, we have $D(h) \leq 0$. Since

$$\begin{aligned} |x_i|^2 &= |X_i|^2 + \theta^2 h^2 |f(X_i, U_i)|^2 - 2\langle X_i, \theta h f(X_i, U_i)\rangle \\ &\geq |X_i|^2 - 2\langle X_i, \theta h f(X_i, U_i)\rangle \\ &\geq |X_i|^2 + \theta h\big(\alpha_1|X_i|^2 - \alpha_2|U_i|^2\big), \end{aligned}$$

we get

$$|X_i|^2 \leq \frac{|x_i|^2 + \theta h\alpha_2|U_i|^2}{1 + \theta h\alpha_1} \tag{3.19}$$

and

$$D(h)|x_i|^2 \geq D(h)\left[|X_i|^2 + \theta h\left(\alpha_1|X_i|^2 - \alpha_2|U_i|^2\right)\right].$$

Substituting these into (3.18), we have

$$
\begin{aligned}
A^{nh}|x_n|^2 &\le |\varphi(0)|^2 + D(h)\sum_{i=0}^{n-1} A^{(i+1)h}\left[|X_i|^2 + \theta h\left(\alpha_1|X_i|^2 - \alpha_2|U_i|^2\right)\right] \\
&\quad + \alpha_2 h\sum_{i=0}^{n-1} A^{(i+1)h}|U_i|^2 + \sum_{i=0}^{n-1} A^{(i+1)h}M_i \\
&\le |\varphi(0)|^2 + D(h)(1+\theta h\alpha_1)\sum_{i=0}^{n-1} A^{(i+1)h}|X_i|^2 \\
&\quad + \alpha_2 h[1-\theta D(h)]\sum_{i=0}^{n-1} A^{(i+1)h}|U_i|^2 + \sum_{i=0}^{n-1} A^{(i+1)h}M_i.
\end{aligned}
$$

By (A3), we have

$$
\begin{aligned}
|U_i|^2 &\le \zeta^2\left(h\sum_{j=0}^{i-1} e^{-\eta(t_i-t_j)}X_j\right)^2 \\
&\le \zeta^2\left(h\sum_{j=0}^{i-1} e^{-\eta(t_i-t_j)}\right)\left(h\sum_{j=0}^{i-1} e^{-\eta(t_i-t_j)}|X_j|^2\right) \\
&\le \frac{\zeta^2}{\eta}h\sum_{j=0}^{i-1} e^{-\eta(t_i-t_j)}|X_j|^2. \qquad (3.20)
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
A^{nh}|x_n|^2 &\le |\varphi(0)|^2 + D(h)(1+\theta h\alpha_1)\sum_{i=0}^{n-1} A^{(i+1)h}|X_i|^2 \\
&\quad + \alpha_2 h[1-\theta D(h)]\sum_{i=0}^{n-1} A^{(i+1)h}\frac{\zeta^2}{\eta}h\sum_{j=0}^{i-1} e^{-\eta(t_i-t_j)}|X_j|^2 + \sum_{i=0}^{n-1} A^{(i+1)h}M_i. \qquad (3.21)
\end{aligned}
$$

Due to $1 < A < e^{\alpha_1'}$, we have

$$
\begin{aligned}
\sum_{i=0}^{n-1} A^{(i+1)h}\sum_{j=0}^{i-1} e^{-\eta(t_i-t_j)}|X_j|^2 &= \sum_{j=0}^{n-2} A^{(j+1)h}|X_j|^2 \sum_{i=j+1}^{n-1} A^{(i-j)h}e^{-\eta(t_i-t_j)} \\
&\le \frac{A^h}{e^{\eta h}-A^h}\sum_{j=0}^{n-1} A^{(j+1)h}|X_j|^2.
\end{aligned}
$$

Consequently, one gets

$$
A^{nh}|x_n|^2 \le |\varphi(0)|^2 + F(h,A)\sum_{i=0}^{n-1} A^{(i+1)h}|X_i|^2 + \sum_{i=0}^{n-1} A^{(i+1)h}M_i,
$$

where

$$
F(h,A) := D(h)(1+\theta h\alpha_1) + h^2[1-\theta D(h)]\alpha_2\frac{\beta^2}{\eta}\frac{A^h}{e^{\eta h}-A^h}.
$$

Hence, we have

$$
F(h,1) = -\alpha_1' h(1+\theta h\alpha_1) + h^2(1+\theta\alpha_1' h)\alpha_2\frac{\beta^2}{\eta}\frac{1}{e^{\eta h}-1}
$$

$$\leq -\alpha_1' h(1+\theta h\alpha_1) + h(1+\theta\alpha_1' h)\alpha_2\frac{\beta^2}{\eta^2}$$
$$\leq -h\left[\alpha_1' - \frac{\beta^2}{\eta^2}\alpha_2 + \theta\alpha_1' h\left(\alpha_1 - \frac{\beta^2}{\eta^2}\alpha_2\right)\right].$$

For $\alpha_1 > (\beta^2/\eta^2)\alpha_2$, we can choose $\alpha_1' \in (0, \alpha_1)$ such that $\alpha_1' - (\beta^2/\eta^2)\alpha_2 \geq 0$. Consequently, $F(h, 1) < 0$ and $F(h, A)$ is continuous with respect to A, there is $\varepsilon > 0$ such that for any $A \in (1, (1+\varepsilon) \wedge e^{\eta\wedge\alpha_1'})$, we have

$$F(h, A) < 0.$$

Since Δw_i is independent of $\mathcal{F}_{t_i}$ and $g(X_i, U_i)$ is $\mathcal{F}_{t_i}$-measurable, we have $g(X_i, U_i)$ is independent of Δw_i. It is known that $\mathbb{E}(\Delta w_i) = 0$ and $\mathbb{E}(\Delta w_i)^2 = h$. Notice that $w(t)$ is independent of $\tilde{N}(t, \cdot)$, we have

$$\mathbb{E}\left\langle \int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}} \gamma(X_n, U_n, \xi)\tilde{N}(dz, d\xi), g(X_n, U_n)\Delta w_l\right\rangle = 0,$$
$$\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\int_{\mathbf{Z}} \gamma(X_i, U_i, \xi)\tilde{N}(dz, d\xi)\right) = 0,$$
$$\mathbb{E}\left|\int_{t_i}^{t_{i+1}}\int_{\mathbf{Z}} \gamma(X_i, U_i, \xi)\tilde{N}(dz, d\xi)\right|^2 = h\mathbb{E}\int_{\mathbf{Z}} |\gamma(X_i, U_i, \xi)|^2\nu(d\xi).$$

Hence, we have $\mathbb{E}(M_i) = 0$. Therefore,

$$\mathbb{E}\big(A^{nh}|x_n|^2\big) \leq |\varphi(0)|^2 := c_1 < \infty.$$

By (3.19) and (3.20), one has

$$|X_n|^2 \leq \frac{1}{1+\theta h\alpha_1}\left(|x_n|^2 + \theta h^2\alpha_2\frac{\zeta^2}{\eta}\sum_{j=0}^{n-1} e^{-\eta(t_n - t_j)}|X_j|^2\right). \tag{3.22}$$

Then

$$\mathbb{E}\big(A^{nh}|X_n|^2\big) \leq \frac{1}{1+\theta h\alpha_1}\mathbb{E}\big(A^{nh}|x_n|^2\big) + \frac{\theta h^2\alpha_2\zeta^2}{\eta(1+\theta h\alpha_1)}\sum_{j=0}^{n-1} e^{-\eta(t_n-t_j)}A^{(n-j)h}\mathbb{E}\big(A^{jh}|X_j|^2\big)$$
$$\leq \frac{c_1}{1+\theta h\alpha_1} + \frac{\theta h^2\alpha_2\zeta^2}{\eta(1+\theta h\alpha_1)}\sum_{j=0}^{n-1}(e^{-\eta}A)^{(n-j)h}\mathbb{E}\big(A^{jh}|X_j|^2\big). \tag{3.23}$$

Define $u_n := \mathbb{E}(A^{nh}|X_n|^2)$. We can see that

$$u_n \leq c_2 + \sum_{j=0}^{n-1} l_j u_j,$$

where

$$c_2 = \frac{c_1}{1+\theta h\alpha_1}, \quad l_j := \frac{\theta h^2\alpha_2\zeta^2}{\eta(1+\theta h\alpha_1)}(e^{-\eta}A)^{(n-j)h} \geq 0.$$

By the discrete Gronwall inequality, we have

$$\mathbb{E}\big(A^{nh}|X_n|^2\big) \le c_2 \exp\left[\sum_{j=0}^{n-1} l_j\right] \le c_2 \exp\left[\frac{\theta h^2 \alpha_2 \zeta^2}{\eta(1+\theta h \alpha_1)} \frac{A^h}{e^{\eta h} - A^h}\right] =: c_3 < \infty.$$

Thus, we get

$$\frac{\log \mathbb{E}|X_n|^2}{nh} \le \frac{c_3}{nh} - \log A \quad \to \quad -\log A < 0.$$

Namely, the SST method is mean square exponentially stable with rate $\log A$. □

4. Numerical Experiments

In this section, we support the results obtained in Theorems 3.2 and 3.3 numerically with some examples. We use discrete Brownian paths over $[0,1]$ with $\Delta t = 2^{-12}$. The SST method with step size $h = \Delta t$ is taken as an approximation of the analytic solution and we compare it with the numerical approximation using $h = 2^5\Delta t, h = 2^6\Delta t, h = 2^7\Delta t$ and $h = 2^8\Delta t$ over $M = 4000$ sample paths. Then the mean-square error is denoted as follows:

$$Error_h := \frac{1}{M}\sum_{i=1}^{M} \big|X_h^i(T) - X_{\Delta t}^i(T)\big|^q, \quad q \in [1,2), \tag{4.1}$$

where $X_h^i(T)$ denotes the numerical solution of the SST method along the i-th sample path at $t = T$ with step size h, and the strong convergence order is defined numerically by

$$Order = \frac{1}{\log 2} \log \frac{Error_h}{Error_{h/2}}.$$

Consider the following SVIDE:

$$\begin{aligned} Y(t) = \varphi(t) &+ \int_0^t \left(-aY^3(z) - bY(z) + c\int_0^z \kappa(z-s)Y(s)ds\right) dz \\ &+ \int_0^t \left(eY(z) + k\int_0^z \kappa(z-s)Y(s)ds\right) dw(z) \\ &+ \int_0^t \int_{\mathbf{Z}} \left(lY(z) + n\int_0^z \kappa(z-s)Y(s)\,ds\right) \xi \tilde{N}(dz, d\xi) \end{aligned} \tag{4.2}$$

with initial data $\varphi(t) = 1$ and $\lambda = 1/2$.

Firstly, we discuss a particular type.

Example 4.1. In SVIDE (4.2), we take $a = b = e = 1$ and $c = k = l = n = 0$. We can see SVIDE (4.2) reduce to SDE which is studied by many authors.

From Table 4.1 and Fig. 4.1, it can be observed that if $\theta \in [1/2, 1]$, the SST method is convergent. We can see the error gets smaller as θ stays the same and the step size gets smaller. The error gets smaller when the step size is constant and θ gets bigger if $\theta \in [1/2, 1]$.

See from Fig. 4.2, we can see the SST method is mean square exponentially stable when $\theta \in (1/2, 1]$.

Table 4.1: Strong convergence order of the SST method for Example 4.1.

Stepsize	$\theta = 0.5$		$\theta = 0.75$		$\theta = 1$	
	Error	Order	Error	Order	Error	Order
$2^5\Delta t$	0.0094	-	0.0090	-	0.0088	-
$2^6\Delta t$	0.0145	0.6306	0.0134	0.5725	0.0129	0.5566
$2^7\Delta t$	0.0227	0.6487	0.0213	0.6714	0.0198	0.6176
$2^8\Delta t$	0.0388	0.7690	0.0340	0.6751	0.0319	0.6897

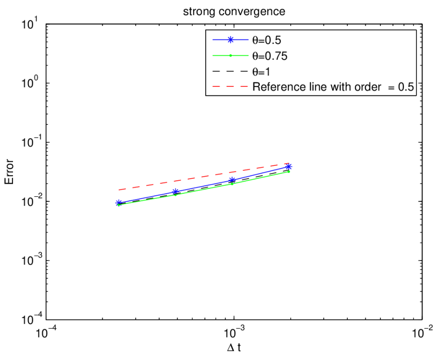

Fig. 4.1. Absolute errors of the SST method of Example 4.1 in average.

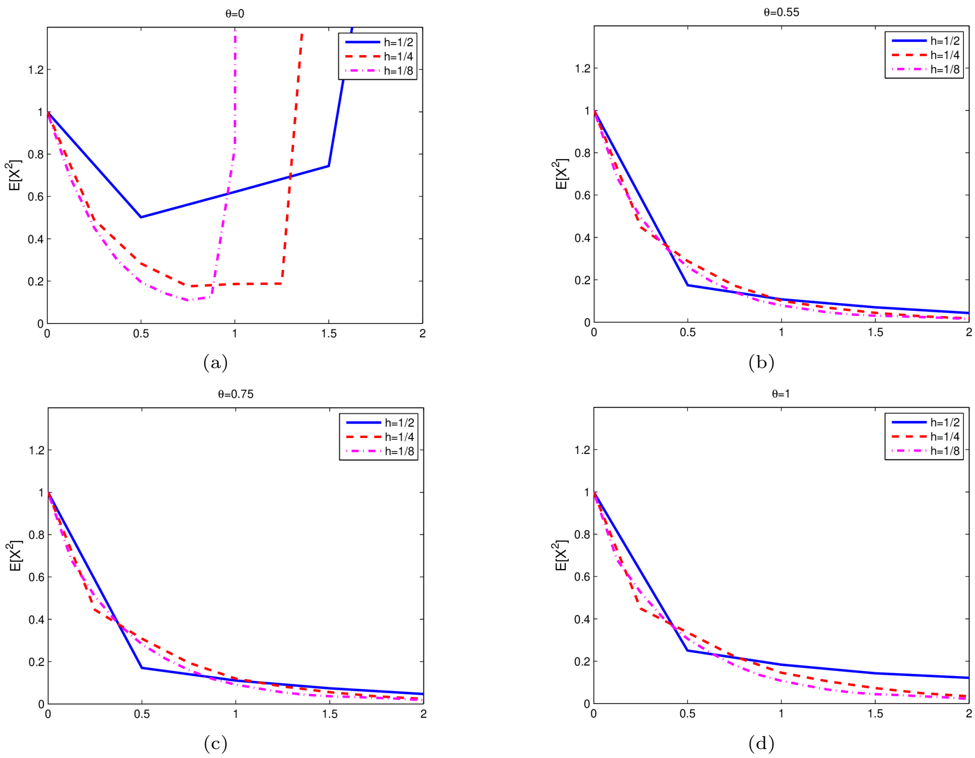

Fig. 4.2. Mean square exponential stability of the SST method of Example 4.1.

Secondly, we consider a more generalized type.

Example 4.2. In SVIDE (4.2), we take a=b=e=1, c=k=0.5 and l=n=0.25 and $\kappa(t) = e^{-2t}$.

The strong convergence results of the SST method of Example 4.2 are shown in Table 4.2. From Table 4.2 and Fig. 4.3, if $\theta \in [1/2, 1]$, we can see that as θ remains unchanged, the step size gets smaller and the error gets smaller. When the step size is constant, the error becomes smaller as θ gets bigger if $\theta \in [1/2, 1]$. Observe Fig. 4.4, we can see the SST method is mean square exponential stable when $\theta \in (1/2, 1]$.

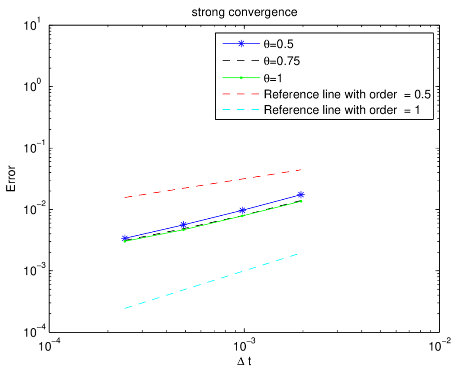

Fig. 4.3. Absolute errors of the SST method of Example 4.2 in average.

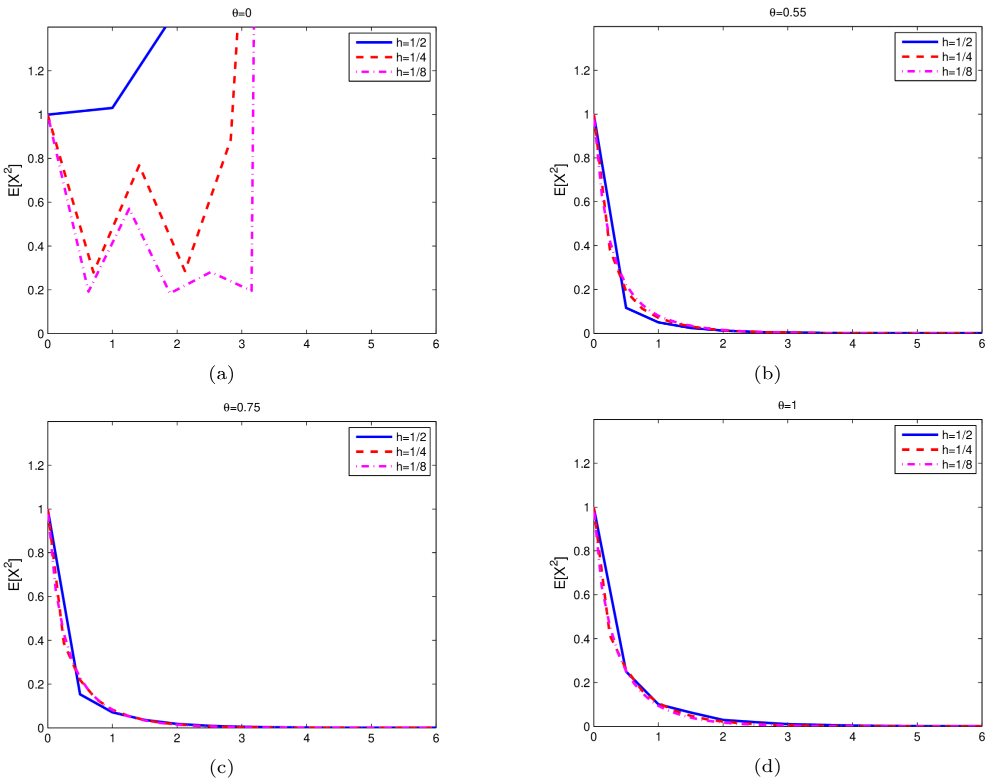

Fig. 4.4. Mean square exponential stability of the SST method of Example 4.2.

Table 4.2: Strong convergence order of the SST method for Example 4.2.

Stepsize	$\theta = 0.5$		$\theta = 0.75$		$\theta = 1$	
	Error	Order	Error	Order	Error	Order
$2^5\Delta t$	0.0034	-	0.0031	-	0.0030	-
$2^6\Delta t$	0.0056	0.7250	0.0049	0.6360	0.0047	0.6241
$2^7\Delta t$	0.0096	0.7871	0.0079	0.7050	0.0078	0.7485
$2^8\Delta t$	0.0174	0.8481	0.0140	0.8207	0.0136	0.7922

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