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# A NEW SECOND ORDER NUMERICAL SCHEME FOR SOLVING DECOUPLED MEAN-FIELD FBSDES WITH JUMPS*

Yabing Sun
*College of Science, National University of Defense Technology, Changsha, Hunan 410073, China*
*Email: sunybly@163.com*
Weidong Zhao[1)]
*School of Mathematics & Finance Institute, Shandong University, Jinan, Shandong 250100, China*
*Email: wdzhao@sdu.edu.cn*

**Abstract**

In this paper, we consider the numerical solution of decoupled mean-field forward backward stochastic differential equations with jumps (MFBSDEJs). By using finite difference approximations and the Gaussian quadrature rule, and the weak order 2.0 Itô-Taylor scheme to solve the forward mean-field SDEs with jumps, we propose a new second order scheme for MFBSDEJs. The proposed scheme allows an easy implementation. Some numerical experiments are carried out to demonstrate the stability, the effectiveness and the second order accuracy of the scheme.

*Mathematics subject classification:* 60H10, 60H35, 65C05, 65C30.
*Key words:* Mean-field forward backward stochastic differential equation with jumps, Finite difference approximation, Gaussian quadrature rule, Second order.

## 1. Introduction

To characterise the jumps in a Lévy process on a given probability space $(\Omega, \mathcal{F}, P)$, we introduce the Poisson random measure $\mu$ on $\mathsf{E} \times [0,T]$

$$\begin{aligned} \mu: \quad \Omega \times \mathcal{E} \times [0,T] &\;\to\; \mathbb{N}, \\ (\omega, A, [0,t]) &\;\to\; \mu(A \times [0,t]), \end{aligned}$$

where $\mathsf{E} = \mathbb{R}^q \backslash \{0\}$ and $\mathcal{E}$ is its Borel field. For given $t \in [0,T]$ and $A \in \mathcal{E}$, $\mu(A \times [0,t])$ is a random variable counting the number of jumps occurring in $[0,t]$ whose jump sizes belong to $A$. We usually suppress $\omega$ in $\mu$ for simplicity.

We call the measure $\nu :\ \mathsf{E} \times [0,T]$ defined by $\nu(A \times [0,t]) = \mathbb{E}[\mu(A \times [0,t])]$ the intensity measure of $\mu$. Suppose that $\nu(de, dt) = \lambda(de)dt$ with $\lambda$ being a Lévy measure on $(\mathsf{E}, \mathcal{E})$ satisfying $\int_{\mathsf{E}} (1 \wedge |e|^2)\lambda(de) < +\infty$, then the compensated Poisson random measure is defined as

$$\tilde{\mu}(de, dt) = (\mu - \nu)(de, dt) = \mu(de, dt) - \lambda(de)dt$$

such that $\{\tilde{\mu}(A \times [0,t])\}_{0 \le t \le T}$ is a martingale for any $A \in \mathcal{E}$ with $\lambda(A) < \infty$. Moreover, let $F$ and $\rho$ be the distribution and the probability density function of the jump size $e$, respectively, then it holds that

$$\lambda(de) = \lambda F(de) = \lambda \rho(e) de,$$

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1) Corresponding author

where $\lambda = \lambda(\mathsf{E})$ is the intensity of $\mu$. For more details of the Poisson random measure, the readers are referred to [6, 18].

Then we can get a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ by letting $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ be the filtration generated by the mutually independent $m$-dimensional Brownian motion $W_t$ and the Poisson random measure $\mu$, that is $\mathcal{F}_t = \mathcal{F}^0_{t+} \vee \mathcal{F}_0$, where $\mathcal{F}^0_{t+} = \bigcap_{s\ge t} \mathcal{F}^0_s$ with

$$\mathcal{F}^0_s = \sigma\big\{W_r,\ \mu(A\times[0,r])\,\big|\,A\in\mathcal{E},\ r\le s\big\},\quad s\in[0,T],$$

and the $\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$ satisfies:

- The Brownian motion $W_t$ and the measure $\mu$ are independent of $\mathcal{F}_0$.
- $\mathcal{N}_p \subset \mathcal{F}_0$ with $\mathcal{N}_p$ being the set of all $P$-null subset of $\mathcal{F}$.

Now we consider decoupled mean-field forward backward stochastic differential equations with jumps (MFBSDEJs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$

$$\begin{aligned}
X_t^{0,X_0} &= X_0 + \int_0^t \mathbb{E}\big[b\big(t, X_t^{0,x_0}, x\big)\big]\Big|_{x=X_s^{0,X_0}} ds + \int_0^t \mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big]\Big|_{x=X_s^{0,X_0}} dW_s \\
&\quad + \int_0^t \int_{\mathsf{E}} \mathbb{E}\big[c\big(s, X_{s-}^{0,x_0}, x, e\big)\big]\Big|_{x=X_{s-}^{0,X_0}} \tilde{\mu}(de, ds), \\
Y_t^{0,X_0} &= \mathbb{E}\big[\Phi\big(X_T^{0,x_0}, x\big)\big]\Big|_{x=X_T^{0,X_0}} + \int_t^T \mathbb{E}\big[f\big(s, \Theta_s^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_s^{0,X_0}} ds - \int_t^T Z_s^{0,X_0} dW_s \\
&\quad - \int_t^T \int_{\mathsf{E}} U_s^{0,X_0}(e)\tilde{\mu}(de, ds),
\end{aligned} \tag{1.1}$$

where $t\in[0,T], x_0, X_0 \in \mathcal{F}_0$ is the initial values of mean-field forward stochastic differential equations with jumps (MSDEJs) and $\mathbb{E}[\Phi(X_T^{0,x_0}, x)]|_{x=X_T^{0,X_0}} \in \mathcal{F}_T$ with $\Phi : \Omega_d = \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^p$ is the terminal condition of mean-field backward stochastic differential equations with jumps (MBSDEJs); $b : [0,T]\times\Omega_d \to \mathbb{R}^d, \sigma : [0,T]\times\Omega_d \to \mathbb{R}^{d\times m}$, and $c : [0,T]\times\Omega_d\times\mathsf{E} \to \mathbb{R}^d$ are the drift, diffusion and jump coefficients of MSDEJs, respectively; $f : [0,T]\times\Omega_f \to \mathbb{R}^p$ is the so called generator of MBSDEJs with $\Omega_f = \mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m}\times\mathbb{R}^p\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m}\times\mathbb{R}^p$; the term $\Theta_s^{0,x} = (X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x}, \Gamma_s^{0,x})$ with $x = x_0$ or $X_0$, and

$$\Gamma_s^{0,x} = \int_{\mathsf{E}} U_s^{0,x}(e)\eta(e)\lambda(de)$$

for some Borel function $\eta : \mathsf{E} \to \mathbb{R}$ satisfying $\sup_{e\in\mathsf{E}} |\eta(e)| < \infty$. We call a quadruplet $(X_t^{0,X_0}, Y_t^{0,X_0}, Z_t^{0,X_0}, U_t^{0,X_0})$ an $L^2$-adapted solution of (1.1) if it is $\mathcal{F}_t$-adapted, square integrable and satisfies (1.1). In general, initial values $x_0$ and $X_0$ are different, and

$$\big(X_t^{0,x_0}, Y_t^{0,x_0}, Z_t^{0,x_0}, U_t^{0,x_0}\big) = \big(X_t^{0,X_0}, Y_t^{0,X_0}, Z_t^{0,X_0}, U_t^{0,X_0}\big)\big|_{X_0=x_0}.$$

In this paper, we shall numerically solve the solutions $(X_t^{0,X_0}, Y_t^{0,X_0}, Z_t^{0,X_0}, \Gamma_t^{0,X_0})$ instead of $(X_t^{0,X_0}, Y_t^{0,X_0}, Z_t^{0,X_0}, U_t^{0,X_0})$. Here the MFBSDEJs (1.1) is called decoupled because the coefficients of MSDEJs do not depend on the solutions of MBSDEJs.

In 2009, Buckdahn *et al.* [4] first studied the existence and uniqueness of the solutions of mean-field forward backward stochastic differential equations (MFBSDEs) in a general Markovian setting. Then based on those researches, Li [12] further proved the existence and uniqueness of the solutions of decoupled MFBSDEJs, and gave a probability interpretation of the solutions of the nonlocal parabolic partial integral-differential equations (PIDEs). By comprising Lévy jump processes, mean-field forward and backward SDEs with jumps can model the event-driven stochastic phenomena much more accurately, and the theory of MFBSDEJs have found many applications in diverse research areas such as the mean-field problems with delay [1], nonlocal diffusion problems [8, 23], mean-field games [16, 27, 29] and stochastic optimal control [9, 13–15, 17, 28], to name a few. Therefore it is important and interesting to solve MFBSDEJs numerically.

Notice that the solutions of MFBSDEJs depend on the distributions of the forward MSDEJs which makes its structure very complicated and brings a challenge to construct accurate numerical schemes. Moreover, when designing high order schemes, the random jump times of the measure $\mu$ can be coupled with the Brownian motion which is difficult to deal with in practice. Because of these reasons, up to now, there are few works on the numerical methods for MFBSDEJs.

Some numerical methods for mean-field forward and backward SDEs have been studied in recent years, see e.g. [3,5,11,19,21,22,24]. A class of explicit $\theta$-schemes for mean-field backward stochastic differential equations were constructed in [24] and its error estimates were theoretically proved by using the mean-field Itô formula and Itô-Taylor expansion developed in [19]. After that, by solving mean-filed stochastic differential equations with Itô-Taylor schemes, the authors proposed an explicit second order one-step scheme [22] and an explicit high order multi-step scheme [21] for decoupled MFBSDEs. By using the full-history recursive multilevel Picard approximations, the authors constructed some efficient schemes for solving high dimensional MSDEs [2, 10].

For solving MFBSDEs with jumps, the authors in [23] developed the associated Itô formula and Itô-Taylor expansions, and proposed Itô-Taylor type schemes for MSDEJs. Then by combining with Itô-Taylor schemes, the authors in [20] derived an explicit second order scheme for decoupled MFBSDEJs and rigorously analyzed its stability and second order accuracy. However the conditional expectations for solving the solutions $Z_t^{0,X_0}$ and $\Gamma_t^{0,X_0}$ in the scheme proposed in [20] are complicated to calculate, which makes the scheme inefficient in application.

In this paper, we shall design a new second order numerical scheme for solving decoupled MFBSDEJs. By the nonlinear Feynman-Kac formula in (2.1) given in Section 2, the solutions of $Z_t^{0,X_0}$ and $\Gamma_t^{0,X_0}$ can be represented by the derivative and the integral of $Y_t^{0,X_0}$, respectively. By using finite difference approximation and the Gaussian quadrature rule to approximate $Z_t^{0,X_0}$ and $\Gamma_t^{0,X_0}$, respectively, we propose a new second order scheme for solving the decoupled MFBSDEJs (1.1). By adopting the finite difference approximation and the Gaussian quadrature rule, the proposed scheme avoids to solve the complicated conditional expectations used to solve $Z_t^{0,X_0}$ and $\Gamma_t^{0,X_0}$ in [20] and thus is much simplified in structure, which makes it easier to implement in practice. Several numerical examples are performed to verify the accuracy, the convergence and the stability of the proposed scheme. The numerical results show that the scheme can be convergent with second order when the weak order 2.0 Itô-Taylor scheme is used to solve the associated MSDEJs in (1.1). It is also shown that the accuracy of the scheme is not sensitive to the step size used in the finite difference approximation. This observation allows us to use the small step size in finite difference approximation to guarantee the required accuracy of the scheme. Moreover, the numerical results indicate that the new second order scheme is more efficient and accurate than the one in [20], especially for multi-dimensional cases.

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries on the nonlinear Feynman-Kac formula, numerical derivatives and integrals, and general Itô-Taylor schemes for solving MSDEJs. In Section 3, we discuss the design of a new second order scheme for solving decoupled MFBSDEJs, and present the details of the approximations of expectations and conditional expectations in the scheme in Section 4. In Section 5, numerical experiments are carried out to verify the effectiveness of the scheme and we finally give some conclusions in Section 6.

We close this section by listing some notation that will be used in what follows:

- $C_b^k(\Omega_d)$: The set of continuous differential functions $\phi(x,y)$ with uniformly bounded partial derivatives up to the $k$-th order.
- $C_b^{l,k}([0,T]\times\Omega_d)$: The set of continuous differential functions $\phi(t,x,y)$ with uniformly bounded partial derivatives up to the $l$-th order with respect to the time variable and up to the $k$-th order with respect to the spatial variables. Moreover, we can define $C_b^{l,k}([0,T]\times\Omega_f)$ in a similar way.

## 2. Preliminaries

### 2.1. The nonlinear Feynman-Kac formula

We recall the nonlinear Feynman-Kac formula in this subsection. To proceed, we make the following assumptions on the coefficients of MFBSDEJs.

**Assumption 2.1.** *Assume that $b,\sigma\in C_b^{1,2}([0,T]\times\Omega_d)$, $\Phi\in C_b^2(\Omega_d)$ and $f\in C_b^{1,2}([0,T]\times\Omega_f)$. Moreover, $c(\cdot,\cdot,\cdot,e)\in C_b^{1,2}([0,T]\times\Omega_d)$ with the bound of $K(1\wedge|e|)$ for all its derivatives of first and second order, where $K$ is a positive constant.*

Now we give the nonlinear Feynman-Kac formula [12] in the following lemma.

**Lemma 2.1.** *Under Assumption* 2.1*, the MFBSDEJs* (1.1) *has a unique solution, and the solution $(Y_t^{0,X_0},Z_t^{0,X_0},\Gamma_t^{0,X_0})$ can be represented as*

$$\begin{aligned}
Y_t^{0,X_0} &= u\big(t,X_t^{0,X_0}\big),\\
Z_t^{0,X_0} &= \nabla_x u\big(t,X_t^{0,X_0}\big)\mathbb{E}\big[\sigma\big(t,X_t^{0,x_0},x\big)\big]\Big|_{x=X_t^{0,X_0}},\\
\Gamma_t^{0,X_0} &= \int_{\mathsf{E}}\left(u\left(t,X_{t-}^{0,X_0}+\mathbb{E}\big[c\big(t,X_{t-}^{0,x_0},x,e\big)\big]\Big|_{x=X_{t-}^{0,X_0}}\right)-u\big(t-,X_{t-}^{0,X_0}\big)\right)\eta(e)\lambda(de),
\end{aligned}\tag{2.1}$$

*where $u(t,x)$ is the unique classical solution of the nonlocal PIDE*

$$\begin{aligned}
\mathcal{A}[u](t,x)+\mathbb{E}\Big[f\Big(t,X_t^{0,x_0},u\big(t,X_t^{0,x_0}\big),\nabla_x u\big(t,X_t^{0,x_0}\big)\mathbb{E}\big[\sigma\big(t,X_t^{0,x_0},x\big)\big]\Big|_{x=X_t^{0,x_0}},\\
\mathcal{B}[u]\big(t-,X_{t-}^{0,x_0}\big),x,u(t,x),\nabla_x u(t,x)\mathbb{E}\big[\sigma\big(t,X_t^{0,x_0},x\big)\big],\mathcal{B}[u](t,x)\Big)\Big]=0
\end{aligned}\tag{2.2}$$

*with the terminal condition $u(T,x)=\mathbb{E}[\Phi(X_T^{0,x_0},x)]$. Here the differential-integral operator $\mathcal{A}$ and integral operator $\mathcal{B}$ are defined by*

$$\mathcal{A}[u](t,x)=\frac{\partial u}{\partial t}(t,x)+\sum_{i=1}^{d}\frac{\partial u}{\partial x_i}(t,x)\mathbb{E}\big[b_i\big(t,X_t^{0,x_0},x\big)\big]$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 u}{\partial x_i\partial x_j}(t,x)\Big(\mathbb{E}\big[\sigma\big(t,X_t^{0,x_0},x\big)\big]\mathbb{E}\big[\sigma^\top\big(t,X_t^{0,x_0},x\big)\big]\Big)_{ij}\\
&+\int_{\mathsf{E}}\Bigg(u\big(t,x+\mathbb{E}\big[c\big(t,X_{t-}^{0,x_0},x,e\big)\big]\big)-u(t,x)\\
&\qquad-\sum_{i=1}^{d}\frac{\partial u}{\partial x_i}(t,x)\mathbb{E}\big[c_i\big(t,X_{t-}^{0,x_0},x,e\big)\big]\Bigg)\lambda(de),
\end{aligned}
\tag{2.3}
$$

$$
\mathcal{B}[u](t,x)=\int_{\mathsf{E}}\Big(u\big(t,x+\mathbb{E}\big[c\big(t,X_{t-}^{0,x_0},x,e\big)\big]\big)-u(t,x)\Big)\eta(e)\lambda(de). \tag{2.4}
$$

**Remark 2.1.** It is known that when the functions $b,\sigma,c,f$ and $\Phi$ satisfy the conditions in Assumption 2.1, the PIDE (2.2) has a unique smooth solution $u(t,x)$, which is also bounded and smooth with bounded derivatives [12].

**Remark 2.2.** In this paper, we shall construct our numerical scheme for the MFBSDEJs (1.1) based on the nonlinear Feynman-Kac formula (2.1). The formulas in (2.1) indicate that once $Y_t^{0,X_0}$ is known, we can approximate $Z_t^{0,X_0}$ and $\Gamma_t^{0,X_0}$ by using some numerical methods for approximating derivatives and integrals, respectively.

### 2.2. Numerical derivatives and integrals

In this subsection, we recall the finite difference approximation and the Gaussian quadrature rule for approximating derivatives and integrals, respectively. For simplicity, we consider the one-dimensional case and all the results below can be generated to the multi-dimensional cases in a natural way.

(i) **Numerical derivatives.** For a given function $g:\mathbb{R}\to\mathbb{R}$, we define the following difference quotient operators:

$$
\begin{aligned}
D_h^{-1}g(x)&=\frac{1}{h}\big(g(x)-g(x-h)\big),\\
D_h^{0}g(x)&=\frac{1}{2h}\big(g(x+h)-g(x-h)\big),\\
D_h^{1}g(x)&=\frac{1}{12h}\big(-g(x+2h)+8g(x+h)-8g(x-h)+g(x-2h)\big),
\end{aligned}
\tag{2.5}
$$

where $h>0$ is a small positive real number. It is easy to deduce

$$
\begin{aligned}
D_h^{-1}g(x)-g'(x)&=\mathcal{O}(h),\quad g\in C_b^2,\\
D_h^{0}g(x)-g'(x)&=\mathcal{O}(h^2),\quad g\in C_b^3,\\
D_h^{1}g(x)-g'(x)&=\mathcal{O}(h^4),\quad g\in C_b^5,
\end{aligned}
\tag{2.6}
$$

which show that $D_h^{-1}g(x), D_h^0g(x)$ and $D_h^1g(x)$ approximate the first order derivative $g'(x)$ with errors $\mathcal{O}(h),\mathcal{O}(h^2)$ and $\mathcal{O}(h^4)$, respectively.

(ii) **Numerical integrals.** Let $L$ be a positive integer and $\{x_k\}_{k=1}^{L}$ be the Gaussian quadrature points. Suppose that $g$ has a continuous derivative of order $2L$ on $[a,b]$ and $\omega$ is a positive weight function defined and integrable on $(a,b)$, then there exists a number $\eta\in(a,b)$ such that (see [25, Eq. (10.6)])

$$\int_a^b \omega(x)g(x)dx = \sum_{k=1}^{L} \omega_k g(x_k) + K_L g^{(2L)}(\eta), \tag{2.7}$$

where the quadrature weights $\{\omega_k\}_{k=1}^L$ and the constant $K_L$ are defined as

$$\omega_k = \int_a^b \omega(x) \left( \prod_{i=1, i\neq k}^{L} \frac{x - x_i}{x_k - x_i} \right)^2 dx,$$

$$K_L = \frac{1}{(2L)!} \int_a^b \omega(x) \left[(x-x_1)\cdots(x-x_L)\right]^2 dx.$$

When $g^{(2L)}$ is bounded, it holds that (see [25, Eq. (10.18)])

$$\left|K_L g^{(2L)}(\eta)\right| \le \frac{C(b-a)^{2L}}{(2L)!}, \tag{2.8}$$

where the constant $C$ is given as

$$C = \max_{x\in(a,b)} \left|g^{(2L)}(x)\right| \cdot \int_a^b \omega(x)dx.$$

The sum $\sum_{k=1}^{L} \omega_k g(x_k)$ is called the Gaussian quadrature for the integral $\int_a^b \omega(x)g(x)dx$.

### 2.3. Itô-Taylor schemes for MSDEJs

In this subsection, we introduce the general Itô-Taylor scheme for solving the forward MSDEJs [23].

Assume that $\lambda(\mathsf{E}) < \infty$. Then $N_t = \mu(\mathsf{E}\times[0,t])$ is a Poisson process with the parameter $\lambda(\mathsf{E})$ counting the number of jumps occurring in $[0,t]$, and the random measure $\mu$ can generate a sequence of pairs $\{(\tau_i, e_i)\}_{i=1}^{N_T}$ with $\{\tau_i\}_{i=1}^{N_T}$ representing the jump times and $\{e_i\}_{i=1}^{N_T}$ the corresponding jump sizes. To proceed, we introduce the following partition on the time interval $[0,T]$:

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

and define

$$\Delta t_n = t_{n+1} - t_n, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n}, \quad \Delta N_n = N_{t_{n+1}} - N_{t_n}.$$

Let $X_n^{X_0}$ ($X_n^{x_0}$) denote the numerical approximations of the solutions $X_t^{0,X_0}$ ($X_t^{0,x_0}$) at time $t = t_n, n = 0,1,\ldots,N$, then the general Itô-Taylor scheme for MSDEJs developed in [23] can be written as

$$X_{n+1}^{X_0} = X_n^{X_0} + \mathbb{E}\big[\varphi\big(t_n, \Delta t_n, X_n^{x_0}, V\big)\big]\Big|_{V=(\boldsymbol{x},\boldsymbol{w},\boldsymbol{m},\boldsymbol{\tau},\boldsymbol{e})}, \tag{2.9}$$

where $\varphi$ is a method dependent function, $\boldsymbol{x}=X_n^{X_0}, \boldsymbol{w}=\Delta W_n, \boldsymbol{m}=\Delta N_n$, and $\boldsymbol{\tau} = (\tau_1,\ldots,\tau_{\Delta N_n})$ and $\boldsymbol{e} = (e_1,\ldots,e_{\Delta N_n})$ with $(\tau_i, e_i)$ the $i$-th pair of jump time and jump size occurring in $(t_n, t_{n+1}]$. For instance, for the Euler scheme [23]

$$\begin{aligned} X_{n+1}^{X_0} = X_n^{X_0} &+ \left(\mathbb{E}\big[b\big(t_n, X_n^{x_0}, x\big)\big]\Big|_{x=X_n^{X_0}} - \int_{\mathsf{E}} \mathbb{E}\big[c\big(t_n, X_n^{x_0}, x, e\big)\big]\Big|_{x=X_n^{X_0}} \lambda(de)\right)\Delta t_n \\ &+ \mathbb{E}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big]\Big|_{x=X_n^{X_0}} \Delta W_n + \sum_{k=1}^{\Delta N_n} \mathbb{E}\big[c\big(t_n, X_n^{x_0}, x, e_k\big)\big]\Big|_{x=X_n^{X_0}}, \end{aligned}$$

the function $\varphi$ is

$$\varphi\big(t_n, \Delta t_n, x', x, w, m, \tau, e\big) = \left(b\big(t_n, x', x\big) - \int_{\mathsf{E}} c\big(t_n, x', x, e\big)\lambda(de)\right)\Delta t_n + \sigma\big(t_n, x', x\big)w + \sum_{k=1}^{m} c\big(t_n, x', x, e_k\big).$$

To solve the MSDEJ in (1.1) by the above general Itô-Taylor scheme (2.9), one needs the following two steps:

Step 1. Solve the MSDEJ with $X_0 = x_0$ to obtain $\{X_n^{x_0}\}_{n=0}^N$.

Step 2. Using $\{X_n^{x_0}\}$, solve the MSDEJ with $X_0 \neq x_0$ to obtain $\{X_n^{X_0}\}_{n=0}^N$.

## 3. The New Numerical Schemes for MFBSDEJs

In this section, based on the nonlinear Feynman-Kac formula (2.1), we develop a new numerical scheme for solving the decoupled MFBSDEJs (1.1) by combining with the finite difference approximation and the Gaussian quadrature rule. For notational simplicity, we consider the one-dimensional case.

### 3.1. Reference equations

Let $\Theta_s^{t,x} = (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, \Gamma_s^{t,x})$ be the adapted solution of the MFBSDEJs (1.1) with the MSDEJ starting from the point $(t, x)$. Then we have for $n = 0, 1, \ldots, N-1$,

$$\begin{aligned} Y_{t_n}^{t_n,x} = {} & Y_{t_{n+1}}^{t_n,x} + \int_{t_n}^{t_{n+1}} \mathbb{E}\big[f\big(s, \Theta_s^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_s^{t_n,x}} ds - \int_{t_n}^{t_{n+1}} Z_s^{t_n,x} dW_s \\ & - \int_{t_n}^{t_{n+1}} \int_{\mathsf{E}} U_s^{t_n,x}(e)\tilde{\mu}(de, ds). \end{aligned} \tag{3.1}$$

By Lemma 2.1, it holds that

$$\begin{aligned} Y_{t_n}^{t_n,x} &= u(t_n, x), \\ Z_{t_n}^{t_n,x} &= \nabla_x u(t_n, x)\mathbb{E}\big[\sigma\big(t_n, X_{t_n}^{0,x_0}, x\big)\big], \\ \Gamma_{t_n}^{t_n,x} &= \int_{\mathsf{E}} \Big(u\big(t_n, x + \mathbb{E}\big[c\big(t_n, X_{t_n-}^{0,x_0}, x, e\big)\big]\big) - u(t_n, x)\Big)\eta(e)\lambda(de), \end{aligned} \tag{3.2}$$

where $u$ is the smooth solution of (2.2). Then we get the approximation

$$\begin{aligned} Z_{t_n}^{t_n,x} &= \nabla_x u(t_n, x)\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big] \\ &= D_h^i u(t_n, x)\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big] + R_z^{n,X_0} \\ &= D_h^i Y_{t_n}^{t_n,x}\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big] + R_z^{n,X_0}, \end{aligned} \tag{3.3}$$

where $D_h^i, i = -1, 0, 1$, are finite difference operators defined by (2.5) and

$$R_z^{n,X_0} = \big(\nabla_x u(t_n, x) - D_h^i u(t_n, x)\big)\,\mathbb{E}\big[\sigma\big(t_n, X_{t_n}^{0,x_0}, x\big)\big]. \tag{3.4}$$

Let $\rho$ be the probability density function of $e$. Then for a given function $g : \mathsf{E} \to \mathbb{R}$, we let $G_\rho^L[g(\cdot)]$ denote the approximation of $\int_{\mathsf{E}} g(e)\rho(e)de$ using $L$-points Gaussian quadrature rule (2.7) chosen by $\rho(e)$. Take $\eta(e) = 1$ and define

$$c_{t_n}^x(e) = \mathbb{E}\big[c\big(t_n, X_{t_n-}^{0,x_0}, x, e\big)\big].$$

Then we have

$$\begin{aligned}
\Gamma_{t_n}^{t_n,x} &= \lambda \int_{\mathsf{E}} \left(u\left(t_n, x + c_{t_n}^x(e)\right) - u\left(t_n, x\right)\right)\rho(e)de \\
&= \lambda \int_{\mathsf{E}} u\left(t_n, x + c_{t_n}^x(e)\right)\rho(e)de - \lambda u\left(t_n, x\right) \\
&= \lambda G_\rho^L\left[u\left(t_n, x + c_{t_n}^x(\cdot)\right)\right] - \lambda u\left(t_n, x\right) + R_\gamma^{n,X_0} \\
&= \lambda \left(G_\rho^L\left[Y_{t_n}^{t_n, x + c_{t_n}^x(\cdot)}\right] - Y_{t_n}^{t_n,x}\right) + R_\gamma^{n,X_0}, \qquad (3.5)
\end{aligned}$$

where

$$R_\gamma^{n,X_0} = \lambda \left(\int_{\mathsf{E}} u\left(t_n, x + c_{t_n}^x(e)\right)\rho(e)de - G_\rho^L\left[u\left(t_n, x + c_{t_n}^x(\cdot)\right)\right]\right). \qquad (3.6)$$

Now we turn to the approximation of $Y_{t_n}^{t_n,x}$. By taking the conditional expectation

$$\mathbb{E}_{t_n}^x[\cdot] := \mathbb{E}\big[\,\cdot \mid \mathcal{F}_{t_n}, X_{t_n}^{0,X_0} = x\big]$$

on both sides of (3.1), we obtain

$$\begin{aligned}
Y_{t_n}^{t_n,x} &= \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(s, \Theta_s^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_s^{t_n,x}}\right]ds \\
&= \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n \mathbb{E}\big[f\big(t_n, \Theta_{t_n}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{t_n}^{t_n,x}} \\
&\quad + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{t_{n+1}}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right] + R_{y_1}^{n,X_0}, \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
R_{y_1}^{n,X_0} = &\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\big(s, \Theta_s^{0,x_0}, \theta\big)\right]\Big|_{\theta=\Theta_s^{t_n,x}}\right]ds \\
&- \frac{1}{2}\Delta t_n \mathbb{E}\big[f\big(t_n, \Theta_{t_n}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{t_n}^{t_n,x}} \\
&- \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{t_{n+1}}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right].
\end{aligned}$$

Similarly, we have

$$Y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{t_{n+1}}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right] + R_{yr}^{n,X_0}, \qquad (3.8)$$

where

$$\begin{aligned}
R_{yr}^{n,X_0} = &\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(s, \Theta_s^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_s^{t_n,x}}\right]ds \\
&- \Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{t_{n+1}}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right].
\end{aligned}$$

By removing $R_{yr}^{0,X_0}$ in (3.8), we define the prediction value

$$\bar{Y}_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^{x}\left[Y_{t_{n+1}}^{t_n,x}\right] + \Delta t_n \mathbb{E}_{t_n}^{x}\left[\mathbb{E}\big[f\big(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\big)\big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right], \tag{3.9}$$

and let

$$\bar{Z}_{t_n}^{t_n,x} = D_h^i \bar{Y}_{t_n}^{t_n,x}\mathbb{E}\big[\sigma\big(t_n,X_{t_n}^{0,x_0},x\big)\big], \quad i=-1,0,1, \tag{3.10}$$

$$\bar{\Gamma}_{t_n}^{t_n,x} = \lambda\left(G_\rho^L\left[\bar{Y}_{t_n}^{t_n,x+c_{t_n}^x(\cdot)}\right] - \bar{Y}_{t_n}^{t_n,x}\right). \tag{3.11}$$

Note that

$$\left(Y_{t_n}^{0,x_0}, Z_{t_n}^{0,x_0}, \Gamma_{t_n}^{0,x_0}\right) = \left(Y_{t_n}^{t_n,X_{t_n}^{0,x_0}}, Z_{t_n}^{t_n,X_{t_n}^{0,x_0}}, \Gamma_{t_n}^{t_n,X_{t_n}^{0,x_0}}\right).$$

Thus, we define

$$\left(\bar{Y}_{t_n}^{0,x_0}, \bar{Z}_{t_n}^{0,x_0}, \bar{\Gamma}_{t_n}^{0,x_0}\right) = \left(\bar{Y}_{t_n}^{t_n,X_{t_n}^{0,x_0}}, \bar{Z}_{t_n}^{t_n,X_{t_n}^{0,x_0}}, \bar{\Gamma}_{t_n}^{t_n,X_{t_n}^{0,x_0}}\right).$$

Let

$$\bar{\Theta}_{t_n}^{0,x_0} = \left(X_{t_n}^{0,x_0}, \bar{Y}_{t_n}^{0,x_0}, \bar{Z}_{t_n}^{0,x_0}, \bar{\Gamma}_{t_n}^{0,x_0}\right),$$
$$\bar{\Theta}_{t_n}^{t_n,x} = \left(X_{t_n}^{t_n,x}, \bar{Y}_{t_n}^{t_n,x}, \bar{Z}_{t_n}^{t_n,x}, \bar{\Gamma}_{t_n}^{t_n,x}\right),$$

then, by (3.7), we get

$$\begin{aligned} Y_{t_n}^{t_n,x} = {} & \mathbb{E}_{t_n}^{x}\left[Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n \mathbb{E}\big[f\big(t_n,\bar{\Theta}_{t_n}^{0,x_0},\theta\big)\big]\Big|_{\theta=\bar{\Theta}_{t_n}^{t_n,x}} \\ & + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{x}\left[\mathbb{E}\big[f\big(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\big)\big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right] + R_y^{n,X_0}, \end{aligned} \tag{3.12}$$

where $R_y^{n,X_0} = R_{y_1}^{n,X_0} + R_{y_2}^{n,X_0}$ with

$$R_{y_2}^{n,X_0} = \frac{1}{2}\Delta t_n\left(\mathbb{E}\left[f\big(t_n,\Theta_{t_n}^{0,x_0},\theta\big)\right]\Big|_{\theta=\Theta_{t_n}^{t_n,x}} - \mathbb{E}\left[f\big(t_n,\bar{\Theta}_{t_n}^{0,x_0},\theta\big)\right]\Big|_{\theta=\bar{\Theta}_{t_n}^{t_n,x}}\right).$$

### 3.2. The time semidiscrete scheme

For $n=0,\ldots,N$, we let

$$\Theta_n^{x_0} = \big(X_n^{x_0}, Y_n^{x_0}, Z_n^{x_0}, \Gamma_n^{x_0}\big),$$
$$\Theta_n^{X_0} = \big(X_n^{X_0}, Y_n^{X_0}, Z_n^{X_0}, \Gamma_n^{X_0}\big)$$

denote the numerical approximations of the solutions of (1.1) at time $t_n$ with the initial values of $x_0$ and $X_0$, respectively. Moreover, we define the function $c_n^{X_0}:\mathsf{E}\to\mathbb{R}$ as

$$c_n^{X_0}(e) = \mathbb{E}\big[c\big(t_n,X_n^{x_0},x,e\big)\big]\Big|_{x=X_n^{X_0}}.$$

Then by letting $(t_n,x)=(t_n,X_n^{X_0})$ in (3.3), (3.5), (3.9)-(3.12), and removing the truncation errors $R_z^{0,X_0}, R_\gamma^{0,X_0}$ and $R_y^{0,X_0}$ from (3.3), (3.5) and (3.12), we get our time semidiscrete scheme for solving the decoupled MFBSDEJs (1.1).

**Scheme 3.1.** *Step 1. Given initial value $x_0$, solve $X_n^{x_0}, n = 1, \dots, N$, by the Itô-Taylor scheme* (2.9).

*Step 2. Given initial value $X_0$ and terminal conditions $Y_N^{X_0}, Z_N^{X_0}, \Gamma_N^{X_0}$, for $n = N-1, \dots, 0$, solve the random variables $Y_n^{X_0}, Z_n^{X_0}$ and $\Gamma_n^{X_0}$ by*

$$
\begin{aligned}
\bar{Y}_n^{X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\right],\\
\bar{Z}_n^{X_0} &= D_h^i \bar{Y}_n^{X_0} \mathbb{E}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big]\Big|_{x=X_n^{X_0}},\\
\bar{\Gamma}_n^{X_0} &= \lambda\left(G_\rho^L\left[\bar{Y}_n^{X_n^{X_0}+c_n^{X_0}(\cdot)}\right] - \bar{Y}_n^{X_0}\right),\\
Y_n^{X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\right] + \frac{1}{2}\Delta t_n \mathbb{E}\big[f\big(t_n, \bar{\Theta}_n^{x_0}, \theta\big)\big]\Big|_{\theta=\bar{\Theta}_n^{X_0}}\\
&\quad + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\right],\\
Z_n^{X_0} &= D_h^i Y_n^{X_0} \mathbb{E}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big]\Big|_{x=X_n^{X_0}},\\
\Gamma_n^{X_0} &= \lambda\left(G_\rho^L\left[Y_n^{X_n^{X_0}+c_n^{X_0}(\cdot)}\right] - Y_n^{X_0}\right),
\end{aligned}
$$

*where $D_h^i, i = 0, \pm 1$, are finite difference operators defined by* (2.5)*, $X_n^{X_0}$ is solved by the Itô-Taylor scheme* (2.9) *and*

$$\bar{\Theta}_n^x = \big(X_n^x, \bar{Y}_n^x, \bar{Z}_n^x, \bar{\Gamma}_n^x\big), \quad x = x_0 \ \text{or}\ X_0.$$

As a comparison, we present the scheme given in [20] in the following Scheme 3.2. To this end, for $n = 0, \dots, N-1$, we define $\Delta\tilde{W}_n$ and $\Delta\tilde{\mu}_n^*$ by

$$\Delta\tilde{W}_n = \int_{t_n}^{t_{n+1}} p(r)dW_r, \quad \Delta\tilde{\mu}_n^* = \int_{t_n}^{t_{n+1}}\int_{\mathsf{E}} p(r)\tilde{\mu}(de, dr),$$

where $p(r) = 2 - 3(r - t_n)/\Delta t_n$.

**Scheme 3.2.** *Step 1. Given initial value $x_0$, solve $X_n^{x_0}$ for $n = 1, \dots, N$ by the Itô-Taylor scheme* (2.9).

*Step 2. Given initial value $X_0$ and terminal conditions $Y_N^{X_0}, Z_N^{X_0}, \Gamma_N^{0,X_0}$, for $n = N-1, \dots, 0$, solve random variables $Y_n^{X_0}, Z_n^{X_0}$ and $\Gamma_n^{X_0}$ by*

$$
\begin{aligned}
\frac{1}{2}\Delta t_n Z_n^{X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\Delta\tilde{W}_n\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\Delta\tilde{W}_n\right],\\
\frac{1}{2}\Delta t_n \Gamma_n^{X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\Delta\tilde{\mu}_n^*\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\Delta\tilde{\mu}_n^*\right],\\
\bar{Y}_n^{X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\right],\\
Y_n^{X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\right] + \frac{1}{2}\Delta t_n \mathbb{E}\big[f\big(t_n, \hat{\Theta}_n^{x_0}, \theta\big)\big]\Big|_{\theta=\hat{\Theta}_n^{X_0}}\\
&\quad + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\right],
\end{aligned}
$$

*where $X_n^{X_0}$ is solved by the Itô-Taylor scheme* (2.9) *and*

$$\hat{\Theta}_n^x = \big(X_n^x, \bar{Y}_n^x, Z_n^x, \Gamma_n^x\big), \quad x = x_0 \ \text{or}\ X_0.$$

**Remark 3.1.** Compared with Scheme 3.2 for solving $Z_n^{X_0}$ and $\Gamma_n^{X_0}$, Scheme 3.1 avoids calculating some complicated conditional expectations coupled with the random variables defined by the increments of the Brownian motion and the Poisson random measure, which makes it simpler in structure and easier to implement in practice. On the other hand, Scheme 3.1 still suffers from the curse of dimensionality and thus can be only used to solve multi-dimensional problems such as two or three dimensions as shown in the Example 5.3.

### 3.3. The fully discrete scheme

To propose a fully discrete scheme based on Scheme 3.1, the partition of the space $\mathbb{R}^d$ is needed. To this end, we introduce a general uniform space partition $\mathcal{S}_{\Delta x}^n$ at time level $t = t_n$ such that $\mathcal{S}_{\Delta x}^n = \mathcal{S}_{\Delta x}$ with

$$\mathcal{S}_{\Delta x} = \mathcal{S}_{1,\Delta x} \times \mathcal{S}_{2,\Delta x} \times \cdots \times \mathcal{S}_{d,\Delta x},$$

where $\mathcal{S}_{j,\Delta x}$ is the partition of the one-dimensional real axis $\mathbb{R}$

$$\mathcal{S}_{j,\Delta x} = \left\{x_i^j : x_i^j = i\Delta x, i = 0, \pm 1, \ldots, \pm\infty\right\}$$

for $j = 1, 2, \ldots, d$, and $\mathcal{S}_{\Delta x,x} \subset \mathcal{S}_{\Delta x}$ denotes the set of some neighbor grids near $x$.

Then by Scheme 3.1, for each $x \in \mathcal{S}_{\Delta x}^n$, we solve $Y_n^{X_0} = Y_n^{X_0}(x), Z_n^{X_0} = Z_n^{X_0}(x)$ and $\Gamma_n^{X_0} = Z_n^{X_0}(x)$ by

$$\begin{aligned}
\bar{Y}_n^{X_0} &= \mathbb{E}_{t_n}^x\left[Y_{n+1}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\big[f\big(t_{n+1}, \Theta_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\Theta_{n+1}^{X_0}}\right], \\
\bar{Z}_n^{X_0} &= D_h^i \bar{Y}_n^{X_0} \mathbb{E}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big], \\
\bar{\Gamma}_n^{X_0} &= \lambda\left(G_\rho^L\left[\bar{Y}_n^{x+c_n^x(\cdot)}\right] - \bar{Y}_n^{X_0}\right), \\
Y_n^{X_0} &= \mathbb{E}_{t_n}^x\left[Y_{n+1}^{X_0}\right] + \frac{1}{2}\Delta t_n \mathbb{E}\big[f\big(t_n, \bar{\Theta}_n^{x_0}, \theta\big)\big]\Big|_{\theta=\bar{\Theta}_n^{X_0}} \\
&\quad + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f(t_{n+1}, \Theta_{n+1}^{x_0}, \theta)\right]\Big|_{\theta=\Theta_{n+1}^{X_0}}\right], \\
Z_n^{X_0} &= D_h^i Y_n^{X_0} \mathbb{E}\big[\sigma(t_n, X_n^{x_0}, x)\big], \\
\Gamma_n^{X_0} &= \lambda\left(G_\rho^L\left[Y_n^{x+c_n^x(\cdot)}\right] - Y_n^{X_0}\right),
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
c_n^x(e) &= \mathbb{E}\big[c\big(t_n, X_n^{x_0}, x, e\big)\big], \\
\bar{\Theta}_n^{X_0} &= \big(x, \bar{Y}_n^{X_0}, \bar{Z}_n^{X_0}, \bar{\Gamma}_n^{X_0}\big), \\
X_{n+1}^{X_0} &= x + \mathbb{E}\big[\varphi\big(t_n, \Delta t_n, X_n^{x_0}, V\big)\big]\big|_{V=(x,\boldsymbol{w},\boldsymbol{m},\boldsymbol{\tau},\boldsymbol{e})}
\end{aligned}$$

with $\boldsymbol{w}, \boldsymbol{m}, \boldsymbol{\tau}, \boldsymbol{e}$ being defined in (2.9).

Generally, $X_{n+1}^{X_0}$ does not belong to $\mathcal{S}_{\Delta x}^{n+1}$ for $x \in \mathcal{S}_{\Delta x}^n$. Thus we need to approximate the values of $Y_{n+1}^{X_0}$, $Z_{n+1}^{X_0}$ and $\Gamma_{n+1}^{X_0}$ at $X_{n+1}^{X_0}$ using the values of them on $\mathcal{S}_{\Delta x}^{n+1}$. Let $I_X^n$ be a local interpolation operator such that $I_X^n g$ is the interpolation value of the function $g$ at space points $X \in \mathbb{R}^d$ by using the values of $g$ only on $\mathcal{S}_{\Delta x,X}^n$. Then we can define

$$\hat{\Theta}_{n+1}^{X_0} = \left(X_{n+1}^{X_0}, I_{X_{n+1}^{X_0}}^{n+1} Y_{n+1}^{X_0}, I_{X_{n+1}^{X_0}}^{n+1} Z_{n+1}^{X_0}, I_{X_{n+1}^{X_0}}^{n+1} \Gamma_{n+1}^{X_0}\right). \tag{3.14}$$

Similarly, $X_n^{x_0}$ does not belong to $\mathcal{S}_{\Delta x}^n$ in general, thus we define

$$\begin{aligned}\hat{\bar{\Theta}}_n^{x_0} &= \left(X_n^{x_0}, I^n_{X_n^{x_0}}\bar{Y}_n^{X_0}, I^n_{X_n^{x_0}}\bar{Z}_n^{X_0}, I^n_{X_n^{x_0}}\bar{\Gamma}_n^{X_0}\right),\\ \hat{\Theta}_n^{x_0} &= \left(X_n^{x_0}, I^n_{X_n^{x_0}}Y_n^{X_0}, I^n_{X_n^{x_0}}Z_n^{X_0}, I^n_{X_n^{x_0}}\Gamma_n^{X_0}\right).\end{aligned} \tag{3.15}$$

Moreover, we define the operators $\hat{G}_\rho^L$ and $\hat{D}_h^1$ by

$$\hat{G}_\rho^L g(x) = \sum_{k=1}^{L}\omega_k I^n_{x_k} g, \quad \hat{D}_h^{-1} g(x) = \frac{I_x^n g - I^n_{x-h} g}{h},$$

where $\{x_k,\, k=1,\ldots,L\}$ are the Gaussian quadrature points. The operators $\hat{D}_h^0$ and $\hat{D}_h^1$ can be defined similarly.

To apply Scheme 3.1 in practice, one also needs to approximate the expectation $\mathbb{E}[\cdot]$ and the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$. Let $\hat{\mathbb{E}}[\cdot]$ and $\hat{\mathbb{E}}_{t_n}^x[\cdot]$ denote the approximation operators of $\mathbb{E}[\cdot]$ and $\mathbb{E}_{t_n}^x[\cdot]$, respectively, which will be made clear in the next section. Then by using the operators $I_X^n$, $\hat{\mathbb{E}}[\cdot]$ and $\hat{\mathbb{E}}_{t_n}^x[\cdot]$, we can rewrite the reference equations (3.3), (3.5), (3.9)-(3.12) in the form of

$$\begin{aligned}
\bar{Y}_{t_n}^{t_n,x} &= \hat{\mathbb{E}}_{t_n}^x\left[I^{n+1}_{X_{t_{n+1}}^{t_n,x}} Y_{t_{n+1}}^{t_n,x}\right] + \Delta t_n \hat{\mathbb{E}}_{t_n}^x\left[\hat{\mathbb{E}}\big[f\big(t_{n+1}, \hat{\Theta}_{t_{n+1}}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\hat{\Theta}_{t_{n+1}}^{t_n,x}}\right] + R_{yr}^{n,I,\mathbb{E}},\\
\bar{Z}_{t_n}^{t_n,x} &= \hat{D}_h^i \bar{Y}_{t_n}^{t_n,x}\hat{\mathbb{E}}\left[\sigma\big(t_n, X_{t_n}^{0,x_0}, x\big)\right] + \bar{R}_z^{n,I,\mathbb{E}},\\
\bar{\Gamma}_{t_n}^{t_n,x} &= \lambda\left(\hat{G}_\rho^L\left[\bar{Y}_{t_n}^{t_n,x+c_{t_n}^x(\cdot)}\right] - \bar{Y}_{t_n}^{t_n,x}\right) + \bar{R}_\gamma^{n,I,\mathbb{E}},\\
Y_{t_n}^{t_n,x} &= \hat{\mathbb{E}}_{t_n}^x\left[I^{n+1}_{X_{t_{n+1}}^{t_n,x}} Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n \hat{\mathbb{E}}\big[f\big(t_n, \hat{\bar{\Theta}}_{t_n}^{0,x_0}, \theta\big)\big]\Big|_{\theta=\bar{\Theta}_{t_n}^{t_n,x}}\\
&\quad + \frac{1}{2}\Delta t_n \hat{\mathbb{E}}_{t_n}^x\left[\hat{\mathbb{E}}\big[f(t_{n+1}, \hat{\Theta}_{t_{n+1}}^{0,x_0}, \theta)\big]\Big|_{\theta=\hat{\Theta}_{t_{n+1}}^{t_n,x}}\right] + R_y^{n,X_0} + R_y^{n,I,\mathbb{E}},\\
Z_{t_n}^{t_n,x} &= \hat{D}_h^i Y_{t_n}^{t_n,x}\hat{\mathbb{E}}\big[\sigma\big(t_n, X_{t_n}^{0,x_0}, x\big)\big] + R_z^{n,X_0} + R_z^{n,I,\mathbb{E}},\\
\Gamma_{t_n}^{t_n,x} &= \lambda\left(\hat{G}_\rho^L\left[Y_{t_n}^{t_n,x+c_{t_n}^x(\cdot)}\right] - Y_{t_n}^{t_n,x}\right) + R_\gamma^{n,X_0} + R_\gamma^{n,I,\mathbb{E}},
\end{aligned} \tag{3.16}$$

where the terms $R_{yr}^{n,I,\mathbb{E}}, R_y^{n,I,\mathbb{E}}, \bar{R}_z^{n,I,\mathbb{E}}, \bar{R}_\gamma^{n,I,\mathbb{E}}, R_z^{n,I,\mathbb{E}}$ and $R_\gamma^{n,I,\mathbb{E}}$ are the local truncation errors come from the interpolations and the approximations of expectations. Here the notations $\hat{\bar{\Theta}}_{t_n}^{0,x_0}$, $\hat{\Theta}_{t_n}^{0,x_0}$ and $\hat{\Theta}_{t_{n+1}}^{t_n,x}$ are defined as

$$\begin{aligned}
\hat{\bar{\Theta}}_{t_n}^{0,x_0} &= \left(X_{t_n}^{0,x_0}, I^n_{X_{t_n}^{0,x_0}}\bar{Y}_{t_n}^{0,x_0}, I^n_{X_{t_n}^{0,x_0}}\bar{Z}_{t_n}^{0,x_0}, I^n_{X_{t_n}^{0,x_0}}\bar{\Gamma}_{t_n}^{0,x_0}\right),\\
\hat{\Theta}_{t_n}^{0,x_0} &= \left(X_{t_n}^{0,x_0}, I^n_{X_{t_n}^{0,x_0}}Y_{t_n}^{0,x_0}, I^n_{X_{t_n}^{0,x_0}}Z_{t_n}^{0,x_0}, I^n_{X_{t_n}^{0,x_0}}\Gamma_{t_n}^{0,x_0}\right),\\
\hat{\Theta}_{t_{n+1}}^{t_n,x} &= \left(X_{t_{n+1}}^{t_n,x}, I^n_{X_{t_{n+1}}^{t_n,x}}Y_{t_{n+1}}^{t_n,x}, I^n_{X_{t_{n+1}}^{t_n,x}}Z_{t_{n+1}}^{t_n,x}, I^n_{X_{t_{n+1}}^{t_n,x}}\Gamma_{t_{n+1}}^{t_n,x}\right).
\end{aligned}$$

By removing the nine error terms $R_y^{n,X_0}$, $R_z^{n,X_0}$, $R_\gamma^{n,X_0}$, $R_{yr}^{n,I,\mathbb{E}}$, $R_y^{n,I,\mathbb{E}}$, $R_z^{n,I,\mathbb{E}}$, $R_\gamma^{n,I,\mathbb{E}}$, $\bar{R}_z^{n,I,\mathbb{E}}$ and $\bar{R}_\gamma^{n,I,\mathbb{E}}$ from (3.16), we obtain our fully discrete scheme for solving the decoupled MFBSDEJs (1.1) as below.

**Scheme 3.3.** *Step 1. Given initial value $x_0$, solve $X_n^{x_0}, n=1,\ldots,N$, by the Itô-Taylor scheme* (2.9).

*Step 2. Given initial value $X_0$ and terminal conditions $Y_N^{X_0}, Z_N^{X_0}$ and $\Gamma_N^{X_0}$, for $n = N-1, \ldots, 0$ and for $x \in \mathcal{S}_{\Delta x}^n$, solve the random variables $Y_n^{X_0} = Y_n^{X_0}(x), Z_n^{X_0} = Z_n^{X_0}(x)$ and $\Gamma_n^{X_0} = \Gamma_n^{X_0}(x)$ by*

$$\bar{Y}_n^{X_0} = \hat{\mathbb{E}}_{t_n}^x \left[ I_{X_{n+1}^{X_0}}^{n+1} Y_{n+1}^{X_0} \right] + \Delta t_n \hat{\mathbb{E}}_{t_n}^x \left[ \hat{\mathbb{E}}\big[ f\big(t_{n+1}, \hat{\Theta}_{n+1}^{x_0}, \theta\big)\big]\Big|_{\theta=\hat{\Theta}_{n+1}^{X_0}} \right], \tag{3.17}$$

$$\bar{Z}_n^{X_0} = \hat{D}_h^i \bar{Y}_n^{X_0} \hat{\mathbb{E}}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big], \tag{3.18}$$

$$\bar{\Gamma}_n^{X_0} = \lambda \left( \hat{G}_\rho^L \left[ \bar{Y}_n^{x+c_n^x(\cdot)} \right] - \bar{Y}_n^{X_0} \right), \tag{3.19}$$

$$\begin{aligned} Y_n^{X_0} = {} & \hat{\mathbb{E}}_{t_n}^x \left[ I_{X_{n+1}^{X_0}}^{n+1} Y_{n+1}^{X_0} \right] + \frac{1}{2} \Delta t_n \hat{\mathbb{E}}\big[ f\big(t_n, \hat{\bar{\Theta}}_n^{x_0}, \theta\big)\big]\Big|_{\theta=\bar{\Theta}_n^{X_0}} \\ & + \frac{1}{2} \Delta t_n \hat{\mathbb{E}}_{t_n}^x \left[ \hat{\mathbb{E}} \left[ f\big(t_{n+1}, \hat{\Theta}_{n+1}^{x_0}, \theta\big)\right]\Big|_{\theta=\hat{\Theta}_{n+1}^{X_0}} \right], \end{aligned} \tag{3.20}$$

$$Z_n^{X_0} = \hat{D}_h^i Y_n^{X_0} \hat{\mathbb{E}}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big], \tag{3.21}$$

$$\Gamma_n^{X_0} = \lambda \left( \hat{G}_\rho^L \left[ Y_n^{x+c_n^x(\cdot)} \right] - Y_n^{X_0} \right), \tag{3.22}$$

*where $X_{n+1}^{X_0}$ is solved by the Itô-Taylor scheme (2.9) with $X_n^{X_0} = x$, and the notations $\hat{\bar{\Theta}}_n^{x_0}, \hat{\Theta}_n^{x_0}$ and $\hat{\Theta}_n^{X_0}$ are defined in (3.14) and (3.15).*

Among the local truncation errors $R_y^{n,X_0}, R_z^{n,X_0}, R_\gamma^{n,X_0}, R_{yr}^{n,I,\mathbb{E}}, R_y^{n,I,\mathbb{E}}, R_z^{n,I,\mathbb{E}}, R_\gamma^{n,I,\mathbb{E}}, \bar{R}_z^{n,I,\mathbb{E}}$, $\bar{R}_\gamma^{n,I,\mathbb{E}}$, the six terms $R_{yr}^{n,I,\mathbb{E}}, R_y^{n,I,\mathbb{E}}, R_z^{n,I,\mathbb{E}}, R_\gamma^{n,I,\mathbb{E}}, \bar{R}_z^{n,I,\mathbb{E}}$ and $\bar{R}_\gamma^{n,I,\mathbb{E}}$ are local errors generated by the interpolations and the approximations of expectations, which can be sufficiently small by carefully choosing the numerical interpolation and integral methods used in Scheme 3.3. And the three terms $R_y^{n,X_0}, R_z^{n,X_0}$ and $R_\gamma^{n,X_0}$ are defined in (3.4), (3.6) and (3.12).

Without loss of generality, we set $L \geq 3$. Assume that $b, \sigma \in C_b^{L,2L}([0,T] \times \mathbb{R}^2), \Phi \in C_b^{2L}(\mathbb{R}^2)$ and $f \in C_b^{L,2L}([0,T] \times \mathbb{R}^8)$. Moreover, assume that $c(\cdot, \cdot, \cdot, e) \in C_b^{L,2L}([0,T] \times \mathbb{R}^2)$ with the upper bound of the form $K(1 \wedge |e|)$. Then the solution of the PIDE (2.2) satisfies $u \in C_b^{L,2L}([0,T] \times \mathbb{R})$. Thus, by combining with the estimates (2.6) and (2.8) and the Itô-Taylor expansion [23], we get

$$\begin{aligned} &\big|R_z^{n,X_0}\big| \leq Ch^r, \\ &\big|R_\gamma^{n,X_0}\big| \leq \frac{C}{(2L)!}, \\ &\big|R_y^{n,X_0}\big| \leq C\bigg( (\Delta t_n)^3 + h^r \Delta t_n + \frac{\Delta t_n}{(2L)!} \bigg), \end{aligned} \tag{3.23}$$

where $r = 1, 2, 4$ when the difference operators $D_h^i$ for $i = -1, 0, 1$ are used in the scheme, respectively. Here $C > 0$ is a constant depending on $T, \rho$, and the upper bounds of derivatives of $b, \sigma, c, f$ and $\Phi$.

Assume that the Itô-Taylor schemes used to solve the MSDEJ in (3.1) are accurate enough, then by (3.23), we come to the following conclusions:

1. If $h = \mathcal{O}((\Delta t_n)^{1/r})$ and $1/(2L)! = \mathcal{O}(\Delta t_n)$, we have

$$R_y^{n,X_0} = \mathcal{O}\left((\Delta t_n)^2\right), \quad R_z^{n,X_0} = \mathcal{O}\left(\Delta t_n\right), \quad R_\gamma^{n,X_0} = \mathcal{O}\left(\Delta t_n\right),$$

   which indicates that Scheme 3.3 is first order accurate in time.

2. If $h = \mathcal{O}((\Delta t_n)^{2/r})$ and $1/(2L)! = \mathcal{O}((\Delta t_n)^2)$, we have

$$R_y^{n,X_0} = \mathcal{O}\left((\Delta t_n)^3\right), \quad R_z^{n,X_0} = \mathcal{O}\left((\Delta t_n)^2\right), \quad R_\gamma^{n,X_0} = \mathcal{O}\left((\Delta t_n)^2\right),$$

which indicates that Scheme 3.3 is second order accurate in time.

## 4. The Approximations of Mathematical Expectations

In this section, by using the Monte-Carlo method and the Gauss quadrature rules, we show how to approximate the expectation $\mathbb{E}[\cdot]$ and the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ in Scheme 3.1 and further give the definitions of operators $\hat{\mathbb{E}}[\cdot]$ and $\hat{\mathbb{E}}_{t_n}^x[\cdot]$ in Scheme 3.3.

We use the Monte Carlo method with the sample times $M_c$ to approximate the expectation $\mathbb{E}[\cdot]$ in the scheme and the definition of the corresponding operator $\hat{\mathbb{E}}[\cdot]$ can be found in [22]. Note that in high order Itô-Taylor schemes, the Brownian motion and the jump times are coupled together [23], which will bring us some technique difficulties to approximate the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$.

For simplicity of presentation, for two given functions $g_1 : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathsf{E} \to \mathbb{R}$, we define

$$\begin{aligned} &g_1^{X_n^{x_0}}(t_n, x) = \mathbb{E}\big[g_1\big(t_n, X_n^{x_0}, x\big)\big], \\ &g_2^{X_n^{x_0}}(t_n, x, e) = \mathbb{E}\big[g_2\big(t_n, X_n^{x_0}, x, e\big)\big]. \end{aligned}$$

Moreover, we define the operators $L^0, L^1$ and $L_e^{-1}$ by

$$\begin{aligned} L^0 g_1^{X_n^{x_0}}(t_n, x) &= \frac{\partial g_1^{X_n^{x_0}}}{\partial t}(t_n, x) + \frac{\partial g_1^{X_n^{x_0}}}{\partial x}(t_n, x) b^{X_n^{x_0}}(t_n, x) \\ &\quad + \frac{1}{2}\frac{\partial^2 g_1^{X_n^{x_0}}}{\partial x^2}\left(\sigma^{X_n^{x_0}}(t_n, x)\right)^2, \\ L^1 g_1^{X_n^{x_0}}(t_n, x) &= \frac{\partial g_1^{X_n^{x_0}}}{\partial x}(t_n, x)\sigma^{X_n^{x_0}}(t_n, x), \\ L_e^{-1} g_1^{X_n^{x_0}}(t_n, x) &= g_1^{X_n^{x_0}}\left(t_n, x + c^{X_n^{x_0}}(t_n-, x, e)\right) - g_1^{X_n^{x_0}}(t_n-, x). \end{aligned}$$

### 4.1. The Euler scheme

In this subsection, we show how to approximate $\mathbb{E}_{t_n}^x[Y_{n+1}^{X_0}]$ when the Euler scheme is used to solve MSDEJ, that is,

$$\begin{aligned} X_{n+1}^{X_0} = X_n^{X_0} &+ \tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta W_n \\ &+ \sum_{k=1}^{\Delta N_n} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right), \end{aligned} \tag{4.1}$$

where the function

$$\tilde{b}(t, x', x) = b(t, x', x) - \int_{\mathsf{E}} c(t, x', x, e)\lambda(de), \quad \forall\, (t, x', x) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

For convenience of presentation, we write $Y_n^{X_0} = Y_n$ and let

$$\tilde{b}_n = \tilde{b}^{X_n^{x_0}}(t_n, x), \quad \sigma_n = \sigma^{X_n^{x_0}}(t_n, x), \quad c_n(e_k) = c^{X_n^{x_0}}(t_n, x, e_k).$$

Then we get

$$\begin{aligned}
\mathbb{E}_{t_n}^x[Y_{n+1}] &= \mathbb{E}\left[Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n \Delta W_n + \sum_{k=1}^{\Delta N_n} c_n(e_k)\right)\right] \\
&= \mathbb{E}\left[\sum_{m=0}^{\infty} Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n \Delta W_n + \sum_{k=1}^{m} c_n(e_k)\right)\mathbb{I}_{\{\Delta N_n = m\}}\right] \\
&= \sum_{m=0}^{\infty} \mathbb{E}\left[Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n \Delta W_n + \sum_{k=1}^{m} c_n(e_k)\right)\right]\mathbb{P}\{\Delta N_n = m\} \\
&= \mathbb{E}_{t_n,M_y}^x[Y_{n+1}] + \mathcal{O}\big((\Delta t_n)^{M_y+1}\big),
\end{aligned} \tag{4.2}$$

where $M_y$ is the number of the truncated jumps and

$$\mathbb{E}_{t_n,M_y}^x[Y_{n+1}] = \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!}\mathbb{E}\left[Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n \Delta W_n + \sum_{k=1}^{m} c_n(e_k)\right)\right]$$

is the approximation of $\mathbb{E}_{t_n}^x[Y_{n+1}]$. Since $\{e_1, \cdots, e_m\}$ are independent and identically distributed, we have

$$\begin{aligned}
\mathbb{E}_{t_n,M_y}^x[Y_{n+1}] &= \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!} \\
&\quad \times \int_{\mathbb{R}}\int_{\mathsf{E}}\cdots\int_{\mathsf{E}} Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n\sqrt{\Delta t_n}s + \sum_{k=1}^{m} c_n(e_k)\right) \\
&\qquad\qquad \times \frac{\exp(-s^2/2)}{\sqrt{2\pi}}\prod_{k=1}^{m}\rho(e_k)de_1\cdots de_m ds \\
&= \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!\sqrt{\pi}}\sum_{j=1}^{L_1}\sum_{j_1=1}^{L_2}\cdots\sum_{j_m=1}^{L_2} w_j v_{j_1}\cdots v_{j_m} \\
&\quad \times Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n\sqrt{2\Delta t_n}p_j + \sum_{k=1}^{m}\mathbb{E}\big[c\big(t_n, X_n^{x_0}, x, q_{j_k}\big)\big]\right) + R_\rho,
\end{aligned}$$

where $\{p_j\}_{j=1}^{L_1}$ are the roots of the Hermite polynomial of degree $L_1$ and $\{w_j\}_{j=1}^{L_1}$ the corresponding weights, $\{q_{j_k}\}_{j_k=1}^{L_2}$ are the points of the Gaussian quadrature rules chosen by $\rho$ for $k = 1, \ldots, m$, and $\{v_{j_k}\}_{j_k=1}^{L_2}$ the corresponding weights, and $R_\rho$ is the corresponding error of the Gaussian quadrature rules used in the above approximation, which can be sufficiently small when $L_1$ and $L_2$ are large enough. Then we can define the approximation $\hat{\mathbb{E}}_{t_n}^x[Y_{n+1}]$ by

$$\hat{\mathbb{E}}_{t_n}^x[Y_{n+1}] := \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!\sqrt{\pi}}\sum_{j=1}^{L_1}\sum_{j_1=1}^{L_2}\cdots\sum_{j_m=1}^{L_2} w_j v_{j_1}\cdots v_{j_m} I_{\mathbf{X_j}}^{n+1}Y_{n+1}, \tag{4.3}$$

where $I_X^n$ denotes the cubic spline interpolation operator and

$$\mathbf{X_j} = x + \hat{\tilde{b}}_n\Delta t_n + \hat{\sigma}_n\sqrt{2\Delta t_n}p_j + \sum_{k=1}^{m}\hat{\mathbb{E}}\big[c\big(t_n, X_n^{x_0}, x, q_{j_k}\big)\big],$$

where

$$\hat{\tilde{b}}_n = \hat{\mathbb{E}}\big[\tilde{b}\big(t_n, X_n^{x_0}, x\big)\big], \quad \hat{\sigma}_n = \hat{\mathbb{E}}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big].$$

**Remark 4.1.** The definition (4.3) shows that the local errors generated by the approximations of conditional expectations can be neglected when the interpolation step size $\Delta x$ is sufficiently small, and the sample times of Monte-Carlo method $M_c$, the numbers of Gaussian quadrature points $L_1$ and $L_2$, and the number of the truncated jumps $M_y$ are sufficiently large. We take $\Delta x$ and $M_y$ as an example to illustrate this assertion. Since the cubic spline interpolation method is fourth order accurate, to balance the time discrete truncation error and the interpolation error, that is, $(\Delta x)^4 = (\Delta t)^3$, we require $\Delta x = (\Delta t)^{3/4}$ to guarantee the second order of temporal convergence rate of the scheme. On the other hand, the Eq. (4.2) indicates that the error generated by the jump truncation is $\mathcal{O}((\Delta t)^{M_y+1})$ and thus it is enough to take $M_y = 2$ so as not to affect the temporal convergence rate.

### 4.2. The Milstein scheme

In this subsection, we show how to approximate $\mathbb{E}_{t_n}^x[Y_{n+1}]$ when we take the Itô-Taylor scheme (2.9) as the Milstein scheme

$$\begin{aligned} X_{n+1}^{X_0} = X_n^{X_0} &+ \tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta W_n \\ &+ \sum_{k=1}^{\Delta N_n} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right) + \frac{1}{2}L^1\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\left((\Delta W_n)^2 - \Delta t_n\right) \\ &+ \sum_{k=1}^{\Delta N_n} L^1 c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right)\left(W_{\tau_k} - W_{t_n}\right) \\ &+ \sum_{k=1}^{\Delta N_n} L_{e_k}^{-1}\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\left(W_{t_{n+1}} - W_{\tau_k}\right) \\ &+ \sum_{k=1}^{\Delta N_n}\sum_{j=N_{t_n}+1}^{N_{\tau_k-}} L_{e_j}^{-1} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right). \end{aligned} \tag{4.4}$$

Notice that the Milstein scheme (4.4) is much more complex than the Euler scheme (4.1), in which the random jump times and the Brownian motion are coupled together. Hence, the approximation of $\mathbb{E}_{t_n}^x[Y_{n+1}]$ in this case will be much more complicated as shown in the below.

To proceed, we let $\tau_0 = t_n$ and $\tau_{\Delta N_n+1} = t_{n+1}$, and define

$$\Delta W_{\tau_i} = W_{\tau_i} - W_{\tau_{i-1}}, \quad i = 1, \ldots, \Delta N_n + 1.$$

Then by the same procedures used in (4.2), we deduce

$$\begin{aligned} \mathbb{E}_{t_n}^x[Y_{n+1}] &= \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!} \\ &\quad\times \mathbb{E}\Bigg[Y_{n+1}\Bigg(x + \tilde{b}_n\Delta t_n + \sigma_n\Delta W_n + \sum_{k=1}^m c_n(e_k) \\ &\qquad + \sum_{k=1}^m L^1 c_n(e_k)\left(W_{\tau_k} - W_{t_n}\right) + \sum_{k=1}^m L_{e_k}^{-1}\sigma_n(W_{t_{n+1}} - W_{\tau_k}) \\ &\qquad + \frac{1}{2}L^1\sigma_n\left((\Delta W_n)^2 - \Delta t_n\right) + \sum_{k=1}^m\sum_{j=1}^{k-1} L_{e_j}^{-1}c_n(e_k)\Bigg)\Bigg] + \mathcal{O}\left((\Delta t_n)^{M_y+1}\right) \\ &= \mathbb{E}_{t_n,M_y}^x[Y_{n+1}] + \mathcal{O}\left((\Delta t_n)^{M_y+1}\right). \end{aligned}$$

Then by the definition $\Delta W_{\tau_i} = W_{\tau_{i+1}} - W_{\tau_i}$ and the fact that the jump sizes are independent with the jump times and the Brownian motion, we get

$$\begin{aligned}\mathbb{E}^x_{t_n,M_y}[Y_{n+1}] = &\sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!} \\ &\times \int_{\mathsf{E}}\cdots\int_{\mathsf{E}} \mathbb{E}\Bigg[Y_{n+1}\Bigg(x+\tilde{b}_n\Delta t_n+\sigma_n\sum_{i=1}^{m+1}\Delta W_{\tau_i}+\sum_{k=1}^{m}c_n(e_k)+\sum_{k=1}^{m}L^1c_n(e_k)\sum_{i=1}^{k}\Delta W_{\tau_i} \\ &\qquad +\sum_{k=1}^{m}L^{-1}_{e_k}\sigma_n\sum_{i=k+1}^{m+1}\Delta W_{\tau_i}+\frac{1}{2}L^1\sigma_n\Bigg(\Bigg(\sum_{i=1}^{m+1}\Delta W_{\tau_i}\Bigg)^2-\Delta t_n\Bigg) \\ &\qquad +\sum_{k=1}^{m}\sum_{j=1}^{k-1}L^{-1}_{e_j}c_n(e_k)\Bigg)\Bigg]\prod_{k=1}^{m}\rho(e_k)de_1\cdots de_m. \end{aligned} \tag{4.5}$$

Since the random jump times and the Brownian motion are coupled together, it is difficult to handle the expectation contained in the integrand. To settle this problem, we consider to use the Monte Carlo method to approximate it. To this end, we sample the jump times $\{\tau_1,\cdots,\tau_m\}$ for $M_c$ times and denote the $s$-th sample as

$$\{\tau_1^s,\cdots,\tau_m^s\},\quad s=1,\ldots,M_c.$$

Let $\tau_0^s = t_n$ and $\tau_{m+1}^s = t_{n+1}$, and define

$$\Delta\tau_i^s = \tau_i^s - \tau_{i-1}^s,\quad i=1,\ldots,m+1.$$

Then we can generate the $s$-th increments $\{\Delta W^s_{\tau_1},\cdots,\Delta W^s_{\tau_{m+1}}\}$ of the Brownian motion by

$$\Delta W^s_{\tau_i} \sim \sqrt{\Delta\tau_i^s}N(0,1),\quad i=1,\ldots,m+1. \tag{4.6}$$

Now based on (4.6), we apply the Monte-Carlo method to (4.5) thus obtaining

$$\begin{aligned}\mathbb{E}^x_{t_n,M_y}[Y_{n+1}] = &\frac{1}{M_c}\sum_{s=1}^{M_c}\sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!} \\ &\times \int_{\mathsf{E}}\cdots\int_{\mathsf{E}} Y_{n+1}\Bigg(x+\tilde{b}_n\Delta t_n+\sigma_n\sum_{i=1}^{m+1}\Delta W^s_{\tau_i}+\sum_{k=1}^{m}c_n(e_k)+\sum_{k=1}^{m}L^1c_n(e_k)\sum_{i=1}^{k}\Delta W^s_{\tau_i} \\ &\qquad +\sum_{k=1}^{m}L^{-1}_{e_k}\sigma_n\sum_{i=k+1}^{m+1}\Delta W^s_{\tau_i}+\frac{1}{2}L^1\sigma_n\Bigg(\Bigg(\sum_{i=1}^{m+1}\Delta W^s_{\tau_i}\Bigg)^2-\Delta t_n\Bigg) \\ &\qquad +\sum_{k=1}^{m}\sum_{j=1}^{k-1}L^{-1}_{e_j}c_n(e_k)\Bigg)\prod_{k=1}^{m}\rho(e_k)de_1\cdots de_m + \mathcal{O}\left(\frac{1}{\sqrt{M_c}}\right), \end{aligned}$$

which can be approximated by the appropriate Gaussian quadrature rules chosen by the probability density function $\rho$. Then the approximation $\hat{\mathbb{E}}^x_{t_n}[Y_{n+1}]$ can be defined by the same way as in (4.3). Moreover, we can obtain the same conclusions on the local errors for approximating conditional expectations as in Remark 4.1.

### 4.3. The weak order 2.0 Itô-Taylor scheme

In this subsection, we consider to use the following weak order 2.0 Itô-Taylor scheme to solve the MSDEJ in (1.1):

$$
\begin{aligned}
X_{n+1}^{X_0} =& X_n^{X_0} + \tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta W_n \\
&+ \sum_{k=1}^{\Delta N_n} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right) + \frac{1}{2}L^0\tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)(\Delta t_n)^2 \\
&+ \sum_{k=1}^{\Delta N_n} L_{e_k}^{-1}\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\left(W_{t_{n+1}} - W_{\tau_k}\right) \\
&+ \sum_{k=1}^{\Delta N_n} L^1 c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right)\left(W_{\tau_k} - W_{t_n}\right) \\
&+ \frac{1}{2}\Big(L^1\tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right) + L^0\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Big)\Delta W_n \Delta t_n \\
&+ \frac{1}{2}L^1\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\left((\Delta W_n)^2 - \Delta t_n\right) + \sum_{k=1}^{\Delta N_n}\sum_{j=N_{t_n}+1}^{N_{\tau_k-}} L_{e_j}^{-1}c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right) \\
&+ \sum_{k=1}^{\Delta N_n} L^0 c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_k\right)(\tau_k - t_n) + \sum_{k=1}^{\Delta N_n} L_{e_k}^{-1}\tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)(t_{n+1} - \tau_k). \qquad (4.7)
\end{aligned}
$$

The approximation of $\mathbb{E}_{t_n}^x[Y_{n+1}]$ for the weak order 2.0 Itô-Taylor scheme (4.7) is similar to the one for the Milstein scheme (4.4) as shown in the above subsection. So we omit it here.

We close this section by giving some remarks about the above methods used to approximate the conditional expectations:

- For the simulations of the jump times of Poisson random measure, the readers are referred to [6, Chapter 6].
- The Eq. (4.2) indicates that taking the truncated jump number $M_y = 2$ is enough to guarantee the second order accuracy of Scheme 3.3.
- Gaussian quadrature rules are used to approximate the solution $\Gamma_t$ as well as the conditional expectations in Scheme 3.3, which are chosen by the distribution of the jump size $e$.
- Note that the approximations of $\mathbb{E}_{t_n}^x[Y_{n+1}\Delta\tilde{W}_n]$ and $\mathbb{E}_{t_n}^x[Y_{n+1}\Delta\tilde{\mu}_n^*]$ (see [20,32]) in Scheme 3.2 used to solve $Z_n$ and $\Gamma_n$ are much more complicated than the one of $\mathbb{E}_{t_n}^x[Y_{n+1}]$, since they are coupled with the increments of the Brownian motion and the Poisson random measure. Thus Scheme 3.3 is easier to implement than the fully discrete version of Scheme 3.2 in practice.

## 5. Numerical Experiments

In this section, we shall present several numerical examples to show the performance of Scheme 3.3.

In the following examples, we set $T = 1.0$ and take uniform partition in time with time step $\Delta t = T/N$, where the integer $N$ is the partition number. The initial values will be chosen in a bounded domain $[x_L, x_R]$ with $x_L \leq 0 \leq x_R$, i.e. $x_0, X_0 \in [x_L, x_R]$. In all our tests, we take $x_0 = 0$ and adopt the maximum norm to measure the errors between the true solutions and the numerical ones. We shall also use $\mathcal{E}_Y, \mathcal{E}_Z$ and $\mathcal{E}_\Gamma$ to represent the maximum errors of $Y, Z$ and $\Gamma$, that is

$$
\begin{aligned}
\mathcal{E}_Y &= \max_{X_0 \in [x_L, x_R]} \left| Y_0^{0,X_0} - Y_0^{X_0} \right|, \\
\mathcal{E}_Z &= \max_{X_0 \in [x_L, x_R]} \left| Z_0^{0,X_0} - Z_0^{X_0} \right|, \\
\mathcal{E}_\Gamma &= \max_{X_0 \in [x_L, x_R]} \left| \Gamma_0^{0,X_0} - \Gamma_0^{X_0} \right|.
\end{aligned}
$$

For periodic cases, without loss of generality, we assume that the period is $[-\pi, \pi]$ and we take $[x_L, x_R] = [-\pi, \pi]$. For nonpriodic cases, we take $[x_L, x_R] = [-10, 10]$.

Our main goal of the numerical experiments is to demonstrate the second order temporal accuracy of the fully discrete Scheme 3.3. Hence in the approximations of expectations and conditional expectations, we set the sample times of Monte-Carlo method and the numbers of Gaussian quadrature points to be large enough such that the errors resulted from the use of Monte-Carlo method and Gaussian quadrature rules are negligible. Besides, we take $I_X^n$ as the cubic spline interpolation method in Scheme 3.3. Since the cubic spline interpolation method is fourth order accurate and the finite difference approximation of $D_h^i$ is $r$th order accurate with $r = 1, 2, 4$ for $i = -1, 0, 1$, respectively, to balance the time discrete truncation errors and the space discrete truncation errors, we require $\Delta x = (\Delta t)^{3/4}$ and $h = (\Delta t)^{2/r}$.

In what follows, the convergence rate (CR) with respect to $\Delta t$ is obtained by the least square fitting.

**Example 5.1.** In this example, we consider a nonlinear MFBSDEJs driven by multi-dimensional Brownian moiton and jump size.

Let $W_t = (W_t^1, \cdots, W_t^m)$ be an $m$-dimensional Brownian motion and $\mathbf{e} = (e_1, \cdots, e_q)$ a $q$-dimensional jump size with $W_t^i$ and $e_j$ being independent with each other. Then we consider the following MFBSDEJs driven by $W_t$ and $\mathbf{e}$:

$$
\begin{aligned}
dX_t^{0,X_0} &= b dt + \sum_{i=1}^{m} \sigma_i dW_t^i + \int_{\mathsf{E}} \sum_{i=1}^{q} \arcsin(e_i) \tilde{\mu}(d\mathbf{e}, dt), \\
-dY_t^{0,X_0} &= \left( \frac{1}{2} \left( Y_t^{0,X_0} \sum_{i=1}^{m} \sigma_i^2 - 2\Gamma_t^{0,X_0} \right) - \frac{(1+b)}{m} \sum_{i=1}^{m} \frac{Z_{i,t}^{0,X_0}}{\sigma_i} \right. \\
&\qquad \left. + \mathbb{E}\left[ \left( \left( \frac{2^q}{\pi^q} - 1 \right) Y_t^{0,x_0} - \frac{1}{\lambda} \Gamma_t^{0,x_0} \right)^3 \right] \right) dt \\
&\qquad - \sum_{i=1}^{m} Z_{i,t}^{0,X_0} dW_t^i - \int_{\mathsf{E}} U_t^{0,X_0}(\mathbf{e}) \tilde{\mu}(d\mathbf{e}, dt), \\
Y_T^{0,X_0} &= \sin\left(T + X_T^{0,X_0}\right) - \cos\left(T + X_T^{0,X_0}\right).
\end{aligned}
\tag{5.1}
$$

We take the jump space $\mathsf{E} = [-1, 1]^q$ and define the Lévy measure by

$$
\lambda(d\mathbf{e}) = \lambda \rho(\mathbf{e}) d\mathbf{e} = \lambda \frac{1}{\pi^q} \prod_{i=1}^{q} \frac{1}{\sqrt{1 - e_i^2}} \mathcal{X}_{[-1,1]}(e_i) de_i,
$$

where $\lambda = \lambda(\mathsf{E})$ is the jump intensity and

$$
\rho(\mathbf{e}) = \frac{1}{\pi^q} \prod_{i=1}^{q} \frac{1}{\sqrt{1 - e_i^2}} \mathcal{X}_{[-1,1]}(e_i)
$$

is the probability density of $\mathbf{e}$. Then the analytic solution yields

$$
\begin{aligned}
Y_t^{0,X_0} &= \sin\left(t + X_t^{0,X_0}\right) - \cos\left(t + X_t^{0,X_0}\right),\\
Z_{i,t}^{0,X_0} &= \sigma_i\Big(\cos\left(t + X_t^{0,X_0}\right) + \sin\left(t + X_t^{0,X_0}\right)\Big), \quad i = 1, \ldots, m,\\
\Gamma_t^{0,X_0} &= \lambda\left(\left(\frac{2}{\pi}\right)^q - 1\right)\Big(\sin\left(t + X_t^{0,X_0}\right) - \cos\left(t + X_t^{0,X_0}\right)\Big).
\end{aligned}
$$

Note that the true solution of the MSDEJ in (5.1) is

$$
X_t^{0,X_0} = X_0 + bt + \sum_{i=1}^{m} \sigma_i W_t^i + \sum_{k=1}^{N_t} \sum_{i=1}^{q} \arcsin\left(e_i^k\right).
$$

Hence, the errors of the Itô-Taylor schemes used to solve the MSDEJ vanish, and the orders of convergence rate of Scheme 3.3 are independent with Itô-Taylor schemes.

In our experiments, we take $\lambda = 2.0$ and set $b = 1$ and $\sigma_i = 1/m$ for $i = 1, \ldots, m$, and carry out the following two tests:

1. To test the effect of finite difference approximations on convergence rates of Scheme 3.3, we set $m = q = 2$ and choose $D_h^i, i = 0, \pm 1$ in the scheme with $h = (\Delta t)^{1/2}, \Delta t$ and $(\Delta t)^2$, respectively.

2. To compare the effectiveness of Schemes 3.2 and 3.3, we set $m = q = 3$ and use Schemes 3.2 and 3.3 to solve (5.1), respectively. In the use of Scheme 3.3, we choose the finite difference approximation of $D_h^1$ with $h = (\Delta t)^{1/2}$.

It is worth noting that when $h = (\Delta t)^2$ with $\Delta t = 1/256$, the error of $D_h^1$ becomes

$$
h^4 = (\Delta t)^8 = \frac{1}{2^{64}} \approx 0. \tag{5.2}
$$

In Tables 5.1-5.4, we have listed the numerical results of our experiments, in which the unit of the running time (RT) is second. We also plot the errors of the two schemes against $\Delta t$ and the running times for $m = q = 3$ in Figs. 5.1-5.4.

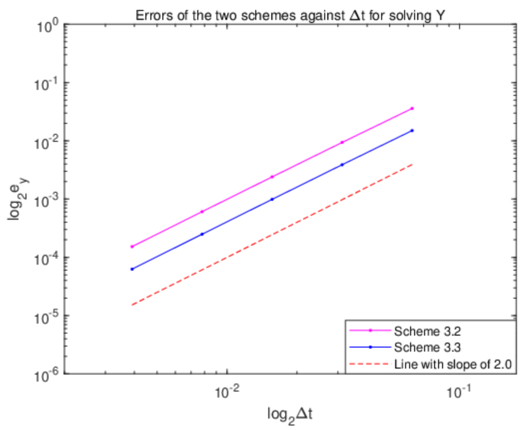


Fig. 5.1. Errors of the two schemes against $\Delta t$ for solving $Y$ with $m = q = 3$.

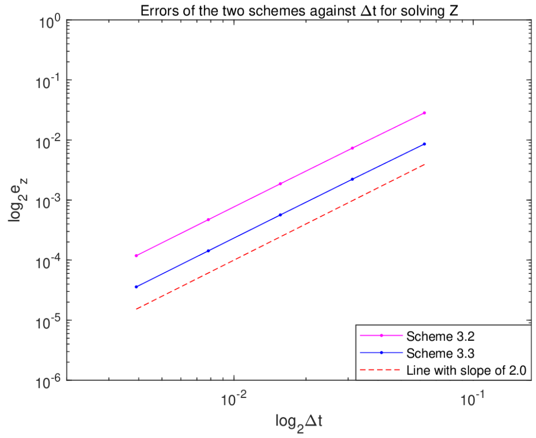


Fig. 5.2. Errors of the two schemes against $\Delta t$ for solving $Z$ with $m = q = 3$.

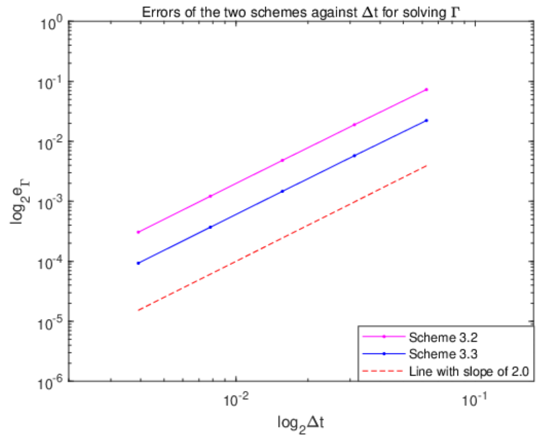


Fig. 5.3. Errors of the two schemes against $\Delta t$ for solving $\Gamma$ with $m = q = 3$.

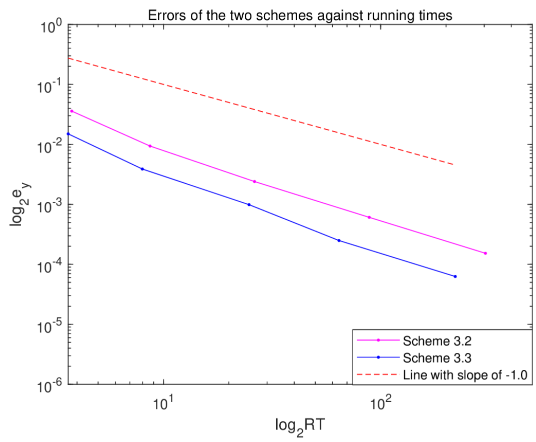


Fig. 5.4. Errors of the two schemes for the solution $Y$ against the running times with $m = q = 3$.

Table 5.1: Errors and convergence rates of Schemes 3.3 for $D_h^{-1}$ with $m = q = 2$.

| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | CR |
|---|---|---|---|---|---|---|
| $h = (\Delta t)^{1/2}$ | | | | | | |
| $\mathcal{E}_Y$ | 2.967E-01 | 2.238E-01 | 1.645E-01 | 1.191E-01 | 8.552E-02 | 0.450 |
| $\mathcal{E}_Z$ | 2.298E-01 | 1.742E-01 | 1.285E-01 | 9.337E-02 | 6.720E-02 | 0.445 |
| $\mathcal{E}_\Gamma$ | 3.529E-01 | 2.662E-01 | 1.956E-01 | 1.417E-01 | 1.017E-01 | 0.450 |
| $h = \Delta t$ | | | | | | |
| $\mathcal{E}_Y$ | 7.360E-02 | 4.042E-02 | 2.123E-02 | 1.088E-02 | 5.511E-03 | 0.937 |
| $\mathcal{E}_Z$ | 5.834E-02 | 3.200E-02 | 1.678E-02 | 8.596E-03 | 4.351E-03 | 0.939 |
| $\mathcal{E}_\Gamma$ | 8.755E-02 | 4.808E-02 | 2.525E-02 | 1.295E-02 | 6.555E-03 | 0.937 |
| $h = (\Delta t)^2$ | | | | | | |
| $\mathcal{E}_Y$ | 7.786E-03 | 1.999E-03 | 5.067E-04 | 1.275E-04 | 3.200E-05 | 1.982 |
| $\mathcal{E}_Z$ | 4.549E-03 | 1.207E-03 | 3.115E-04 | 7.908E-05 | 2.250E-05 | 1.925 |
| $\mathcal{E}_\Gamma$ | 9.263E-03 | 2.378E-03 | 6.029E-04 | 1.517E-04 | 3.807E-05 | 1.982 |

Table 5.2: Errors and convergence rates of Schemes 3.3 for $D_h^0$ with $m = q = 2$.

| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | CR |
|---|---|---|---|---|---|---|
| $h = (\Delta t)^{1/2}$ | | | | | | |
| $\mathcal{E}_Y$ | 3.538E-02 | 1.592E-02 | 7.614E-03 | 3.738E-03 | 1.854E-03 | 1.060 |
| $\mathcal{E}_Z$ | 2.372E-02 | 1.145E-02 | 5.732E-03 | 2.883E-03 | 1.448E-03 | 1.006 |
| $\mathcal{E}_\Gamma$ | 4.209E-02 | 1.893E-02 | 9.056E-03 | 4.446E-03 | 2.206E-03 | 1.060 |
| $h = \Delta t$ | | | | | | |
| $\mathcal{E}_Y$ | 1.328E-02 | 3.402E-03 | 8.620E-04 | 2.169E-04 | 5.442E-05 | 1.983 |
| $\mathcal{E}_Z$ | 8.846E-03 | 2.265E-03 | 5.746E-04 | 1.443E-04 | 3.770E-05 | 1.972 |
| $\mathcal{E}_\Gamma$ | 1.580E-02 | 4.046E-03 | 1.025E-03 | 2.580E-04 | 6.472E-05 | 1.983 |
| $h = (\Delta t)^2$ | | | | | | |
| $\mathcal{E}_Y$ | 1.248E-02 | 3.217E-03 | 8.177E-04 | 2.061E-04 | 5.176E-05 | 1.979 |
| $\mathcal{E}_Z$ | 8.831E-03 | 2.276E-03 | 5.792E-04 | 1.464E-04 | 4.905E-05 | 1.894 |
| $\mathcal{E}_\Gamma$ | 1.485E-02 | 3.827E-03 | 9.726E-04 | 2.452E-04 | 6.157E-05 | 1.979 |

From the data in Tables 5.1-5.4, we draw the following conclusions:

- The convergence rates in Tables 5.1-5.3 show that Scheme 3.3 is second order accurate when the parameter $h \leq (\Delta t)^2, \Delta t$ and $(\Delta t)^{1/2}$ for the finite difference approximations of $D_h^i$ with $i = -1, 0, 1$, respectively. This is due to the reason that the errors of these three approximations are $h, h^2$ and $h^4$, respectively.

- The errors in Tables 5.2-5.3 indicate that the accuracy of Scheme 3.3 is not sensitive to $h$ when it is small enough. Namely, our scheme is stable with respect to $h$. The estimate (5.2) also implies that the finite difference approximations can be regarded as the derivatives of the approximated functions for small $\Delta t$.

Table 5.3: Errors and convergence rates of Schemes 3.3 for $D_h^1$ with $m = q = 2$.

| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | CR |
|---|---|---|---|---|---|---|
| $h = (\Delta t)^{\frac{1}{2}}$ | | | | | | |
| $\mathcal{E}_Y$ | 1.262E-02 | 3.250E-03 | 8.254E-04 | 2.080E-04 | 5.221E-05 | 1.980 |
| $\mathcal{E}_{Z_1}$ | 8.815E-03 | 2.269E-03 | 5.762E-04 | 1.452E-04 | 3.644E-05 | 1.980 |
| $\mathcal{E}_\Gamma$ | 1.501E-02 | 3.866E-03 | 9.818E-04 | 2.474E-04 | 6.210E-05 | 1.980 |
| $h = \Delta t$ | | | | | | |
| $\mathcal{E}_Y$ | 1.248E-02 | 3.217E-03 | 8.177E-04 | 2.061E-04 | 5.176E-04 | 1.979 |
| $\mathcal{E}_{Z_1}$ | 8.832E-03 | 2.277E-03 | 5.790E-04 | 1.461E-04 | 4.314E-05 | 1.932 |
| $\mathcal{E}_\Gamma$ | 1.485E-02 | 3.826E-03 | 9.726E-04 | 2.451E-04 | 6.158E-05 | 1.979 |
| $h = (\Delta t)^2$ | | | | | | |
| $\mathcal{E}_Y$ | 1.248E-02 | 3.217E-03 | 8.177E-04 | 2.061E-04 | 5.176E-04 | 1.979 |
| $\mathcal{E}_{Z_1}$ | 8.831E-03 | 2.276E-03 | 5.792E-04 | 1.464E-04 | 4.904E-05 | 1.894 |
| $\mathcal{E}_\Gamma$ | 1.485E-02 | 3.826E-03 | 9.726E-04 | 2.452E-04 | 6.157E-05 | 1.979 |

Table 5.4: Errors and convergence rates of Schemes 3.2 and 3.3 with $m = q = 3$.

| | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | CR |
|---|---|---|---|---|---|---|
| Scheme 3.2 | | | | | | |
| $\mathcal{E}_Y$ | 3.589E-02 | 9.383E-03 | 2.402E-03 | 6.078E-04 | 1.529E-04 | 1.970 |
| $\mathcal{E}_Z$ | 2.829E-02 | 7.328E-03 | 1.867E-03 | 4.715E-04 | 1.185E-04 | 1.976 |
| $\mathcal{E}_\Gamma$ | 7.296E-02 | 1.890E-02 | 4.815E-03 | 1.216E-03 | 3.057E-04 | 1.976 |
| RT | 3.78 | 8.67 | 26.25 | 88.60 | 303.79 | |
| Scheme 3.3 | | | | | | |
| $\mathcal{E}_Y$ | 1.499E-02 | 3.884E-03 | 9.888E-04 | 2.495E-04 | 6.270E-05 | 1.977 |
| $\mathcal{E}_Z$ | 8.573E-03 | 2.222E-03 | 5.655E-04 | 1.426E-04 | 3.583E-05 | 1.977 |
| $\mathcal{E}_\Gamma$ | 2.225E-02 | 5.763E-03 | 1.467E-03 | 3.702E-04 | 9.300E-05 | 1.977 |
| RT | 3.64 | 7.99 | 24.77 | 64.29 | 220.59 | |

- The data in Table 5.4 shows that Scheme 3.3 consumes much less time than Scheme 3.2 to solve the MFBSDEJs (5.1) because of its simpler structure. Moreover, the accuracy of Scheme 3.3 outperforms Scheme 3.2. For instance, to yield the numerical errors around $10^{-4}$, Scheme 3.3 needs the time step size $\Delta t = 1/128$, while $\Delta t = 1/256$ is used for Scheme 3.2 reaching the same magnitude of the numerical errors. In other words, given a requirement of accuracy, Scheme 3.3 allows larger time step sizes than Scheme 3.2, and thus, is less time-consuming and more efficient.

According to the above conclusions, for simplicity, we choose the finite difference operator $D_h^1$ with the optimal step size $h = (\Delta t)^{1/2}$ in the following examples.

**Example 5.2.** Consider the MFBSDEJs with nonperiodic solution

$$
\begin{aligned}
dX_t^{0,X_0} &= \frac{\mathbb{E}\big[X_t^{0,x_0}\big]}{1+\exp(t)}dt + \frac{1}{1+\exp\big(t-\big(X_t^{0,X_0}\big)^2/2\big)}dW_t + \int_{\mathsf{E}} \frac{e}{2}\tilde{\mu}(de,dt),\\
-dY_t^{0,X_0} &= \Bigg(-Y_t^{0,X_0}\bigg(1-\frac{X_t^{0,X_0}\mathbb{E}\big[X_t^{0,x_0}\big]}{1+\exp(t)}\bigg) + \frac{\exp\big(t-\big(X_t^{0,X_0}\big)^2/2\big)}{2\big(1+Y_t^{0,X_0}\big)^2}\\
&\qquad + \frac{X_t^{0,X_0}Z_t^{0,X_0}}{2\big(1+Y_t^{0,X_0}\big)} - \Gamma_t^{0,X_0} + \frac{1}{3}\mathbb{E}\left[\left(Y_t^{0,x_0} - \exp\left(t-\frac{1}{2}\big(X_t^{0,x_0}\big)^2\right)\right)^3\right]\Bigg)dt \qquad (5.3)\\
&\qquad - Z_t^{0,X_0}dW_t - \int_{\mathsf{E}} U_t^{0,X_0}(e)\tilde{\mu}(de,dt),\\
Y_T^{0,X_0} &= \exp\left(T-\frac{1}{2}\big(X_T^{0,X_0}\big)^2\right).
\end{aligned}
$$

Define the Lévy measure by

$$
\lambda(de) = \lambda\rho(e)de = \lambda\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e^2}{2}\right)de,
$$

where $\lambda = \lambda(\mathsf{E})$ is the jump intensity and

$$
\rho(e) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e^2}{2}\right)
$$

is the probability density of $e$. Then the analytic solution is

$$
\begin{aligned}
Y_t^{0,X_0} &= \exp\left(t-\frac{1}{2}\big(X_t^{0,X_0}\big)^2\right),\\
Z_t^{0,X_0} &= -\frac{X_t^{0,X_0}\exp\big(t-\big(X_t^{0,X_0}\big)^2/2\big)}{1+\exp\big(t-\big(X_t^{0,X_0}\big)^2/2\big)},\\
\Gamma_t^{0,X_0} &= \lambda\left(\frac{2}{\sqrt{5}}\exp\left(t-\frac{2}{5}\big(X_t^{0,X_0}\big)^2\right) - \exp\left(t-\frac{1}{2}\big(X_t^{0,X_0}\big)^2\right)\right).
\end{aligned}
$$

We take $\lambda = 0.5$ and choose the Euler scheme, the Milstein scheme and the weak order 2.0 Itô-Taylor scheme to solve the MSDEJ, respectively. The numerical results are shown in Table 5.5 as below.

The numerical results in Table 5.5 show that Scheme 3.3 is stable and accurate for solving the MFBSDEJs (5.3) with nonperiodic solution. Moreover, Scheme 3.3 is convergent with first order when the Euler scheme or the Milstein scheme are used, and second order when the weak order 2.0 Itô-Taylor scheme is used to solve the MSDEJ. This is mainly due to the fact that the weak convergence orders of these three schemes are $1.0, 1.0$ and $2.0$, respectively (see [23]).

Note that when solving the MFBSDEJs with periodic solutions, the space domains are fixed as $[-\pi,\pi]$ for all the time levels. However, in the nonperiodic cases, the space domains grow larger with the iteration of the time, the speed of which is problem dependent and can be rapid (see [7]). Therefore it is much more time consuming to solve the MFBSDEJs with nonperiodic solutions in general.

Table 5.5: Errors and convergence rates of Scheme 3.3 with $\lambda = 0.5$.

| | | $N=4$ | $N=8$ | $N=16$ | $N=32$ | $N=64$ | CR |
|---|---|---|---|---|---|---|---|
| Eul | $\mathcal{E}_Y$ | 1.019E-02 | 4.050E-03 | 1.895E-03 | 1.074E-03 | 5.649E-04 | 1.026 |
| | $\mathcal{E}_Z$ | 8.923E-03 | 4.543E-03 | 2.507E-03 | 1.333E-03 | 6.873E-04 | 0.917 |
| | $\mathcal{E}_\Gamma$ | 8.347E-04 | 5.469E-04 | 3.058E-04 | 1.605E-04 | 8.275E-05 | 0.844 |
| Mil | $\mathcal{E}_Y$ | 1.436E-02 | 4.908E-03 | 2.213E-03 | 9.955E-04 | 5.479E-04 | 1.173 |
| | $\mathcal{E}_Z$ | 1.356E-02 | 5.258E-03 | 2.752E-03 | 1.275E-03 | 5.556E-04 | 1.126 |
| | $\mathcal{E}_\Gamma$ | 1.575E-03 | 7.190E-04 | 3.499E-04 | 1.710E-04 | 8.369E-05 | 1.054 |
| W-2.0 | $\mathcal{E}_Y$ | 1.221E-02 | 2.784E-03 | 6.971E-04 | 1.801E-04 | 4.605E-05 | 2.005 |
| | $\mathcal{E}_Z$ | 8.301E-03 | 1.946E-03 | 5.573E-04 | 1.651E-04 | 3.561E-05 | 1.930 |
| | $\mathcal{E}_\Gamma$ | 5.527E-04 | 1.392E-04 | 4.714E-05 | 1.249E-05 | 2.975E-06 | 1.855 |

**Example 5.3.** In this example, we consider a two-dimensional problem

$$
\begin{aligned}
\begin{pmatrix} dX_{1,t}^{0,X_0} \\ dX_{2,t}^{0,X_0} \end{pmatrix} &= \begin{pmatrix} \mathbb{E}\left[\sin\left(X_{2,t}^{0,x_0}\right)\right] \\ \mathbb{E}\left[\sin\left(X_{1,t}^{0,x_0}\right)\right] \end{pmatrix} dt + \begin{pmatrix} 0 & \frac{1}{3}\mathbb{E}\left[\cos^2\left(X_{1,t}^{0,x_0}\right)\right] \\ \frac{1}{3}\mathbb{E}\left[\cos^2\left(X_{2,t}^{0,x_0}\right)\right] & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} + \int_{\mathsf{E}} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tilde{\mu}(d\mathbf{e}, dt), \\
-dY_t^{0,X_0} &= \Bigg( \frac{1}{18} Y_t^{0,X_0} \bigg( \Big( \mathbb{E}\left[\cos^2\left(X_{1,t}^{0,x_0}\right)\right] \Big)^2 + \Big( \mathbb{E}\left[\cos^2\left(X_{2,t}^{0,x_0}\right)\right] \Big)^2 \bigg) - \left(Y_t^{0,X_0} + \Gamma_t^{0,X_0}\right) \\
&\quad - \exp(t)\cos\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0}\right)\mathbb{E}\left[\sin\left(X_{1,t}^{0,x_0}\right) + \sin\left(X_{2,t}^{0,x_0}\right)\right] \\
&\quad + \frac{1}{9}\mathbb{E}\left[Z_{1,t}^{0,x_0}\right]\mathbb{E}\left[\cos^2\left(X_{1,t}^{0,x_0}\right)\right] - \frac{1}{9}\mathbb{E}\left[Z_{2,t}^{0,x_0}\right]\mathbb{E}\left[\cos^2\left(X_{2,t}^{0,x_0}\right)\right] \Bigg) dt \\
&\quad - Z_{1,t}^{0,X_0} dW_t^1 - Z_{2,t}^{0,X_0} dW_t^2 - \int_{\mathsf{E}} U_t^{0,X_0}(\mathbf{e})\tilde{\mu}(d\mathbf{e}, dt), \\
Y_T^{0,X_0} &= \exp(T)\sin\left(X_{1,T}^{0,X_0} + X_{2,T}^{0,X_0}\right).
\end{aligned} \tag{5.4}
$$

Take $\mathsf{E} = [-\delta, \delta]^2$ and define the Lévy measure by

$$
\lambda(d\mathbf{e}) = \lambda\rho(\mathbf{e})d\mathbf{e} = \lambda\frac{1}{4\delta^2}\mathcal{X}_{[-\delta,\delta]^2}(e_1, e_2)de_1 de_2,
$$

where $\lambda = \lambda(\mathsf{E})$ is the jump intensity and

$$
\rho(\mathbf{e}) = \frac{1}{4\delta^2}\mathcal{X}_{[-\delta,\delta]^2}(e_1, e_2)
$$

is the probability density of $\mathbf{e}$. Then the analytic solution is

$$
\begin{aligned}
Y_t^{0,X_0} &= \exp(t)\sin\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0}\right), \\
Z_{1,t}^{0,X_0} &= \frac{1}{3}\exp(t)\cos\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0}\right)\mathbb{E}\left[\cos^2\left(X_{2,t}^{0,x_0}\right)\right], \\
Z_{2,t}^{0,X_0} &= \frac{1}{3}\exp(t)\cos\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0}\right)\mathbb{E}\left[\cos^2\left(X_{1,t}^{0,x_0}\right)\right], \\
\Gamma_t^{0,X_0} &= \frac{\lambda}{4\delta^2}\exp(t)\Big(2\sin\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0}\right) - \sin\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0} + 2\delta\right) \\
&\qquad - \sin\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0} - 2\delta\right) - 4\delta^2\sin\left(X_{1,t}^{0,X_0} + X_{2,t}^{0,X_0}\right)\Big).
\end{aligned}
$$

Table 5.6: Errors and convergence rates of Scheme 3.3 with $\lambda = 1.0$.

| | | $N=4$ | $N=8$ | $N=16$ | $N=32$ | $N=64$ | CR |
|---|---|---|---|---|---|---|---|
| Eul | $\mathcal{E}_Y$ | 3.558E-02 | 1.738E-02 | 8.509E-03 | 4.253E-03 | 2.123E-03 | 1.017 |
| | $\mathcal{E}_Z$ | 9.392E-02 | 4.717E-02 | 2.356E-02 | 1.173E-02 | 5.844E-03 | 1.002 |
| | $\mathcal{E}_\Gamma$ | 1.249E-02 | 5.782E-03 | 2.834E-03 | 1.409E-03 | 7.049E-04 | 1.033 |
| Mil | $\mathcal{E}_Y$ | 5.378E-02 | 2.894E-02 | 1.409E-02 | 6.072E-03 | 3.089E-03 | 1.050 |
| | $\mathcal{E}_Z$ | 8.488E-02 | 5.251E-02 | 2.810E-02 | 1.528E-02 | 7.250E-03 | 0.888 |
| | $\mathcal{E}_\Gamma$ | 2.971E-02 | 8.555E-03 | 2.645E-03 | 1.507E-03 | 5.510E-04 | 1.401 |
| W-2.0 | $\mathcal{E}_Y$ | 1.757E-02 | 2.223E-03 | 5.667E-04 | 2.021E-04 | 6.104E-05 | 1.980 |
| | $\mathcal{E}_Z$ | 8.416E-03 | 9.422E-04 | 2.069E-04 | 8.032E-05 | 2.499E-05 | 2.034 |
| | $\mathcal{E}_\Gamma$ | 1.848E-03 | 2.621E-04 | 4.639E-05 | 1.661E-05 | 5.102E-06 | 2.098 |

We take $\lambda = 4\delta^2$ and set $\delta = 0.5$, i.e. $\lambda = 1.0$. We implement Scheme 3.3 to solve the two-dimensional MFBSDEJs (5.4) and test the Euler scheme, the Milstein scheme and the weak order 2.0 Itô-Taylor scheme for solving the MSDEJ, respectively. The numerical results are listed in Table 5.6.

The results in Table 5.6 clearly show that Scheme 3.3 also works well and is an order two scheme for solving the multi-dimensional MFBSDEJs if the associated multi-dimensional MSDEJ is solved by the weak order 2.0 Itô-Taylor scheme.

## 6. Conclusions

In this work, we proposed a new second order accurate scheme for solving decoupled MF-BSDEJs. The key features are that the finite difference approximations and the Gaussian quadrature rule are respectively used to approximate the derivative and the integral in the solution representations, which dramatically simplify the structure of the proposed scheme. Numerical results showed that the proposed scheme is stable, efficient, and can be of second order accuracy for solving MFBSDEJs when the weak order 2.0 Itô-Taylor scheme is used to solve the MSDEJ.

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