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NUMERICAL METHODS FOR APPROXIMATING STOCHASTIC SEMILINEAR TIME-FRACTIONAL RAYLEIGH-STOKES EQUATIONS*

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Abstract

This paper investigates a semilinear stochastic fractional Rayleigh-Stokes equation featuring a Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ in time and a fractional time-integral noise. The study begins with an examination of the solution's existence, uniqueness, and regularity. The spatial discretization is then carried out using a finite element method, and the error estimate is analyzed. A convolution quadrature method generated by the backward Euler method is employed for the time discretization resulting in a fully discrete scheme. The error estimate for the fully discrete solution is considered based on the regularity of the solution, and a strong convergence rate is established. The paper concludes with numerical tests to validate the theoretical findings.

Mathematics subject classification: 65M60, 65M12, 65M15.
Key words: Riemann-Liouville fractional derivative, Stochastic Rayleigh-Stokes equation, Finite element method, Convolution quadrature, Error estimates.

1. Introduction

We investigate the following semilinear stochastic fractional order Rayleigh-Stokes problem:

$$u_t(t,x) + \big(1+\partial_t^{1-\alpha}\big)Au(t,x) = f\big(u(t,x)\big) + \partial_t^{-\gamma}\dot{W}(t), \quad t\in(0,T], \quad x\in\mathcal{D}, \tag{1.1a}$$

$$u(t,x) = 0, \quad t\in(0,T], \quad x\in\partial\mathcal{D}, \tag{1.1b}$$

$$u(0,x) = u_0, \quad x\in\mathcal{D}, \tag{1.1c}$$

where $0<\alpha<1$, $0\le\gamma\le1$ and $T>0$ is a fixed time. Here, $A=-\Delta$ denotes the negative Laplace operator with its domain $D(A)=H^2(\mathcal{D})\cap H_0^1(\mathcal{D})$ and $\mathcal{D}\subset\mathbb{R}^d, d\le3$, is an open convex polygonal domain with a boundary $\partial\mathcal{D}$. The operator $\partial_t^{-\gamma}$ denotes the Riemann-Liouville time-fractional integral operator defined by

$$\partial_t^{-\gamma}\varphi(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\varphi(s)\,ds,$$

where $\Gamma(\cdot)$ is the usual Gamma function. The operator $\partial_t^{1-\alpha} := \partial_t\partial_t^{-\alpha}$ denotes the Riemann-Liouville time-fractional derivative, where $\partial_t=\partial/\partial t$. In the model (1.1), the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the global Lipschitz condition:

$$|f(t)-f(s)| \le L|t-s|, \quad \forall\, t,s\in\mathbb{R}, \quad L>0. \tag{1.2}$$

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The noise $\{W(t)\}_{t\geq 0}$ is an $L^2(\mathcal{D})$-valued Wiener process with a covariance operator Q with respect to a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, $\dot{W}(t) := dW(t)/dt$ is its formal derivative. The initial data u_0 is an $\mathcal{F}_0$-measurable random variable with values in $L^2(\mathcal{D})$.

The stochastic Rayleigh-Stokes problem is a model used to describe the dynamic behavior of non-Newtonian fluids, where the time-fractional derivative $\partial_t^{1-\alpha}$ is utilized to capture fluid elasticity (as noted in [1, 4, 5] and related references). The numerical approximation of linear time-fractional stochastic evolution equations has been extensively studied, with several works including [7, 9, 10, 13, 14, 17]. Jin *et al.* [10] analyzed the strong and weak convergence of a numerical scheme for subdiffusion equations with fractionally integrated Gaussian noise, which was created using the Galerkin finite element method for the spatial aspect and convolution quadrature for the fractional derivative. In [17], the focus was on a stochastic subdiffusion problem driven by integrated space-time white noise, with the L1 scheme and Lubich's first order convolution quadrature formula being used to approximate the time-fractional derivative and time-fractional integral, respectively. The study established a strong convergence rate.

The numerical analysis of semilinear time-fractional stochastic equations has been explored in recent works such as [3,11]. Kang *et al.* [11] investigated a stochastic space and time-fractional subdiffusion problem that included a fractionally integrated additive noise and a globally Lipschitz term $f(u)$. The authors regularized the problem and derived error estimates based on the properties of the Mittag-Leffler functions. More recently, in [3], the authors studied a stochastic time-fractional Allen-Cahn model perturbed by a fractionally integrated Gaussian noise. The Galerkin finite element method was used for the spatial approximation, and a convolution quadrature was used to approximate both the fractional derivative and integral. By utilizing the temporal Hölder continuity property of the solution, strong convergence rates for the error were derived. In both [3, 11], conditions on α and γ were imposed for the well-posedness of the stochastic time-fractional models.

In this study, the solution is represented in an integral form and global existence and uniqueness of solution are discussed. The regularity of the solution in both space and time is established. The main objective of this work is to prove a strong convergence rate in $L^2(\Omega; L^2(\mathcal{D}))$ for the fully discrete scheme using a semigroup type approach. The spatial discretization is performed using a Galerkin finite element method, while the noise is approximated by an L^2-projection. Under the condition $-\alpha(2-r)/2 + \gamma > -1/2$, where r is defined in (4.6), we derive error estimates for the semidiscrete scheme. The fully discrete scheme is then obtained by applying a convolution quadrature generated by the backward Euler method for the fractional derivative and integral. By exploiting the solution regularity and the globally Lipschitz property of the source term f given in (1.2), the error estimate is analyzed and a strong convergence rate for the fully discrete scheme is proved.

The paper is structured as follows. In Section 2, we introduce notations and recall some properties of Wiener processes. In Section 3, the representation of the solution is discussed along with its existence, uniqueness, and regularity. Section 4 deals with the spatial discretization and the error analysis of the resulting semi-discrete scheme. In Section 5, error estimates for the fully discrete scheme are established. Finally, in Section 6, numerical experiments are conducted to validate the theoretical results.

Throughout the paper, we use c and C to denote generic constants that may change from one occurrence to another, but are always independent of the mesh size h and time step size τ. Additionally, we simplify the notation by writing $u(t)$ instead of $u(t, x)$.

2. Preliminaries

This section presents the notations and key characteristics of Wiener processes, which are essential for the subsequent sections.

Let $H := L^2(\mathcal{D})$ be the Hilbert space of square integrable functions with $(\cdot\,,\cdot)$ the usual $L^2(\mathcal{D})$-inner product and induced norm $\|\cdot\|$. For $s \geq 0$, we define the space $\dot{H}^s := \dot{H}^s(\mathcal{D})$ and its induced norm $\|\cdot\|_{\dot{H}^s}$ by

$$\dot{H}^s := \left\{ v \in H : \sum_{j=1}^{\infty} \lambda_j^s (v, \phi_j)^2 < \infty \right\}, \quad \|v\|_{\dot{H}^s} = \left\| A^{\frac{s}{2}} v \right\| = \left(\sum_{j=1}^{\infty} \lambda_j^s (v, \phi_j)^2 \right)^{\frac{1}{2}},$$

where $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ are the Dirichlet eigenpairs of A. The $\{\phi_j\}_{j=1}^{\infty}$ form an orthonormal basis in H. So, we have $\dot{H}^0 = H, \dot{H}^1 = H_0^1(\mathcal{D})$, and $\dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1((\mathcal{D})$, see [16].

Let $L(H)$ denote the space of linear and bounded operators form H into H and let $Q \in L(H)$ be a self-adjoint and positive definite operator. The eigenvalues and eigenfunctions of the operator Q are $\{(\eta_j, e_j)\}_{j=1}^{\infty}$ with $\{e_j\}_{j=1}^{\infty}$ being an orthonormal basis of H. The trace of Q is

$$\mathrm{Tr}(Q) := \sum_{j=1}^{\infty} \eta_j.$$

The stochastic process $W(t)$ is assumed to be a Wiener process expressed as

$$W(t) = \sum_{j=1}^{\infty} \eta_j^{\frac{1}{2}} \beta_j(t) e_j,$$

where $\beta_j(t)$ are independent real-valued Brownian motions. When the series of $\mathrm{Tr}(Q)$ converges, Q is referred to as trace class and W is considered an H-valued Wiener process. On the other hand, if $\mathrm{Tr}(Q)$ is not finite, such as in the case of $Q = I$, W does not take values from H, and is referred to as a cylindrical Wiener process. The rate of decrease in η_j as $\eta_j \to 0$ indicates the regularity of $W(t)$, a faster decrease implies a smoother noise.

Let $\mathcal{L}_2^0$ be the space of Hilbert-Schmidt operators defined by

$$\mathcal{L}_2^0 = \left\{ \psi \in L(H) : \sum_{j=1}^{\infty} \left\| \psi Q^{\frac{1}{2}} e_j \right\|^2 < \infty \right\},$$

equipped with the norm

$$\|\psi\|_{\mathcal{L}_2^0} = \left(\sum_{j=1}^{\infty} \left\| \psi Q^{\frac{1}{2}} e_j \right\|^2 \right)^{\frac{1}{2}}.$$

This definition is independent of the choice of the orthonormal basis in H. For $p \geq 2$, we denote by $L^p(\Omega; H)$ the space of H-valued p-times integrable random variables with norm

$$\|v\|_{L^p(\Omega;H)} = \left(\mathbf{E} \|v\|_H^p \right)^{\frac{1}{p}} = \left(\int_{\Omega} \|v(\omega)\|_H^p d\mathbb{P}(\omega) \right)^{\frac{1}{p}},$$

where $\mathbf{E}$ stands for the expected value. Similarly, the space $L^2(\Omega; V)$ is defined for an arbitrary Banach space V.

We shall use some properties of the Itô-type integrals $\int_0^t \psi(s)dW(s)$. Given $\{\psi(t)\}_{t\in[0,T]}$ a predictable and $\mathcal{L}_2^0$-valued stochastic process with

$$\|\psi\|_{L^p(\Omega;L^2(0,T;\mathcal{L}_2^0))} < \infty, \quad p \geq 2,$$

we have

$$\left\| \int_0^T \psi(s)dW(s) \right\|_{L^2(\Omega;H)}^2 = \mathbf{E} \int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds, \tag{2.1}$$

which is known as Itô's isometry [8, Proposition 4.5]. The Burkholder inequality is given by [12, Proposition 2.12]

$$\left\| \int_0^T \psi(s)dW(s) \right\|_{L^p(\Omega;H)} \leq C_p \|\psi\|_{L^p(\Omega;L^2(0,T;\mathcal{L}_2^0))}. \tag{2.2}$$

For our analysis, we shall make the following assumptions.

Assumption 2.1. *The noise $\{W(t)\}_{t\geq 0}$ is an $L^2(\mathcal{D})$-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with a self-adjoint and positive definite operator $Q \in L(H)$ such that*

$$\left\|A^{\frac{\beta-1}{2}}\right\|_{\mathcal{L}_2^0} < \infty \quad \textit{for some} \quad \beta \in [0,1].$$

Assumption 2.2. *The initial data u_0 belongs to $L^2(\Omega, \dot{H}^\nu)$ for some $\nu \in [0,2]$.*

Remark 2.1. Our results in this paper remain valid when $A = -\Delta$ is replaced by a second order self-adjoint, positive definite elliptic operator. However, we prefer to take a standard setting as above to highlight the key ideas in this work.

3. Existence of Solutions and Regularity

This section begins by deriving the solution representation and then utilizes it to analyse the existence, uniqueness, and regularity of the solution.

3.1. Solution representation

We recall that the mild solution to problem (1.1) is a predictable stochastic process $u : [0,T] \times \Omega \to H$ satisfying the following integral equation:

$$u(t) = E_0(t)u_0 + \int_0^t E_0(t-s)f\big(u(s)\big)\,ds + \int_0^t E_\gamma(t-s)dW(s). \tag{3.1}$$

The solution operator $E_m(t), m \geq 0$, is defined by

$$E_m(t) = \frac{1}{2\pi i} \int_{\Gamma_{\rho,\delta}} e^{zt} z^{-m} \frac{g(z)}{z} \big(g(z)I + A\big)^{-1}\, dz,$$

where

$$g(z) = \frac{z}{1 + z^{1-\alpha}}.$$

The contour of integration is

$$\Gamma_{\rho,\delta} := \{\zeta e^{\pm i\rho} : \zeta \geq \delta\} \cup \{\delta e^{i\psi} : |\psi| \leq \rho\}, \quad \rho \in \left(\frac{\pi}{2}, \pi\right), \quad \delta > 0,$$

oriented with an increasing imaginary part.

The smoothness of E_m is given in the following lemma.

Lemma 3.1 (cf. [4, Theorem 2.1]). *The operator $E_m(t)$ satisfies*

$$\left\|\partial_t^\ell E_m(t)v\right\|_{\dot H^p} \le ct^{-\frac{\alpha(p-q)}{2}+m-\ell}\|v\|_{\dot H^q},$$

where $\ell = 0$ and $0 \le q \le p \le 2$ or $\ell > 0$ and $0 \le p, q \le 2$.

Note that by (1.2), we have

$$\|f(u)\| \le \|f(u) - f(0)\| + \|f(0)\| \le C(1 + \|u\|). \tag{3.2}$$

Now, consider the model (1.1) when $u_0 = 0$ and $f = 0$, whose mild solution is the stochastic convolution

$$W_A(t) = \int_0^t E_\gamma(t-s)dW(s), \quad t \ge 0.$$

Using the bound

$$\left\|A^{\frac{1-\beta}{2}}E_\gamma(t)v\right\| \le ct^{-\frac{\alpha(1-\beta)}{2}+\gamma}\|v\|$$

from Lemma 3.1 and Burkholder's inequality (2.2), we derive the following estimates: For $p \ge 2$,

$$\begin{aligned}\|W_A(t)\|_{L^p(\Omega;H)} &= \left\|\int_0^t A^{\frac{1-\beta}{2}}E_\gamma(t-s)A^{\frac{\beta-1}{2}}dW(s)\right\|_{L^p(\Omega;H)} \\ &\le C(\alpha,\beta,\gamma,p)\left(\int_0^t (t-s)^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)}ds\right)^{\frac12}\left\|A^{\frac{\beta-1}{2}}\right\|_{\mathcal{L}_2^0} \\ &\le C(\alpha,\beta,\gamma,p)t^{-\frac{\alpha(1-\beta)}{2}+\gamma+\frac12}\left\|A^{\frac{\beta-1}{2}}\right\|_{\mathcal{L}_2^0}.\end{aligned}$$

As a result, we prove an important smoothness property of $W_A(t)$ in the next lemma.

Lemma 3.2. *Let $\beta \in [0,1]$ and θ be defined by*

$$\theta = -\frac{\alpha(1-\beta)}{2} + \gamma + 1. \tag{3.3}$$

Then, $\theta > 1/2$, and for $0 < t_1 < t_2$ and any $p \ge 2$, we have

$$\|W_A(t_2) - W_A(t_1)\|_{L^p(\Omega;H)} \le c(t_2-t_1)^{\min\{\theta-\frac12-\epsilon_0,1\}}\left\|A^{\frac{\beta-1}{2}}\right\|_{\mathcal{L}_2^0}, \tag{3.4}$$

where $\epsilon_0 > 0$ if $\theta = 3/2$ and $\epsilon_0 = 0$ otherwise.

Proof. The difference $W_A(t_2) - W_A(t_1)$ for $t_1 < t_2$ may written as

$$\begin{aligned}W_A(t_2) - W_A(t_1) &= \int_0^{t_1}\big(E_\gamma(t_2-s) - E_\gamma(t_1-s)\big)dW(s) \\ &\quad + \int_{t_1}^{t_2} E_\gamma(t_2-s)dW(s) =: I_1 + I_2.\end{aligned}$$

We first consider the case when $1/2 < \theta < 3/2$. For this case, we follow the proof in [10, Theorem A.2] to get

$$\|I_1\|_{L^p(\Omega;H)} = \left\|\int_0^{t_1}\big(E_\gamma(t_2-s) - E_\gamma(t_1-s)\big)dW(s)\right\|_{L^P(\Omega;H)}$$

$$
\begin{aligned}
&= \left\| \int_0^{t_1} \left(\int_{t_1}^{t_2} A^{\frac{1-\beta}{2}} E'_\gamma(t-s)dt \right) A^{\frac{\beta-1}{2}} dW(s) \right\|_{L^P(\Omega;H)} \\
&\le \int_{t_1}^{t_2} \left\| \int_0^{t_1} A^{\frac{1-\beta}{2}} E'_\gamma(t-s) A^{\frac{\beta-1}{2}} dW(s) \right\|_{L^P(\Omega;H)} dt \\
&\le \int_{t_1}^{t_2} \left(\int_0^{t_1} \left(\left\| A^{\frac{1-\beta}{2}} E'_\gamma(t-s) \right\| \|A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0} \right)^2 ds \right)^{\frac{1}{2}} dt \\
&\le \int_{t_1}^{t_2} \left(\int_0^{t_1} (t-s)^{2\theta-4} \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} ds \right)^{\frac{1}{2}} dt \\
&\le c \int_{t_1}^{t_2} (t-t_1)^{\theta-\frac{3}{2}} \|A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0} dt \\
&\le c(t_2-t_1)^{\theta-\frac{1}{2}} \|A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0}.
\end{aligned}
$$

For $3/2 \le \theta$, we obtain by using arguments from [3],

$$
\begin{aligned}
\|I_1\|^2_{L^p(\Omega;H)} &= \left\| \int_0^{t_1} \big(E_\gamma(t_2-s) - E_\gamma(t_1-s) \big) dW(s) \right\|^2_{L^p(\Omega;H)} \\
&= \left\| \int_0^{t_1} \left(\int_{t_1}^{t_2} A^{\frac{1-\beta}{2}} E'_\gamma(t-s)dt \right) A^{\frac{\beta-1}{2}} dW(s) \right\|^2_{L^p(\Omega;H)} \\
&\le c \int_0^{t_1} \left(\int_{t_1}^{t_2} (t-s)^{\theta-2} dt \right)^2 ds \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} \\
&\le c \int_0^{t_1} (t_1-s)^{2\theta-4} \left(\int_{t_1}^{t_2} dt \right)^2 ds \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} \qquad \left(\theta > \frac{3}{2} \right) \\
&\le c t_1^{2\theta-3} (t_2-t_1)^2 \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0}.
\end{aligned}
$$

For $\theta = 3/2$, we see that

$$
\begin{aligned}
\|I_1\|^2_{L^p(\Omega;H)} &\le c \int_0^{t_1} (t_1-s)^{-1+2\epsilon} \left(\int_{t_1}^{t_2} (t-s)^{-\epsilon} dt \right)^2 ds \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} \\
&\le c t_1^{2\epsilon} (t_2-t_1)^{2(1-\epsilon)} \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0}.
\end{aligned}
$$

Finally, for the second term I_2, we have

$$
\begin{aligned}
\|I_2\|^2_{L^p(\Omega;H)} &\le \int_{t_1}^{t_2} \|A^{\frac{1-\beta}{2}} E_\gamma(t_2-s) A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} ds \\
&\le c \int_{t_1}^{t_2} (t_2-s)^{2\theta-2} ds \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} \\
&= c(t_2-t_1)^{2\theta-1} \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0} \le c(t_2-t_1)^2 T^{2\theta-3} \|A^{\frac{\beta-1}{2}}\|^2_{\mathcal{L}_2^0},
\end{aligned}
$$

which completes the proof of (3.4). □

3.2. Existence and regularity of solutions

In this subsection, we discuss existence, uniqueness and regularity of the solution to problem (1.1). To start, we recall a Gronwall-type inequality, which will be a common tool in our analysis, see [6].

Lemma 3.3. *Assume that φ is a nonnegative function in $L^1(0,T)$, which satisfies*

$$\varphi(t) \le \zeta(t) + b\int_0^t (t-s)^{-\alpha}\varphi(s)\,ds, \quad t \in (0,T],$$

where $\zeta(t) \ge 0, b \ge 0$, and $0<\alpha<1$. Then there exists a constant C_T such that

$$\varphi(t) \le \zeta(t) + C_T\int_0^t (t-s)^{-\alpha}\zeta(s)\,ds, \quad t \in (0,T].$$

The well-posedness of problem (1.1) is discussed in the next theorem. Note that the proof is established without assigning conditions on α and γ.

Theorem 3.1. *Let $u_0 \in L^2(\Omega; H)$, $\|A^{(\beta-1)/2}\|_{\mathcal{L}_2^0} < \infty$ with $\beta \in [0,1]$, and f satisfies* (1.2)*. Then problem* (1.1) *has a unique mild solution $u \in C([0,T]; L^2(\Omega; H))$.*

Proof. For $\lambda > 0$, let X_λ denote the space $C([0,T]; L^2(\Omega; H))$ equipped with the norm

$$\|v\|_\lambda^2 = \sup_{0\le t\le T} \mathbf{E}\|e^{-\lambda t}v(t)\|^2,$$

which is equivalent to the standard norm of $C([0,T]; L^2(\Omega; H))$ for a fixed parameter $\lambda > 0$. Define the nonlinear operator $S : X_\lambda \to X_\lambda$ by

$$\begin{aligned} Sv(t) = E_0(t)u_0 &+ \int_0^t E_0(t-s)f\big(v(s)\big)\,ds \\ &+ \int_0^t E_\gamma(t-s)dW(s) =: I_1 + I_2 + I_3. \end{aligned} \tag{3.5}$$

Then, $u \in X_\lambda$ satisfies (3.1) if and only if u is a fixed point of S.

If $v \in X_\lambda$, then by Lemma 3.1,

$$\mathbf{E}\big\|e^{-\lambda t}I_1\big\|^2 = \mathbf{E}\big\|e^{-\lambda t}E_0(t)u_0\big\|^2 \le C_\alpha \mathbf{E}\big[\|u_0\|^2\big] < \infty.$$

For I_2, we have by (3.2) and Lemma 3.1,

$$\mathbf{E}\big\|e^{-\lambda t}I_2\big\|^2 \le \int_0^t \mathbf{E}\big\|e^{-\lambda t}E_0(t-s)f\big(v(s)\big)\big\|^2 ds \le C_\alpha \int_0^t \big(1 + \mathbf{E}\|v\|^2\big)\,ds < \infty.$$

Similarly, for I_3,

$$\begin{aligned} \mathbf{E}\|e^{-\lambda t}I_3\|^2 &\le \mathbf{E}\left[\int_0^t \|e^{-\lambda t}A^{\frac{1-\beta}{2}}E_\gamma(t-s)A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0}^2 ds\right] \\ &\le C_\alpha\left(\mathbf{E}\left[\int_0^t (t-s)^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)}\big\|A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0}^2 ds\right]\right) \\ &\le C_\alpha t^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)+1}\big\|A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0}^2 < \infty, \end{aligned}$$

showing that $S(v)(t) \in X_\lambda$ for all $\lambda \ge 0$.

Next we show that S is a contraction. If $v_1, v_2 \in X_\lambda$, then by (1.2) and Lemma 3.1,

$$\begin{aligned} &\mathbf{E}\big\|e^{-\lambda t}\big(S(v_1)(t) - S(v_2)(t)\big)\big\|^2 \\ &\le \mathbf{E}\left(\int_0^t \big\|e^{-\lambda t}E_0(t-s)\big(f\big(v_1(s)\big) - f\big(v_2(s)\big)\big)\big\|^2 ds\right) \end{aligned}$$

$$\le C_\alpha \mathbf{E}\left(\int_0^t e^{-2\lambda(t-s)}\big\|e^{-\lambda s}\big(f\big(v_1(s)\big)-f\big(v_2(s)\big)\big)\big\|^2 ds\right)$$
$$\le C_\alpha\left(\int_0^t e^{-2\lambda(t-s)}\|v_1(s)-v_2(s)\|_\lambda^2 ds\right)$$
$$\le C_\alpha \frac{1-e^{-2\lambda t}}{2\lambda}\|v_1(t)-v_2(t)\|_\lambda^2.$$

The map $S: X_\lambda \to X_\lambda$ is a contraction for large λ. The Banach fixed point theorem implies that S has a unique fixed point $u \in X_\lambda$, which is also the unique solution of (3.1). □

Regularity properties of the mild solution of (1.1) are obtained in the following theorem.

Theorem 3.2. *Let Assumptions* 2.1 *and* 2.2 *be valid. Then, the following regularity result for the solution u of* (1.1) *holds*

$$\|u\|_{L^2(\Omega;\dot{H}^q)} \le Ct^{-\frac{\alpha(q-\nu)}{2}},$$

where $0 \le \nu \le q \le 2$, C depends on α, q, γ and T, and $-\alpha(q+1-\beta)/2+\gamma > -1/2$.

Proof. Considering (3.1) and using (3.2) and Lemma 3.1, we obtain for $q \in [0,2]$,

$$\begin{aligned}\|u(t)\|_{L^2(\Omega;\dot{H}^q)} \le \|E_0(t)u_0\|_{L^2(\Omega;\dot{H}^q)} &+ \int_0^t \big\|A^{\frac{q}{2}}E_0(t-s)f\big(u(s)\big)\big\|_{L^2(\Omega;H)}ds \\ &+ \int_0^t \big\|A^{\frac{q}{2}}E_\gamma(t-s)dW(s)\big\|_{L^2(\Omega;H)} \\ \le Ct^{-\frac{\alpha(q-\nu)}{2}}\|u_0\|_{L^2(\Omega;\dot{H}^\nu)} &+ C\int_0^t (t-s)^{-\frac{\alpha q}{2}}\big\|f\big(u(s)\big)\big\|_{L^2(\Omega;H)}ds \\ &+ C\left(\int_0^t (t-s)^{2(-\frac{\alpha(q+1-\beta)}{2}+\gamma)}\big\|A^{\frac{\beta-1}{2}}\big\|_{\mathcal{L}_2^0}^2 ds\right)^{\frac{1}{2}},\end{aligned}$$

where the last inequality holds since $-\alpha(q+1-\beta)/2+\gamma > -1/2$. □

Further regularity results of the mild solution are presented in the following theorem.

Theorem 3.3. *Let Assumptions* 2.1 *and* 2.2 *be fulfilled. Then, the mild solution of problem* (1.1) *satisfies*

$$\|u(t)-u(s)\|_{L^2(\Omega;H)} \le c(t-s)^{\min\{1,\theta-\frac{1}{2}\}}t^{\frac{\alpha\nu}{2}-1} \tag{3.6}$$

for $0 < s < t < T$, where θ is given in (3.3).

Proof. In view of (3.1), we get for $h > 0$,

$$\begin{aligned}u(t+h)-u(t) = &\big\{\big(E_0(t+h)-E_0(t)\big)u_0\big\} \\ &+ \left\{\int_t^{t+h} E_0(t+h-s)f\big(u(s)\big)ds \right.\\ &\quad\left.+ \int_0^t \big(E_0(t+h-s)-E_0(t-s)\big)f\big(u(s)\big)ds\right\} \\ &+ W_A(t+h)-W_A(t) =: I_1+I_2+I_3.\end{aligned}$$

For $\nu \in (0,2]$, we apply Lemma 3.1 with $\ell = 1$ so that

$$\begin{aligned}\|I_1\|_{L^2(\Omega;H)} &= \Big\| \int_t^{t+h} E_0'(s)u_0 ds \Big\|_{L^2(\Omega;H)} \\ &\leq c\int_t^{t+h} s^{\frac{\alpha\nu}{2}-1} ds \|u_0\|_{L^2(\Omega;\dot{H}^\nu)} \leq ct^{\frac{\alpha\nu}{2}-1}h.\end{aligned}$$

For I_2, a use of (1.2) and Lemma 3.1 with $\ell = 0$ gives

$$\begin{aligned}\|I_2\|_{L^2(\Omega;H)} &\leq \Big\| \int_t^{t+h} E_0(s) f\big(u(t+h-s)\big) ds \Big\|_{L^2(\Omega;H)} \\ &\quad + \Big\| \int_0^t E_0(s)\big(f\big(u(t+h-s)\big) - f\big(u(t-s)\big)\big) ds \Big\|_{L^2(\Omega;H)} \\ &\leq c\int_t^{t+h} \big\|f\big(u(t+h-s)\big)\big\|_{L^2(\Omega;H)} ds \\ &\quad + c\int_0^t \|u(t+h-s) - u(t-s)\|_{L^2(\Omega;H)} ds \\ &\leq ch + c\int_0^t \|u(t+h) - u(s)\|_{L^2(\Omega;H)} ds.\end{aligned}$$

The bound of the last term follows by Lemma 3.2. For the last term, we have

$$\|I_3\|_{L^2(\Omega;H)} \leq ch^{\min\{\theta-\frac{1}{2}-\epsilon_0,1\}} \big\|A^{\frac{\beta-1}{2}}\big\|_{\mathcal{L}_2^0}.$$

Thus, the desired estimate is obtained by using Gronwall's inequality. □

4. The Semidiscrete Problem

This section focuses on the spatial semidiscrete approximation of problem (1.1) and its error analysis using a piecewise linear Galerkin finite element method (FEM). The procedure begins with a description of the FEM method. The domain $\bar{\mathcal{D}}$ is divided into d-simplexes through a shape-regular and quasi-uniform partition represented by $\mathcal{T}_h$, where h represents the maximum mesh size. The space $V_h \subset \dot{H}^1$ consists of all continuous piecewise linear functions on $\mathcal{T}_h$, and P_h represents the orthogonal L^2-projection from H to V_h. Then the FEM seeks to find an approximate solution $u_h(t) \in V_h$ such that

$$u_{t,h}(t) + \big(1+\partial_t^{1-\alpha}\big)A_h u_h(t) = P_h f\big(u_h(t)\big) + \partial_t^{-\gamma} P_h \dot{W}(t), \quad t \in (0,T], \quad u_h(0) = P_h u_0, \tag{4.1}$$

where $A_h : V_h \to V_h$ denotes the discrete Laplacian

$$(A_h\psi, \chi) = (\nabla\psi, \nabla\chi), \quad \forall\, \psi, \chi \in V_h.$$

As in Eq. (3.1), the semidiscrete solution u_h can be represented by

$$u_h(t) = E_{0,h}(t)P_h u_0 + \int_0^t E_{0,h}(t-s)P_h f\big(u_h(s)\big) ds + \int_0^t E_{\gamma,h}(t-s)P_h dW(s), \quad t > 0, \tag{4.2}$$

where the operator $E_{m,h}(t) : V_h \to V_h$ is the discrete analogues of $E_m(t), m \geq 0$, defined by

$$E_{m,h}(t) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}} e^{zt} z^{-m} \frac{g(z)}{z} \big(g(z)I + A_h\big)^{-1} dz.$$

Since A_h is selfadjoint and positive definite uniformly in h, the estimates in Lemma 3.2 and Theorem 3.2 still hold for A_h, also uniformly in h.

Now, we introduce the operator

$$F_{m,h}(t) := E_{m,h}(t)P_h - E_m(t), \quad m \geq 0.$$

The estimate of $F_{m,h}(t)$ plays an important role in the error analysis. Using [4, Remark 3.2] and Lemma 3.1, one can deduce that for $v \in \dot{H}^\nu$,

$$\left\| A^{\frac{s}{2}} F_{m,h}(t)v \right\| \leq c\, h^{2-s-r} t^{-\frac{\alpha(2-r-\nu)}{2}+m} \|v\|_{\dot{H}^\nu}, \tag{4.3}$$

where $0 \leq s \leq 1, 0 \leq r \leq 2$ with $r+s \leq 2$ and $\nu \in [0,2]$.

Let $e(t) = u_h(t) - u(t)$ denote the error at time t. An error estimate for the semidiscrete problem (4.1) is established in the following theorem.

Theorem 4.1. *Let Assumptions* 2.1 *and* 2.2 *be valid. Let u and u_h be the solutions of* (1.1) *and* (4.1), *respectively. Then, there is a constant $c = c(\alpha, \gamma, T)$ such that*

$$\begin{aligned} &\|e(t)\|_{L^2(\Omega,L^2)} + h\|\nabla e(t)\|_{L^2(\Omega,L^2)} \\ &\leq c\, h^2 t^{-\frac{\alpha(2-\nu)}{2}} \|u_0\|_{\dot{H}^\nu} + c\, h^{1+\beta-r} \left\| A^{\frac{\beta-1}{2}} \right\|_{\mathcal{L}_2^0}, \quad t \in (0,T], \end{aligned} \tag{4.4}$$

where r is defined by (4.6) *and $-\alpha(2-r)/2 + \gamma > -1/2$.*

Proof. From (3.1) and (4.2), we can represent the error as follows:

$$\begin{aligned} e(t) = F_{0,h}(t)u_0 &+ \int_0^t E_{0,h}(t-s)P_h\big[f\big(u_h(s)\big) - f\big(u(s)\big)\big]ds \\ &+ \int_0^t F_{0,h}(t-s)f\big(u(s)\big)ds + \int_0^t F_{\gamma,h}(t-s)dW(s), \quad t > 0. \end{aligned} \tag{4.5}$$

Set $\omega = \alpha(2-\nu)/2$. Then using the bounds (4.3) for the operators F_m, we find that

$$\begin{aligned} \|e(t)\|_{L^2(\Omega,L^2)} \leq\ & c\, h^2 t^{-\omega} \|u_0\|_{L^2(\Omega,\dot{H}^\nu(\Omega))} \\ &+ c\int_0^t \left\| E_{0,h}(t-s)P_h\big[f\big(u_h(s)\big) - f\big(u(s)\big)\big] \right\|_{L^2(\Omega,L^2)} ds \\ &+ \int_0^t \left\| F_{0,h}(t-s)f\big(u(s)\big) \right\|_{L^2(\Omega,L^2)} ds + \left(\mathbf{E}\left[\int_0^t \|F_{\gamma,h}(t-s)\|^2_{\mathcal{L}_2^0} ds \right] \right)^{\frac{1}{2}} \\ \leq\ & c\, h^2 t^{-\omega} + c\int_0^t \|e(s)\| ds + c\, h^2 \int_0^t (t-s)^{-\alpha} \left\| f\big(u(s)\big) \right\|_{L^2(\Omega,L^2)} ds \\ &+ c\, h^{1+\beta-r} \left(\int_0^t (t-s)^{2(-\frac{\alpha(2-r)}{2}+\gamma)} \left\| A^{\frac{\beta-1}{2}} \right\|^2_{\mathcal{L}_2^0} ds \right)^{\frac{1}{2}}. \end{aligned}$$

The last inequality holds with

$$r = \begin{cases} (-1+2\alpha-2\gamma)/\alpha + \epsilon, & \text{if} \quad \gamma - \alpha < -1/2, \\ \epsilon, & \text{if} \quad \gamma - \alpha = -1/2, \\ 0, & \text{if} \quad \gamma - \alpha > -1/2. \end{cases} \tag{4.6}$$

The desired estimate follows now by applying Lemma 3.3. The estimate for $\|\nabla e(t)\|$ is derived by similar arguments, which completes the proof. $\square$

Remark 4.1. For $\beta \in (1,2]$, the error has a convergence rate $\mathcal{O}(h^{2-r})$, where r is given by

$$r = \begin{cases} \big(-1-\alpha(\beta-3)-2\gamma\big)/\alpha+\epsilon, & \text{if} \quad \alpha(\beta-3)/2+\gamma < -1/2, \\ \epsilon, & \text{if} \quad \alpha(\beta-3)/2+\gamma = -1/2, \\ 0, & \text{if} \quad \alpha(\beta-3)/2+\gamma > -1/2. \end{cases}$$

This can be obtained using (4.3) with $s=0$ and $\nu = \beta-1$ to bound $\int_0^t F_{\gamma,h}(t-s)dW(s)$.

5. Completely Discrete Scheme

This section is devoted to the analysis of a fully discrete scheme. The proposed scheme involves using a convolution quadrature generated by the backward Euler method to approximate the temporal fractional derivative and fractional integral.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0,T]$ and $t_n = n\tau, n = 0,\ldots,N$, where a time step size $\tau = T/N$. Assume $g^0 = 0$ and

$$g^k = \tau^{-1} P_h \Delta W^k, \quad \Delta W^k = W(t_k) - W(t_{k-1}), \quad k = 1,\ldots,N.$$

Applying the convolution quadrature, the fully discrete scheme reads: Find u_h^n, $n = 1,2,\ldots,N$, such that

$$\partial_\tau u_h^n + \big(1+\partial_\tau^{1-\alpha}\big) A_h u_h^n = P_h f\big(u_h^{n-1}\big) + \partial_\tau^{-\gamma} g^n, \quad u_h^0 = P_h u_0, \tag{5.1}$$

where ∂_τ^ν for $\nu = 1, 1-\alpha$ and $\nu = -\gamma$ are approximations of the Riemann-Liouville derivative and integral operators, respectively, at time t_n and

$$\partial_\tau^\nu v^n = \tau^{-\nu} \sum_{j=0}^n a_{n-j}^{(\nu)} v^j. \tag{5.2}$$

The quadrature weights $a_j^{(\nu)}$ are determined from the power series expansion (with $\delta(\zeta) = 1-\zeta$)

$$\delta(\zeta)^\nu = \sum_{j=0}^\infty a_j^{(\nu)} \zeta^j.$$

Hence, the numerical scheme (5.1) may written as

$$\begin{aligned} \tau^{-1}\big(u_h^n - u_h^{n-1}\big) + A_h u_h^n + \tau^{\alpha-1} \sum_{j=0}^n a_{n-j}^{(1-\alpha)} A_h u_h^j \\ = f_h\big(u_h^{n-1}\big) + \tau^\gamma \sum_{j=0}^n a_{n-j}^{(-\gamma)} g^j, \quad n \geq 1, \end{aligned} \tag{5.3}$$

where $f_h = P_h f$. We define operators R_j and Q_j such that

$$\sum_{j=0}^\infty R_j \zeta^j = \widetilde{R}(\zeta), \quad \widetilde{R}(\zeta) = 1 + \zeta\big(\tau^{-1}\delta(\zeta) + \big(\tau^{\alpha-1}\delta(\zeta)^{1-\alpha} + I\big)A_h\big)^{-1} \tau^{-1}, \tag{5.4}$$

$$\sum_{j=0}^\infty Q_j \zeta^j = \widetilde{Q}(\zeta), \quad \widetilde{Q}(\zeta) = 1 + \zeta\big(\tau^{-1}\delta(\zeta) + \big(\tau^{\alpha-1}\delta(\zeta)^{1-\alpha} + I\big)A_h\big)^{-1} \tau^{\gamma-1}\delta(\zeta)^{-\gamma}. \tag{5.5}$$

Then, using (5.3) we obtain

$$u_h^n = \bar{u}_h^n + \tau \sum_{j=1}^n R_{n-(j-1)} f_h\big(u_h^{j-1}\big) + \tau \sum_{j=1}^n Q_{n-(j-1)} g^j, \quad n \geq 1, \tag{5.6}$$

which may expressed as

$$u_h^n = \bar{u}_h^n + \sum_{j=0}^{n-1} R_{n-j} \int_{t_j}^{t_{j+1}} f_h\big(u_h^j\big)dt + \sum_{j=0}^{n-1} Q_{n-j} \int_{t_j}^{t_{j+1}} P_h dW(t). \tag{5.7}$$

Here, $\bar{u}_h^n$ is the discrete solution to the homogeneous problem in (5.3).

For the error analysis, we shall use a generalized variant of the standard discrete Gronwall's inequality, see [2] for $p = 2$.

Lemma 5.1. *Let $0 < \alpha < 1, p$ and N be integers, $\tau > 0$ and $t_n = n\tau$ for $0 \le n \le N$. Let $(y_n)_{n=1}^N$ be a non-negative sequence. Assume that there exist $\eta_1, \cdots, \eta_p \in [0, 1)$ and $s_1, \cdots, s_p, b \ge 0$ such that*

$$y_n \le \sum_{j=1}^{p} s_j t_n^{-\eta_j} + b\tau \sum_{j=1}^{n-1} t_{n-j}^{\alpha-1} y_j, \quad 1 \le n \le N. \tag{5.8}$$

Then there exists a constant $C = C(\eta_1, \cdots, \eta_p, \alpha, b, t_N)$ such that

$$y_n \le C \sum_{j=1}^{p} s_j t_n^{-\eta_j}, \quad 1 \le n \le N.$$

The operators R_j and Q_j satisfy the following smoothing properties and error estimates (deduced from [15, Theorem 2.1]).

Lemma 5.2. *For any $\beta \in [0, 1]$, we have*

$$\big\|A_h^{\frac{\beta}{2}} Q_n\big\| \le c t_{n+1}^{-\frac{\alpha\beta}{2}+\gamma}, \quad \big\|A_h^{\frac{\beta}{2}} \big(E_{\gamma,h}(t_n) - Q_n\big) P_h\big\| \le c\tau t_{n+1}^{-\frac{\alpha\beta}{2}+\gamma-1}.$$

Case $\gamma = 0$ gives the estimate for R_n .

To estimate the fully discrete error, we first write the error as

$$u(t_n) - u_h^n := \big(u(t_n) - u_h(t_n)\big) + \big(u_h(t_n) - u_h^n\big),$$

where the estimate of the semidiscrete error term $(u(t_n) - u_h(t_n))$ is given in Theorem 4.1. We split $u_h(t_n) - u_h^n$ as follows:

$$\begin{aligned} u_h(t_n) - u_h^n = {} & E_{0,h}(t_n) P_h u_0 - \bar{u}_h^n \\ & + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_{0,h}(t_n - t) f_h\big(u_h(t)\big) - R_{n-j} f_h\big(u_h^j\big)\big) dt \\ & + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_{\gamma,h}(t_n - t) - Q_{n-j}\big) P_h dW(t). \end{aligned}$$

Estimates of the terms on the right hand side are established in the coming lemmas.

Lemma 5.3 ([4, Remark 4.3]). *We have for $n \ge 1$,*

$$\big\|E_0(t_n) u_0 - \bar{u}_h^n\big\| \le c\big(\tau t_n^{\frac{\alpha\nu}{2}-1} + h^2 t_n^{-\alpha(1-\frac{\nu}{2})}\big) \|u_0\|_{\dot{H}^\nu}, \quad 0 \le \nu \le 2. \tag{5.9}$$

Lemma 5.4. *For $0 \le \beta \le 1$, there holds*

$$\left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_{\gamma,h}(t_n - t) - Q_{n-j}\big) P_h dW(t) \right\|_{L^2(\Omega;H)} \le c\tau^{\mu} t_n^{\max\{\sigma-\frac{1}{2},0\}} \big\|A^{\frac{\beta-1}{2}}\big\|_{\mathcal{L}_0^2},$$

where $\sigma = -\alpha(1-\beta)/2 + \gamma$ and

$$\mu := \begin{cases} \sigma + 1/2, & \text{if} \quad \sigma < 1/2, \\ 1-\epsilon, & \text{if} \quad \sigma = 1/2, \\ 1, & \text{if} \quad \sigma > 1/2. \end{cases}$$

Proof. We have

$$\begin{aligned} &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_{\gamma,h}(t_n - t) - Q_{n-j}\big) P_h dW(t) \\ =& \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_{\gamma,h}(t_n - t) - E_{\gamma,h}(t_n - t_j)\big) P_h dW(t) \\ &+ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_{\gamma,h}(t_n - t_j) - Q_{n-j}\big) P_h dW(t) =: I_1 + I_2. \end{aligned}$$

Then, by using Hölder's inequality and the smoothness of $E_\gamma(t)$,

$$\begin{aligned} \|I_1\|^2_{L^2(\Omega;H)} \le& \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big\|\big(E_{\gamma,h}(t_n - t) - E_{\gamma,h}(t_n - t_j)\big) P_h dW(t)\big\|^2_{L^2(\Omega;H)} \\ \le& \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \big\|\big(E_{\gamma,h}(t_n - t) - E_{\gamma,h}(t_n - t_j)\big) P_h dW(t)\big\|^2_{L^2(\Omega;H)} \\ &+ \int_0^\tau \big\|\big(E_{\gamma,h}(t) - E_{\gamma,h}(\tau)\big) P_h dW(t)\big\|^2_{L^2(\Omega;H)} \\ \le& \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \tau^2 \int_{t_j}^{t} \big\|A^{\frac{1-\beta}{2}} E'_{\gamma,h}(t_n - s)\big\|^2_{L^2(\Omega;H)} ds dt \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} \\ &+ \int_0^\tau \big\|A^{\frac{1-\beta}{2}} E_{\gamma,h}(t)\big\|^2_{L^2(\Omega;H)} \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} dt \\ &+ \int_0^\tau \big\|A^{\frac{1-\beta}{2}} E_{\gamma,h}(\tau)\big\|^2_{L^2(\Omega;H)} ds \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0}. \end{aligned}$$

It follows that

$$\begin{aligned} \|I_1\|^2_{L^2(\Omega;H)} \le& \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \tau^2 \int_{t_j}^{t} (t_n - s)^{2(-\frac{\alpha(1-\beta)}{2}+\gamma-1)} ds dt \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} \\ &+ \int_0^\tau t^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)} \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} dt + \int_0^\tau \tau^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)} ds \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} \\ \le& \int_\tau^{t_n} \tau^2 t^{2(-\frac{\alpha(1-\beta)}{2}+\gamma-1)} dt \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} \\ &+ \int_0^\tau t^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)} \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0} dt + \int_0^\tau \tau^{2(-\frac{\alpha(1-\beta)}{2}+\gamma)} ds \big\|A^{\frac{\beta-1}{2}}\big\|^2_{\mathcal{L}_2^0}, \end{aligned}$$

This implies that $\|I_1\|^2_{L^2(\Omega;H)} \le c\rho^2 \|A^{(\beta-1)/2}\|^2_{\mathcal{L}_2^0}$, where

$$\rho^2 := \begin{cases} \tau^{2\sigma+1}, & \text{if} \quad \sigma < 1/2, \\ \tau^{2-2\epsilon}, & \text{if} \quad \sigma = 1/2, \\ \tau^2 t_n^{2\sigma-1}, & \text{if} \quad \sigma > 1/2, \end{cases}$$

and $\sigma = -\alpha(1-\beta)/2 + \gamma$.

For I_2, we use the second inequality in Lemma 5.2 to get

$$\begin{aligned} \|I_2\|^2_{L^2(\Omega;H)} &\le \tau \sum_{j=0}^{n-1} \big\| A^{\frac{1-\beta}{2}} \big(E_{\gamma,h}(t_n - t_j) - Q_{n-j} \big) \big\|^2_{L^2(\Omega;H)} \big\| A^{\frac{\beta-1}{2}} \big\|^2_{\mathcal{L}_2^0} \\ &\le c\tau^3 \sum_{j=0}^{n-1} t_{n-j+1}^{2(\sigma-1)} \big\| A^{\frac{\beta-1}{2}} \big\|^2_{\mathcal{L}_2^0} \\ &\le c\rho^2 \big\| A^{\frac{\beta-1}{2}} \big\|^2_{\mathcal{L}_2^0}, \end{aligned}$$

which completes the proof. $\square$

Lemma 5.5. *The following holds:*

$$\begin{aligned} &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big\| E_{0,h}(t_n - t) f_h\big(u_h(t)\big) - R_{n-j} f_h\big(u_h^j\big) \big\|_{L^2(\Omega;H)} dt \\ &\le c\tau^{\min\{1, \theta - \frac{1}{2}\}} + c\tau^{1-\epsilon} + c\tau \sum_{j=0}^{n-1} \big\| u_h(t_j) - u_h^j \big\|_{L^2(\Omega;H)}. \end{aligned}$$

Proof. We split the integrand as follows:

$$\begin{aligned} &E_{0,h}(t_n - t) f_h\big(u_h(t)\big) - R_{n-j} f_h\big(u_h^j\big) \\ &= \big(E_{0,h}(t_n - t) - E_{0,h}(t_n - t_j) \big) f_h\big(u_h(t)\big) \\ &\quad + E_{0,h}(t_n - t_j) \big(f_h\big(u_h(t)\big) - f_h\big(u_h(t_j)\big) \big) \\ &\quad + E_{0,h}(t_n - t_j) \big(f_h\big(u_h(t_j)\big) - f_h\big(u_h^j\big) \big) \\ &\quad + \big(E_{0,h}(t_n - t_j) - R_{n-j} \big) f_h\big(u_h^j\big) =: \sum_{k=1}^{4} I_k. \end{aligned}$$

Now, using Lemma 3.1, we obtain

$$\begin{aligned} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|I_1\|_{L^2(\Omega;H)} dt &\le \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \|I_1\|_{L^2(\Omega;H)} dt + \int_{t_{n-1}}^{t_n} \|I_1\|_{L^2(\Omega;H)} dt \\ &\le \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t} \big\| E'_{0,h}(t_n - s) f_h\big(u_h(t)\big) \big\|_{L^2(\Omega;H)} ds dt \\ &\quad + \int_{t_{n-1}}^{t_n} \big\| E_{0,h}(t_n - t) f_h\big(u_h(t)\big) \big\|_{L^2(\Omega;H)} dt \end{aligned}$$

$$
\begin{aligned}
&+\int_{t_{n-1}}^{t_n}\big\|E_{0,h}(\tau)f_h\big(u_h(t)\big)\big\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-2}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t}(t_n-s)^{-1}dsdt+c\int_{t_{n-1}}^{t_n}dt+c\int_{t_{n-1}}^{t_n}dt\\
&\leq c\sum_{j=0}^{n-2}\int_{t_j}^{t_{j+1}}\tau(t_n-t)^{-1}dt+c\tau\\
&\leq c\int_0^{t_{n-1}}\tau(t_n-t)^{-1}dt+c\tau\leq c\tau^{1-\epsilon}.
\end{aligned}
$$

For the second term I_2, we use Lemma 3.1, (1.2) and (3.6) to obtain

$$
\begin{aligned}
\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|I_2\|_{L^2(\Omega;H)}dt &\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big\|E_{0,h}(t_n-t_j)\big(f_h\big(u_h(t)\big)-f_h\big(u_h(t_j)\big)\big)\big\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big\|f_h\big(u_h(t)\big)-f_h\big(u_h(t_j)\big)\big\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|u_h(t)-u_h(t_j)\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-t_j)^{\min\{1,\theta-\frac{1}{2}\}}t^{\frac{\alpha\nu}{2}-1}dt\\
&\leq c\sum_{j=0}^{n-1}\tau^{\min\{1,\theta-\frac{1}{2}\}}\int_{t_j}^{t_{j+1}}t^{\frac{\alpha\nu}{2}-1}dt\\
&\leq c\sum_{j=0}^{n-1}\tau^{\min\{1,\theta-\frac{1}{2}\}}\left(t_{j+1}^{\frac{\alpha\nu}{2}}-t_j^{\frac{\alpha\nu}{2}}\right)\\
&\leq ct_n^{\frac{\alpha\nu}{2}}\tau^{\min\{1,\theta-\frac{1}{2}\}}.
\end{aligned}
$$

To bound I_3, we use Lemma 3.1 and (1.2) so that

$$
\begin{aligned}
\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|I_3\|_{L^2(\Omega;H)}dt &\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big\|E_{0,h}(t_n-t_j)\big(f_h\big(u_h(t_j)\big)-f_h\big(u_h^j\big)\big)\big\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big\|u_h(t_j)-u_h^j\big\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-1}\tau\big\|u_h(t_j)-u_h^j\big\|_{L^2(\Omega;H)}.
\end{aligned}
$$

For the last term, we use Lemmas 3.1 and 5.2 to get

$$
\begin{aligned}
\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|I_4\|_{L^2(\Omega;H)}dt &\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big\|\big(E_{0,h}(t_n-t_j)-R_{n-j}\big)f_h\big(u_h^j\big)\big\|_{L^2(\Omega;H)}dt\\
&\leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}t_{n-j}^{-1}\tau dt\leq c\tau\sum_{j=0}^{n-1}\tau t_{n-j}^{-1}
\end{aligned}
$$

$$= c\tau \sum_{j=1}^{n} \tau t_j^{-1} \le c\tau^{1-\epsilon}.$$

Altogether give the desired estimate. □

Now, we are ready to provide the final bound for the fully discrete error.

Theorem 5.1. *Let Assumptions* 2.1 *and* 2.2 *hold, and* u *and* u_h^n *be the solutions of* (1.1) *and* (5.1)*, respectively. Then,*

$$\left\|u(t_n) - u_h^n\right\|_{L^2(\Omega;H)} \le c\,h^{1+\beta-r} t_n^{-\frac{\alpha(2-\nu)}{2}} + c\tau t_n^{\frac{\alpha\nu}{2}-1} + c\tau^{\min\{\theta-\frac{1}{2},1-\epsilon\}} + c\tau^{\mu} t_n^{\max\{\sigma-\frac{1}{2},0\}},$$

where r *is given by* (4.6) *and* $\epsilon > 0$ *is a small real number.*

Proof. The estimate follows by using the results in previous lemmas and a discrete Gronwall's inequality. □

6. Numerical Experiments

In this section, we validate our theoretical results through the examination of one-dimensional numerical examples. The parameter τ is defined as T/N, and the interval $\mathcal{D} = (0,1)$ is divided into M equal parts with size $h = 1/M$. We consider Eq. (1.1) with $f(u) = \sqrt{u^2+1}$ and the following initial data:

(a) $u_0(x) = 0$,

(b) $u_0 = x\chi_{[0,1/2]}(x) + (1-x)\chi_{(1/2,1]}(x)$, $u_0 \in L^2(\Omega; \dot{H}^{1+\delta})$, $0 \le \delta < 1/2$,

where χ_S is a characteristic function of the set S.

To implement the stochastic process

$$W(t) = \sum_{j=1}^{\infty} \eta_j^{\frac{1}{2}} \beta_j(t) e_j(x),$$

where β_j represents the Brownian motions, we set $\eta_j = j^{-m}$ for $m \ge 0$ and assume that the operator Q possesses the same eigenfunctions as the operator A, meaning $e_j = \sqrt{2}\sin(j\pi x)$. Then,

$$P_h \dot{W}(t_k) \approx \frac{P_h W(t_k) - P_h W(t_{k-1})}{\tau} \approx \sum_{j=1}^{L} \eta_j^{\frac{1}{2}} e_j(x) \frac{\Delta\beta_j^k}{\tau},$$

where L is appropriately chosen to satisfy the desired convergence and $\Delta\beta_j^k = \sqrt{\tau}\mathcal{N}(0,1)$ with $\mathcal{N}$ being the standard normal distribution.

Hence, the approximation of $\partial_t^{-\gamma} P_h \dot{W}(t_k)$ using the backward Euler convolution quadrature is given by

$$\partial_t^{-\gamma} P_h \dot{W}(t_k) \approx \tau^{\gamma} \sum_{k=1}^{n} a_{n-k}^{(-\gamma)} \sum_{j=1}^{L} \eta_j^{\frac{1}{2}} e_j(x) \frac{\Delta\beta_j^k}{\tau}.$$

In all numerical experiments, we set $L = M$. We start by examining the results in the temporal direction. We have fixed $M = 100$ and a final time of $T = 0.01$, and the reference solution is obtained using a much finer temporal mesh of $N = 3200$. Table 6.1 displays the errors for cases (a) and (b) with different values of α, γ, and m, which are calculated based on the average of 100 trajectories. The "Rate" refers to the empirical convergence rate, with the theoretical estimate included in brackets. The level of noise regularity is represented by m, with $m = 0$ signifying $\beta = 1/2$ and $m = 1$ corresponding to $\beta = 1$. According to Theorem 5.1, the temporal convergence is

$$\left\| u(t_n) - u_h^n \right\|_{L^p(\Omega;H)} \leq c\tau^{\min\{-\frac{\alpha(1-\beta)}{2}+\gamma+\frac{1}{2},1-\epsilon\}}.$$

The numerical results align closely with the theoretical predictions.

For the assessment of the spatial convergence rate, we set $N = 500$ and the final time $T = 0.1$. The numerical results for cases (a) and (b) with $\gamma = 0.2$ and different values of m and α are presented in Table 6.2. For $m = 0$ and $m = 1$, our theoretical estimate predicts that the error will behave as $\mathcal{O}(h^{1+\beta})$ for $\alpha = 0.3$ and $\alpha = 0.5$, and as $\mathcal{O}(h^{1+\beta-(-1+2\alpha-2\gamma)/\alpha})$ when $\alpha = 0.9$. The computed errors are slightly higher than our theoretical estimates. For $m = 2$ ($\beta = 3/2$), the numerical computations show a convergence rate of order $\mathcal{O}(h^2)$, which is consistent with the observation in Remark 4.1.

Table 6.1: Temporal convergence rates for cases (a) and (b) with different values of α and γ at $T = 0.01$, $M = 100$ and 100 trajectories.

γ	m	α	case/N	20	40	80	160	320	Rate
$\gamma = 0.1$	$m = 0$	0.3	(a)	4.86e-4	3.36e-4	2.27e-4	1.51e-4	1.06e-4	0.55 (0.525)
			(b)	7.05e-4	4.21e-4	2.60e-4	1.65e-4	1.11e-4	0.66 (0.525)
		0.5	(a)	1.91e-3	1.32e-3	9.32e-4	6.49e-4	4.41e-4	0.53 (0.475)
			(b)	2.19e-3	1.42e-3	9.69e-4	6.64e-4	4.46e-4	0.57 (0.475)
		0.9	(a)	9.44e-3	7.09e-3	5.15e-3	3.65e-3	2.45e-3	0.49 (0.375)
			(b)	9.44e-3	7.10e-3	5.15e-3	3.65e-3	2.45e-3	0.49 (0.375)
	$m = 1$	0.3	(a)	3.51e-4	2.31e-4	1.48e-4	9.49e-5	6.64e-5	0.60 (0.60)
			(b)	6.20e-4	3.43e-4	1.94e-4	1.15e-4	7.42e-5	0.76 (0.60)
		0.5	(a)	1.04e-3	6.87e-4	4.60e-4	3.14e-4	2.04e-4	0.59 (0.60)
			(b)	1.48e-3	8.64e-4	5.31e-4	3.43e-4	2.15e-4	0.70 (0.60)
		0.9	(a)	2.74e-3	1.84e-3	1.20e-3	7.77e-4	4.80e-4	0.63(0.60)
			(b)	2.77e-3	1.85e-3	1.20e-3	7.78e-4	4.81e-04	0.63 (0.60)
$\gamma = 0.5$	$m = 0$	0.3	(a)	7.73e-5	4.51e-5	2.55e-5	1.51e-5	8.27e-6	0.81 (0.925)
			(b)	4.94e-4	2.45e-4	1.21e-4	5.94e-5	2.80e-5	1.03 (0.925)
		0.5	(a)	1.60e-4	9.51e-5	5.38e-5	3.10e-5	1.75e-5	0.80 (0.875)
			(b)	1.10e-3	5.47e-4	2.71e-4	1.32e-4	6.26e-5	1.03 (0.875)
		0.9	(a)	4.27e-4	2.57e-4	1.51e-4	8.88e-5	5.10e-5	0.77 (0.775)
			(b)	6.66e-4	3.61e-4	1.96e-4	1.07e-4	5.81e-5	0.88 (0.775)
	$m = 1$	0.3	(a)	7.46e-5	4.32e-5	2.44e-5	1.45e-5	7.83e-6	0.81 ($\approx$ 1.00)
			(b)	4.93e-4	2.44e-4	1.21e-4	5.92e-5	2.79e-5	1.04 ($\approx$ 1.00)
		0.5	(a)	1.50e-4	8.70e-5	4.81e-5	2.74e-5	1.49e-5	0.83 ($\approx$ 1.00)
			(b)	1.10e-3	5.45e-4	2.70e-4	1.32e-4	6.19e-5	1.04 ($\approx$ 1.00)
		0.9	(a)	2.85e-4	1.61e-4	8.75e-5	4.80e-5	2.61e-5	0.86 ($\approx$ 1.00)
			(b)	5.86e-4	3.01e-4	1.53e-4	7.69e-5	3.81e-5	0.99 ($\approx$ 1.00)

Table 6.2: Spatial convergence rates for $\gamma = 0.2$ with different values of α, at $T = 0.1$, $N = 500$ and 100 trajectories.

m	α	case/M	20	40	80	160	320	Rate
$m = 0$	0.3	(a)	5.91e-4	2.11e-4	7.45e-5	2.51e-5	7.26e-6	1.62 (1.50)
		(b)	6.01e-4	2.13e-4	7.48e-5	2.51e-5	7.27e-6	1.62 (1.50)
	0.5	(a)	9.92e-4	3.58e-4	1.26e-4	4.26e-5	1.24e-5	1.62 (1.50)
		(b)	1.00e-3	3.59e-4	1.27e-4	4.27e-5	1.24e-5	1.62 (1.50)
	0.9	(a)	3.93e-3	1.60e-3	6.11e-4	2.16e-4	6.45e-5	1.54 (1.06)
		(b)	3.94e-3	1.60e-3	6.11e-4	2.16e-4	6.45e-5	1.54 (1.06)
$m = 1$	0.3	(a)	2.60e-4	7.24e-5	1.97e-5	5.12e-6	1.16e-6	1.99 (2.00)
		(b)	2.82e-4	7.75e-5	2.08e-5	5.40e-6	1.24e-6	1.99 (2.00)
	0.5	(a)	4.13e-4	1.17e-4	3.20e5	8.38e-6	1.91e-6	1.98 (2.00)
		(b)	4.34e-4	1.22e-4	3.31e-5	8.64e-6	1.98e-6	1.98 (2.00)
	0.9	(a)	1.26e-3	4.02e-4	1.20e-4	3.36e-5	8.02e-6	1.88 (1.56)
		(b)	1.28e-3	4.06e-4	1.21e-4	3.38e-5	8.06e-6	1.88 (1.56)
$m = 2$	0.3	(a)	1.75e-4	4.47e-5	1.12e-5	2.72e-6	5.94e-7	2.08 (2.00)
		(b)	2.08e-4	5.26e-5	1.32e-5	3.23e-6	7.35e-7	2.05 (2.00)
	0.5	(a)	2.61e-4	6.67e-5	1.68e-5	4.09e-6	8.93e-7	2.07 (2.00)
		(b)	2.94e-4	7.49e-5	1.88e-5	4.60e-6	1.03e-6	2.06 (2.00)
	0.9	(a)	5.75e-4	1.55e-4	4.04e-5	1.00e-5	2.21e-6	2.04 (2.00)
		(b)	6.22e-4	1.66e-4	4.30e-5	1.06e-5	2.35e-6	2.05 (2.00)

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