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ERROR ANALYSIS OF FRACTIONAL COLLOCATION METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH NONCOMPACT OPERATORS*

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Abstract

This paper is concerned with the numerical solution of Volterra integro-differential equations with noncompact operators. The focus is on the problems with weakly singular solutions. To handle the initial weak singularity of the solution, a fractional collocation method is applied. A rigorous *hp*-version error analysis of the numerical method under a weighted H^1-norm is carried out. The result shows that the method can achieve high order convergence for such equations. Numerical experiments are also presented to confirm the effectiveness of the proposed method.

Mathematics subject classification: L05, L60.
Key words: Volterra integro-differential equation, Noncompact operator, Nonsmooth solution, Collocation method, Fractional polynomial, *hp*-version error analysis.

1. Introduction

Volterra integro-differential equations (VIDEs) arise in mathematical models of many different research fields, such as population models [28], viscoelastic phenomena [19], capillarity theory [4]. In this paper, we consider the VIDEs of the form

$$t^{\gamma}u'(t)=a(t)u(t)+g(t)+\int_0^t (t-s)^{-\mu}s^{\mu+\gamma-1}K(t,s)u(s)ds, \quad t\in I:=[0,T] \tag{1.1}$$

with the initial condition $u(0)=u_0$, where $0\le\mu<1, \gamma>0, \mu+\gamma\ge 1$, and $a(t), g(t)$ and $K(t,s)$ are given smooth functions. The Eq. (1.1) can be equivalently written as the following

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cordial Volterra integro-differential equation (CVIDE):

$$u'(t) = a_\gamma(t)u(t) + g_\gamma(t) + (K_{\mu,\gamma}u)(t), \quad t \in I := [0,T], \tag{1.2}$$

where

$$a_\gamma(t) = t^{-\gamma}a(t), \quad g_\gamma(t) = t^{-\gamma}g(t),$$

and

$$(K_{\mu,\gamma}u)(t) = t^{-\gamma}\int_0^t (t-s)^{-\mu}s^{\mu+\gamma-1}K(t,s)u(s)ds.$$

The operator $K_{\mu,\gamma}$ can be viewed as a cordial operator defined in [29, 30], where the author pointed out that such an operator is noncompact if $K(0,0) \neq 0$. One can also see that when $\gamma = 0$, the Eq. (1.2) reduces to a second kind VIDE.

There have been many researches on the numerical solution for several different classes of Volterra integral or integro-differential equations, such as collocation methods [3, 9, 11, 13, 26, 36, 38, 39], discontinuous Galerkin methods [17], block boundary value methods [40], spectral collocation methods [6, 10, 35, 37], spectral Galerkin methods [8, 27], *hp*-version collocation or Galerkin methods [15, 16, 32, 33]. All the above studies focus on the second kind Volterra-type equations.

For third kind VIDEs with form (1.1) or (1.2), however, there are only few works. One can see the study for the case that $\mu = 0$ in [12] and for the case that $K_{\mu,\gamma}$ is compact in [22]. For CVIDEs with noncompact cordial operators, continuous piecewise-polynomial collocation methods were considered in [25], where the convergence and superconvergence of the method were analysed. Based on smooth transformation, the Legendre spectral collocation method was employed in [14]. A related topic is the numerical solution of Volterra integral equations (VIEs) with the integral operator $K_{\mu,\gamma}$, which are also referred as third-kind VIEs [5, 18, 20]. For the latter, collocation methods [1, 31], multistep collocation methods [21] and Legendre Galerkin spectral methods [2] have received attention. Recently, an *hp*-version method, which can provide a flexible choice of locally varying time steps and approximation orders, was developed for solving third-kind VIEs in [34]. To the best of our knowledge, *hp*-version methods have not been considered for the VIDEs of the form (1.2), although such kind of methods has been widely studied for solving second kind VIDEs and VIEs.

In this paper, we apply a fractional collocation method to the Eq. (1.2) with noncompact cordial operator and nonsmooth solution. The method is based on piecewise fractional polynomial collocation with fractional exponent λ $(0 < \lambda \leq 1)$ which is a user-chosen parameter. Different from the classical polynomial collocation method, the approximation spaces for fractional collocation method are constructed using fractional polynomials of the form $\sum_{k=0}^{N} c_k t^{k\lambda}$ instead of standard polynomials. The motivation for using such spaces is that when the solution exhibits weak singularity at the initial point $t = 0$, the present fractional approximation spaces with a suitable λ can match well with this kind of singularity appearing in the solutions. We mention that such spaces have previously been used for the numerical solution of second kind weakly singular VIDEs, see for example [8, 9, 16].

For the proposed fractional collocation method, an *hp*-error estimate is established under a weighted H^1-norm. The error bound explicitly depends on the local time steps, the local approximation orders, and the regularity of $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$. Notice that for typical weakly singular solutions, $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ can have a better regularity than the original solution $u(t)$ for suitable λ. This means that fractional collocation can achieve high order of convergence even for weakly singular solutions.

This paper is organized as follows. In Section 2, we introduce some notations and give our numerical scheme for (1.2). In Section 3, some technical lemmas are derived. An hp-version error estimate of the proposed method is given in Section 4. The effectiveness of the proposed method is demonstrated by numerical experiments in Section 5. Finally, some concluding remarks are given in Section 6.

2. Fractional Collocation Method

In this section, we propose a fractional collocation scheme for solving the Eq. (1.2) numerically. Firstly, we give some notations.

2.1. Preliminaries

Let $M \in \mathbb{N}$. Define a mesh

$$\mathcal{T}_M = \{t_i : 0 = t_0 < t_1 < \cdots < t_M = T\}$$

on the interval I and set $\sigma_i = (t_{i-1}, t_i], h_i = t_i - t_{i-1}$. We introduce the piecewise fractional polynomial space $S_\lambda(\mathcal{T}_M)$,

$$S_\lambda(\mathcal{T}_M) := \left\{w(t) : w(t)|_{t\in\sigma_i} \in P^\lambda_{N_i},\ i = 1, \ldots, M\right\},$$

where $0 < \lambda \le 1$ and $P^\lambda_{N_i} := \text{span}\{1, t^\lambda, \cdots, t^{N_i\lambda}\}$ with $\{N_i\}_{i=1}^M$ being a set of natural numbers. For any function $v(t)$ defined on I, we denote by $v_i(t)$ the function $v(t)$ on σ_i.

Let $\{c^{\alpha,\beta}_{i,k}, \theta^{\alpha,\beta}_{i,k}\}_{k=0}^{N_i}$ be the standard Jacobi-Gauss quadrature nodes and weights on $\Lambda := [-1,1]$. Let $\hat{\sigma}_i := (t^\lambda_{i-1}, t^\lambda_i]$ and $h_{i,\lambda} = t^\lambda_i - t^\lambda_{i-1}$. Define $\{\xi^{\alpha,\beta}_{i,k}\}_{k=0}^{N_i} \in \hat{\sigma}_i$ and $\{\omega^{\alpha,\beta}_{i,k}\}_{k=0}^{N_i}$ as follows:

$$\xi^{\alpha,\beta}_{i,k} := \frac{1}{2}\big(t^\lambda_{i-1} + t^\lambda_i + c^{\alpha,\beta}_{i,k} h_{i,\lambda}\big), \quad \omega^{\alpha,\beta}_{i,k} := \theta^{\alpha,\beta}_{i,k}\left(\frac{h_{i,\lambda}}{2}\right)^{1+\alpha+\beta}, \tag{2.1}$$

and the associated Lagrange basis polynomials $\{\hat{L}^{\alpha,\beta}_{i,k}\}_{k=0}^{N_i}$ on $\hat{\sigma}_i$ are defined by

$$\hat{L}^{\alpha,\beta}_{i,k}(s) = \prod_{j=0,j\neq k}^{N_i} \frac{s - \xi^{\alpha,\beta}_{i,j}}{\xi^{\alpha,\beta}_{i,k} - \xi^{\alpha,\beta}_{i,j}}, \quad s \in \hat{\sigma}_i, \quad k = 0, \ldots, N_i. \tag{2.2}$$

Then, we can define the corresponding interpolation operators $\hat{I}^{\alpha,\beta}_{N_i,i} : C(\hat{\sigma}_i) \to P^1_{N_i}(\hat{\sigma}_i)$ for $i = 1, \ldots, M$ by

$$\big(\hat{I}^{\alpha,\beta}_{N_i,i} z\big)(s) := \sum_{k=0}^{N_i} \hat{L}^{\alpha,\beta}_{i,k}(s) z\big(\xi^{\alpha,\beta}_{i,k}\big), \quad s \in \hat{\sigma}_i.$$

Next, we introduce a nonlinear transformation $\rho(s) := s^{1/\lambda}$ and for any function w, let $\hat{w}(s) := w(\rho(s))$. The collocation points $\{t^{\alpha,\beta}_{i,k}\}_{k=0}^{N_i} \in \sigma_i$ are given by

$$X_i := \left\{t^{\alpha,\beta}_{i,k} : t^{\alpha,\beta}_{i,k} = \rho\big(\xi^{\alpha,\beta}_{i,k}\big),\ k = 0, \ldots, N_i\right\}. \tag{2.3}$$

Define fractional basis functions $\{L^{\lambda,\alpha,\beta}_{i,k}\}_{k=0}^{N_i}$ on σ_i by

$$L^{\lambda,\alpha,\beta}_{i,k}(t) := \prod_{j=0,j\neq k}^{N_i} \frac{t^\lambda - \big(t^{\alpha,\beta}_{i,j}\big)^\lambda}{\big(t^{\alpha,\beta}_{i,k}\big)^\lambda - \big(t^{\alpha,\beta}_{i,j}\big)^\lambda}, \quad t \in \sigma_i, \quad k = 0, \ldots, N_i.$$

The corresponding interpolation operator $I^{\lambda,\alpha,\beta}_{N_i,i}: C(\sigma_i)\to P^{\lambda}_{N_i}(\sigma_i)$ is defined as

$$\big(I^{\lambda,\alpha,\beta}_{N_i,i}w\big)(t):=\sum_{k=0}^{N_i}L^{\lambda,\alpha,\beta}_{i,k}(t)w\big(t^{\alpha,\beta}_{i,k}\big),\quad t\in\sigma_i.$$

By $t=\rho(s)$, one has for $t\in\sigma_i$,

$$\big(I^{\lambda,\alpha,\beta}_{N_i,i}w\big)(t)=\sum_{k=0}^{N_i}L^{\lambda,\alpha,\beta}_{i,k}(t)w\big(t^{\alpha,\beta}_{i,k}\big)=\sum_{k=0}^{N_i}\hat{L}^{\alpha,\beta}_{i,k}(s)w\big(\rho\big(\xi^{\alpha,\beta}_{i,k}\big)\big)=\big(\hat{I}^{\alpha,\beta}_{N_i,i}\hat{w}\big)(s). \tag{2.4}$$

Let $\{c^L_{i,k},\theta^L_{i,k}\}_{k=0}^{N_i+1}$ be the standard Legendre-Gauss-Lobatto quadrature nodes and weights on Λ. Replace $c^{\alpha,\beta}_{i,k}$ in (2.1) with $c^L_{i,k}$. Similarly as in (2.1)-(2.2), we define

$$\xi^L_{i,k}:=\frac{1}{2}\big(t^{\lambda}_{i-1}+t^{\lambda}_i+c^L_{i,k}h_{i,\lambda}\big),\quad \omega^L_{i,k}=\frac{1}{2}\theta^L_{i,k}h_{i,\lambda},$$
$$\hat{L}^L_{i,k}(s):=\prod_{j=0,j\neq k}^{N_i}\frac{s-\xi^L_{i,j}}{\xi^L_{i,k}-\xi^L_{i,j}},\qquad s\in\hat{\sigma}_i,\quad k=0,\ldots,N_i+1,$$

and the interpolation operator $\hat{I}^L_{N_i+1,i}: C(\hat{\sigma}_i)\to P^1_{N_i+1}(\hat{\sigma}_i)$

$$\big(\hat{I}^L_{N_i+1,i}w\big)(s):=\sum_{k=0}^{N_i+1}\hat{L}^L_{i,k}(s)w\big(\xi^L_{i,k}\big).$$

Using $t=\rho(s)$, we further define

$$t^L_{i,k}=\rho\big(\xi^L_{i,k}\big),\quad L^{\lambda,L}_{i,k}(t)=\prod_{j=0,j\neq k}^{N_i}\frac{t^{\lambda}-\big(t^L_{i,j}\big)^{\lambda}}{\big(t^L_{i,k}\big)^{\lambda}-\big(t^L_{i,j}\big)^{\lambda}},\quad t\in\sigma_i,\quad k=0,\ldots,N_i+1,$$

and the interpolation operator $I^{\lambda,L}_{N_i+1,i}: C(\sigma_i)\to P^{\lambda}_{N_i+1}(\sigma_i)$

$$\big(I^{\lambda,L}_{N_i+1,i}w\big)(t)=\sum_{k=0}^{N_i+1}L^{\lambda,L}_{i,k}(t)w(t_{i,k}).$$

Similar to (2.4), one can easily verify that for $t=\rho(s)$

$$\big(I^{\lambda,L}_{N_i+1,i}w\big)(t)=\big(\hat{I}^L_{N_i+1,i}\hat{w}\big)(s). \tag{2.5}$$

Throughout this paper, let c be a generic positive constant which is independent of diameter of the mesh and local approximation orders N_i. Note that it may have different values in different places.

2.2. The fractional collocation scheme

With the above preparations, we now give the fractional collocation scheme for solving (1.2) numerically. For the prescribed collocation points $\{X_i\}_{i=1}^M$, find functions $V\in S_\lambda(\mathcal{T}_M)$ and U such that for $i=1,2,\ldots,M$,

$$\begin{aligned}V(t)=a_\gamma(t)U_i(t)+g_\gamma(t)+\sum_{j=1}^{i-1}t^{-\gamma}\int_{t_{j-1}}^{t_j}(t-s)^{-\mu}s^{\mu+\gamma-1}I^{\lambda,L}_{N_j+1,j}\big(K(t,s)U_j(s)\big)ds\\+t^{-\gamma}\int_{t_{i-1}}^{t}(t-s)^{-\mu}s^{\mu+\gamma-1}I^{\lambda,L}_{N_i+1,i}\big(K(t,s)U_i(s)\big)ds,\quad t\in X_i,\end{aligned} \tag{2.6a}$$

$$U_i(t) = U_{i-1}(t_{i-1}) + \int_{t_{i-1}}^{t} V_i(s)ds, \quad t \in \sigma_i \tag{2.6b}$$

with $U_0(t_0) = u_0$. It follows from (2.6b) that the numerical solution $U(t)$ is continuous. Since $V \in S_\lambda(\mathcal{T}_M)$, we write V in the following local representation:

$$V(t) = \sum_{l=0}^{N_i} L_{i,l}^{\lambda,\alpha,\beta}(t) V\big(t_{i,l}^{\alpha,\beta}\big), \quad t \in \sigma_i.$$

Combining this local representation with (2.6b) gives

$$U_i(t) = U_{i-1}(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}(t) V_{i,l},$$

where $V_{i,l} = V_i(t_{i,l}^{\alpha,\beta})$ and

$$\beta_{i,l}(t) = \int_{t_{i-1}}^{t} L_{i,l}^{\lambda,\alpha,\beta}(s)ds.$$

Then collocation scheme (2.6a) becomes

$$\begin{aligned} V_{i,k} = {} & a_\gamma\big(t_{i,k}^{\alpha,\beta}\big)\left(U_{i-1}(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}\big(t_{i,k}^{\alpha,\beta}\big) V_{i,l}\right) + g_\gamma\big(t_{i,k}^{\alpha,\beta}\big) + \sum_{j=1}^{i-1} \sum_{p=0}^{N_j+1} U_j\big(t_{j,p}^{L}\big) \phi_{k,p}^{i,j} \\ & + \sum_{p=0}^{N_i+1} \left(U_{i-1}(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}\big(t_{i,p}^{L}\big) V_{i,l}\right) \phi_{k,p}^{i,i}, \quad k = 0, \ldots, N_i, \end{aligned} \tag{2.7}$$

where

$$\phi_{k,p}^{i,j} = \begin{cases} \big(t_{i,k}^{\alpha,\beta}\big)^{-\gamma} K\big(t_{i,k}^{\alpha,\beta}, t_{j,p}^{L}\big) \displaystyle\int_{t_{j-1}}^{t_j} \big(t_{i,k}^{\alpha,\beta} - s\big)^{-\mu} s^{\mu+\gamma-1} L_{j,p}^{\lambda,L}(s)ds, & 1 \le j \le i-1, \\ \big(t_{i,k}^{\alpha,\beta}\big)^{-\gamma} K\big(t_{i,k}^{\alpha,\beta}, t_{i,p}^{L}\big) \displaystyle\int_{t_{i-1}}^{t_{i,k}^{\alpha,\beta}} \big(t_{i,k}^{\alpha,\beta} - s\big)^{-\mu} s^{\mu+\gamma-1} L_{i,p}^{\lambda,L}(s)ds, & j = i. \end{cases}$$

The system (2.7) can be equivalently written in the following matrix form:

$$\big(\mathbb{I}_{N_i+1} - A_i B_i - \Phi^{i,i} B_i^L\big) V_i = U_{i-1}(t_{i-1})\big(A_i + \Phi^{i,i}\big) r + G_i + \sum_{j=1}^{i-1} \Phi^{i,j} U_j, \tag{2.8}$$

where $\mathbb{I}_{N_i+1}$ is the identity matrix of order $N_i + 1$,

$$G_i := \big(g_\gamma\big(t_{i,0}^{\alpha,\beta}\big), \cdots, g_\gamma\big(t_{i,N_i}^{\alpha,\beta}\big)\big)^T, \quad r := (1, \cdots, 1)^T$$

and

$$\begin{aligned} & V_i := (V_{i,0}, \cdots, V_{i,N_i})^T, && U_j := \big(U_j\big(t_{j,0}^L\big), \cdots, U_j\big(t_{j,N_j+1}^L\big)\big)^T, \\ & A_i := \mathrm{diag}\big(a_\gamma\big(t_{i,0}^{\alpha,\beta}\big), \cdots, a_\gamma\big(t_{i,N_i}^{\alpha,\beta}\big)\big), && \Phi^{i,j} := \big(\phi_{k,p}^{i,j}\big)_{\substack{k=0,\ldots,N_i \\ p=0,\ldots,N_j+1}}, \\ & B_i := \big(\beta_{i,l}\big(t_{i,k}^{\alpha,\beta}\big)\big)_{\substack{k=0,\ldots,N_i \\ l=0,\ldots,N_i}}, && B_i^L := \big(\beta_{i,l}\big(t_{i,p}^{L}\big)\big)_{\substack{p=0,\ldots,N_i+1 \\ l=0,\ldots,N_i}}. \end{aligned}$$

When V_i is solved by (2.8), the collocation solution on σ_i can be obtained by (2.6b).

Next, we discuss the solvability of the scheme (2.8). For any fixed N_i, by the definition of B_i and B_i^L, we derive

$$\|B_i\|_\infty = \max_{0\le k\le N_i}\sum_{l=0}^{N_i}\big|\beta_{i,l}\big(t_{i,k}^{\alpha,\beta}\big)\big| \le \max_{0\le k\le N_i}\int_{t_{i-1}}^{t_{i,k}^{\alpha,\beta}}\sum_{l=0}^{N_i}\big|L_{i,l}^{\lambda,\alpha,\beta}(s)\big|ds \le ch_i, \tag{2.9}$$

$$\big\|B_i^L\big\|_\infty = \max_{0\le p\le N_i+1}\sum_{l=0}^{N_i}\big|\beta_{i,l}\big(t_{i,p}^{L}\big)\big| \le \max_{0\le p\le N_i+1}\int_{t_{i-1}}^{t_{i,p}^{L}}\sum_{l=0}^{N_i}\big|L_{i,l}^{\lambda,\alpha,\beta}(s)\big|ds \le ch_i. \tag{2.10}$$

For $\Phi^{i,i}$, one has

$$\begin{aligned}\|\Phi^{i,i}\|_\infty &\le \max_{0\le k\le N_i}\sum_{p=0}^{N_i+1}\big(t_{i,k}^{\alpha,\beta}\big)^{-\gamma}\big|K\big(t_{i,k}^{\alpha,\beta},t_{i,p}^{L}\big)\big|\int_{t_{i-1}}^{t_{i,k}^{\alpha,\beta}}\big(t_{i,k}^{\alpha,\beta}-s\big)^{-\mu}s^{\mu+\gamma-1}\big|L_{i,p}^{\lambda,L}(s)\big|ds\\ &\le c\max_{0\le k\le N_i}\big(t_{i,k}^{\alpha,\beta}\big)^{\mu-1}\sum_{p=0}^{N_i+1}\int_{t_{i-1}}^{t_{i,k}^{\alpha,\beta}}\big(t_{i,k}^{\alpha,\beta}-s\big)^{-\mu}\big|L_{i,p}^{\lambda,L}(s)\big|ds\\ &\le c\max_{0\le k\le N_i}\big(t_{i,k}^{\alpha,\beta}\big)^{\mu-1}\int_{t_{i-1}}^{t_{i,k}^{\alpha,\beta}}\big(t_{i,k}^{\alpha,\beta}-s\big)^{-\mu}ds \le c. \end{aligned}\tag{2.11}$$

By (2.9)-(2.11), one obtains that for sufficiently small h, where $h=\max_{1\le i\le M}h_i$,

$$\big\|A_iB_i+\Phi^{i,i}B_i^L\big\|_\infty \le \|A_i\|_\infty\|B_i\|_\infty + \|\Phi^{i,i}\|_\infty\big\|B_i^L\big\|_\infty \le \frac{1}{2}.$$

Then the following inequality holds:

$$\big\|\big(\mathbb{I}_{N_i+1}-A_iB_i-\Phi^{i,i}B_i^L\big)^{-1}\big\|_\infty \le 2,$$

which shows that the system (2.8) has a unique solution. So the collocation scheme (2.6) defines a unique collocation solution for problem (1.1).

3. Some Useful Lemmas

Let $L^2(\hat\sigma_i)$, $H^m(\hat\sigma_i)$ be the usual Sobolev spaces defined on $\hat\sigma_i$. On the interval I, we define the weighted Sobolev space $H^1_{0,\lambda-1}(I)$ by

$$H^1_{0,\lambda-1}(I) := \big\{w(t) : \partial_t^k w(t)\in L^2_{0,\lambda-1}(I),\ 0\le k\le 1\big\}$$

equipped with norm $\|\cdot\|_{H^1_{0,\lambda-1}(I)}$ as follows:

$$\|w\|_{H^1_{0,\lambda-1}(I)} = \big((w,w)_{L^2_{0,\lambda-1}(I)} + (\partial_t w,\partial_t w)_{L^2_{0,\lambda-1}(I)}\big)^{\frac{1}{2}},$$

where

$$\begin{aligned} L^2_{0,\lambda-1}(I) &:= \left\{w(t) : \int_I w^2(t)t^{\lambda-1}dt < \infty\right\},\\ (w,v)_{L^2_{0,\lambda-1}(I)} &= \int_I w(t)v(t)t^{\lambda-1}dt. \end{aligned}$$

Now we give some lemmas that are needed in the convergence analysis.

Lemma 3.1 ([33]). *For $w \in H^m(\hat{\sigma}_i)$ with $1 \le m \le N_i + 2$,*

$$\big\|w - \hat{I}^L_{N_i+1,i} w\big\|_{H^k(\hat{\sigma}_i)} \le c h^{m-k}_{i,\lambda} (N_i+1)^{k-m} \big\|\partial_t^m w\big\|_{L^2(\hat{\sigma}_i)}, \quad k = 0, 1.$$

Lemma 3.2. *For $w \in H^m(\hat{\sigma}_i)$ with $1 \le m \le N_i + 1$ and $\alpha, \beta \le 0$,*

$$\big\|w - \hat{I}^{\alpha,\beta}_{N_i,i} w\big\|_{L^2(\hat{\sigma}_i)} \le c h^m_{i,\lambda} N_i^{-m} \big\|\partial_t^m w\big\|_{L^2(\hat{\sigma}_i)}.$$

Proof. One can see the details for the special case that $\alpha = \beta = -1/2$ in [16, Lemma 2]. For $\alpha, \beta \le 0$, the proof is similar. □

Lemma 3.3 ([7, 23]). *For any $w \in H^1(a, b)$,*

$$\max_{x\in[a,b]} |w(x)| \le \frac{1}{\sqrt{b-a}} \|w\|_{L^2(a,b)} + \sqrt{b-a}\|w'\|_{L^2(a,b)}.$$

In the rest of the paper, let u be the solution of (1.2) and $v := u'$. Let $e(t) := u(t) - U(t)$, $e'(t) := v(t) - V(t)$. For a given $t \in \sigma_i$, we define the piecewise functions

$$\begin{aligned} F(t,s)|_{s\in\sigma_j} &:= K(t,s)u_j(s) - I^{\lambda,L}_{N_j+1,j}\big(K(t,s)u_j(s)\big), \quad 1 \le j \le i,\\ H(t,s)|_{s\in\sigma_j} &:= I^{\lambda,L}_{N_j+1,j}\big(K(t,s)u_j(s) - K(t,s)U_j(s)\big), \quad 1 \le j \le i. \end{aligned}$$

Then the following lemmas hold.

Lemma 3.4. *For fixed $1 \le i \le M$, assume that $K(t, s^{1/\lambda})|_{t\in\sigma_i, s\in\hat{\sigma}_j} \in C^{m_j+1}(\sigma_i \times \hat{\sigma}_j)$, $u(t^{1/\lambda})|_{\hat{\sigma}_j} \in H^{m_j+1}(\hat{\sigma}_j)$ with $1 \le m_j \le N_j + 1$ for $j = 1, \ldots, i$. Then for $t \in \sigma_i$ one has*

$$\max_{s\in[0,t_i]} |F(t,s)|^2 \le c \max_{1\le j\le i} h^{2m_j+1}_{j,\lambda} (N_j+1)^{-2m_j} \big\|\partial_s^{m_j+1}\big(K\big(t, s^{\frac{1}{\lambda}}\big) u_j\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(\sigma_i; L^2(\hat{\sigma}_j))}.$$

Proof. Lemma 3.3 gives

$$\begin{aligned} &\max_{s\in[0,t_i]} |F(t,s)|^2 = \max_{1\le j\le i} \max_{s\in\hat{\sigma}_j} \big|F\big(t,\rho(s)\big)\big|^2 \\ &\le c \max_{1\le j\le i} \Big(h^{-1}_{j,\lambda}\big\|F\big(t,\rho(s)\big)\big\|^2_{L^2(\hat{\sigma}_j)} + h_{j,\lambda}\big\|\partial_s F\big(t,\rho(s)\big)\big\|^2_{L^2(\hat{\sigma}_j)}\Big). \end{aligned} \tag{3.1}$$

By (2.5) and Lemma 3.1, one has for $l = 0, 1$,

$$\begin{aligned} &\big\|\partial_s^{(l)} F\big(t,\rho(s)\big)\big\|^2_{L^2(\hat{\sigma}_j)} \\ &= \int_{t^\lambda_{j-1}}^{t^\lambda_j} \big|\partial_s^{(l)}\big(K\big(t^{\alpha,\beta}_{i,k}, \rho(s)\big)u_j\big(\rho(s)\big) - \hat{I}^L_{N_j+1,j}\big(K\big(t^{\alpha,\beta}_{i,k}, \rho(s)\big)u_j\big(\rho(s)\big)\big)\big)\big|^2 ds \\ &\le c h^{2(m_j+1-l)}_{j,\lambda} (N_j+1)^{-2(m_j+1-l)} \big\|\partial_s^{m_j+1}\big(K\big(t, s^{\frac{1}{\lambda}}\big)u_j\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(\sigma_i; L^2(\hat{\sigma}_j))}. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) yields the desired result. □

Lemma 3.5. *For fixed $1 \le i \le M$, assume that $u(t^{1/\lambda})|_{\hat{\sigma}_j} \in H^1(\hat{\sigma}_j)$ for $j = 1, \ldots, i$ and $K(t, s^{1/\lambda}) \in C^1(I \times \hat{I})$. Then for $t \in \sigma_i$ one has*

$$\max_{s\in[0,t_i]} |H(t,s)|^2 \le c T^{2-2\lambda} \max_{1\le j\le i} \Big(h^{-1}_{j,\lambda} \int_{t_{j-1}}^{t_j} |e(t)|^2 t^{\lambda-1} dt + h_{j,\lambda} \int_{t_{j-1}}^{t_j} |e'(t)|^2 t^{\lambda-1} dt\Big).$$

Proof. By Lemma 3.3, one has

$$\max_{s\in[0,t_i]}|H(t,s)|^2=\max_{1\le j\le i}\max_{s\in\hat\sigma_j}\big|H\big(t,\rho(s)\big)\big|^2$$
$$\le\max_{1\le j\le i}\Big(h_{j,\lambda}^{-1}\big\|H\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}+h_{j,\lambda}\big\|\partial_sH\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}\Big).\tag{3.3}$$

Then (2.5) and [23, Lemma 3.3] yield

$$\big\|H\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}=\int_{t_{j-1}^\lambda}^{t_j^\lambda}\big|\hat I^L_{N_j+1,j}\big(K\big(t^{\alpha,\beta}_{i,k},\rho(s)\big)u_j\big(\rho(s)\big)-K\big(t^{\alpha,\beta}_{i,k},\rho(s)\big)U_j\big(\rho(s)\big)\big)\big|^2ds$$
$$\le\sum_{k=0}^{N_j+1}\Big(K\big(t^{\alpha,\beta}_{i,k},\rho(\xi^L_{j,k})\big)u_j\big(\rho(\xi^L_{j,k})\big)-K\big(t^{\alpha,\beta}_{i,k},\rho(\xi^L_{j,k})\big)U_j\big(\rho(\xi^L_{j,k})\big)\Big)^2\omega^L_{j,k}.$$

Considering

$$\sum_{k=0}^{N_j+1}\omega^L_{j,k}=\frac{h_{j,\lambda}}{2}\sum_{k=0}^{N_j+1}\theta^L_{j,k}=h_{j,\lambda},$$

we have

$$\big\|H\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}\le c\sum_{k=0}^{N_j+1}\Big(u_j\big(\rho(\xi^L_{j,k})\big)-U_j\big(\rho(\xi^L_{j,k})\big)\Big)^2\omega^L_{j,k}$$
$$\le ch_{j,\lambda}\big\|u_j\big(\rho(s)\big)-U_j\big(\rho(s)\big)\big\|^2_{L^\infty(\hat\sigma_j)}.$$

Lemma 3.3 then gives that

$$\big\|H\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}\le c\int_{t_{j-1}^\lambda}^{t_j^\lambda}\big|e\big(\rho(s)\big)\big|^2ds+ch^2_{j,\lambda}\int_{t_{j-1}^\lambda}^{t_j^\lambda}\big|\partial_se\big(\rho(s)\big)\big|^2ds$$
$$=c\lambda\int_{t_{j-1}}^{t_j}|e(t)|^2t^{\lambda-1}dt+c\lambda^{-1}T^{2-2\lambda}h^2_{j,\lambda}\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{\lambda-1}dt.\tag{3.4}$$

By the triangle inequality and Lemma 3.1 with $k=1,m=1$, one obtains

$$\big\|\partial_sH\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}=\int_{t_{j-1}^\lambda}^{t_j^\lambda}\big|\partial_s\hat I^L_{N_j+1,j}\big(K\big(t,\rho(s)\big)u_j\big(\rho(s)\big)-K\big(t,\rho(s)\big)U_j\big(\rho(s)\big)\big)\big|^2ds$$
$$\le c\int_{t_{j-1}^\lambda}^{t_j^\lambda}\big|\partial_s\big(K\big(t,\rho(s)\big)u_j\big(\rho(s)\big)-K\big(t,\rho(s)\big)U_j\big(\rho(s)\big)\big)\big|^2ds$$
$$=c\int_{t_{j-1}}^{t_j}\big|\partial_s\big(K(t,s)u_j(s)-K(t,s)U_j(s)\big)\big|^2\lambda^{-1}s^{1-\lambda}ds,$$

which leads to

$$\big\|\partial_sH\big(t,\rho(s)\big)\big\|^2_{L^2(\hat\sigma_j)}\le c\lambda^{-1}\int_{t_{j-1}}^{t_j}|e(t)|^2t^{1-\lambda}dt+c\lambda^{-1}\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{1-\lambda}dt$$
$$\le c\lambda^{-1}T^{2-2\lambda}\bigg(\int_{t_{j-1}}^{t_j}|e(t)|^2t^{1-\lambda}dt+\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{1-\lambda}dt\bigg).\tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) gives the desired result. □

4. Convergence Analysis

In this section, we derive an *hp*-version error bound for the collocation solution generated by the scheme (2.6).

The triangle inequality gives

$$\int_{t_{i-1}}^{t_i} |e'(t)|^2 t^{\lambda-1}dt \le 2\int_{t_{i-1}}^{t_i} \big|v(t)-I_{N_i,i}^{\lambda,\alpha,\beta}v(t)\big|^2 t^{\lambda-1}dt + 2\int_{t_{i-1}}^{t_i} \big|I_{N_i,i}^{\lambda,\alpha,\beta}v(t)-V(t)\big|^2 t^{\lambda-1}dt. \tag{4.1}$$

We firstly estimate the second term on the right of (4.1).

Lemma 4.1. *For fixed $1\le i\le M$, assume that $u(t^{1/\lambda})|_{\hat\sigma_j}\in H^{m_j+1}(\hat\sigma_j)$ with $1\le m_j\le N_j+1$ for $j=1,\ldots,i$ and $a_\gamma(t)\in C(I), K(t,s^{1/\lambda})\in C^{m+1}(I\times\hat I)$ with $m=\max_{1\le j\le M} m_j$. Then one has*

$$\int_{t_{i-1}}^{t_i} \big|I_{N_i,i}^{\lambda,\alpha,\beta}v(t)-V(t)\big|^2 t^{\lambda-1}dt \le cQ_i + cT^{2-2\lambda}h_{i,\lambda}\max_{1\le j\le i}\left(h_{j,\lambda}^{-1}\int_{t_{j-1}}^{t_j}|e(t)|^2t^{\lambda-1}dt + h_{j,\lambda}\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{\lambda-1}dt\right),$$

where

$$Q_i := h_{i,\lambda}\max_{1\le j\le i} h_{j,\lambda}^{2m_j+1}(N_j+1)^{-2m_j}\big\|\partial_s^{m_j+1}\big(K\big(t,s^{\frac{1}{\lambda}}\big)u_j\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(\sigma_i;L^2(\hat\sigma_j))}.$$

Proof. From the Eq. (1.2), one gets

$$\begin{aligned} I_{N_i,i}^{\lambda,\alpha,\beta}v_i(t) &= I_{N_i,i}^{\lambda,\alpha,\beta}\big(a_\gamma(t)u_i(t)+g_\gamma(t)\big) \\ &\quad + I_{N_i,i}^{\lambda,\alpha,\beta}\left(\sum_{j=1}^{i-1} t^{-\gamma}\int_{t_{j-1}}^{t_j}(t-s)^{-\mu}s^{\mu+\gamma-1}K(t,s)u_j(s)ds\right) \\ &\quad + I_{N_i,i}^{\lambda,\alpha,\beta}\left(t^{-\gamma}\int_{t_{i-1}}^{t}(t-s)^{-\mu}s^{\mu+\gamma-1}K(t,s)u_i(s)ds\right), \quad t\in\sigma_i. \end{aligned} \tag{4.2}$$

Note that $V\in S_\lambda(\mathcal{T}_M)$. According to the collocation scheme (2.6a), one has

$$\begin{aligned} V_i(t) &= I_{N_i,i}^{\lambda,\alpha,\beta}V_i(t) = I_{N_i,i}^{\lambda,\alpha,\beta}\big(a_\gamma(t)U_i(t)+g_\gamma(t)\big) \\ &\quad + I_{N_i,i}^{\lambda,\alpha,\beta}\left(\sum_{j=1}^{i-1} t^{-\gamma}\int_{t_{j-1}}^{t_j}(t-s)^{-\mu}s^{\mu+\gamma-1}I_{N_j+1,j}^{\lambda,L}\big(K(t,s)U_j(s)\big)ds\right) \\ &\quad + I_{N_i,i}^{\lambda,\alpha,\beta}\left(t^{-\gamma}\int_{t_{i-1}}^{t}(t-s)^{-\mu}s^{\mu+\gamma-1}I_{N_i+1,i}^{\lambda,L}\big(K(t,s)U_i(s)\big)ds\right), \quad t\in\sigma_i. \end{aligned} \tag{4.3}$$

Then we subtract (4.3) from (4.2). By the triangle inequality, the following estimate holds:

$$\int_{t_{i-1}}^{t_i}\big|I_{N_i,i}^{\lambda,\alpha,\beta}v(t)-V(t)\big|^2t^{\lambda-1}dt \le 2\big(\Upsilon_i^1+\Upsilon_i^2\big), \tag{4.4}$$

where

$$\Upsilon_i^1 = \int_{t_{i-1}}^{t_i}\big|I_{N_i,i}^{\lambda,\alpha,\beta}\big(a_\gamma(t)u_i(t)-a_\gamma(t)U_i(t)\big)\big|^2t^{\lambda-1}dt,$$

$$\Upsilon_i^2=\int_{t_{i-1}}^{t_i}\left|I_{N_i,i}^{\lambda,\alpha,\beta}\left(\sum_{j=1}^{i-1}t^{-\gamma}\int_{t_{j-1}}^{t_j}(t-s)^{-\mu}s^{\mu+\gamma-1}\big(K(t,s)u_j(s)-I_{N_j+1,j}^{\lambda,L}\big(K(t,s)U_j(s)\big)\big)ds\right.\right.$$
$$\left.\left.+t^{-\gamma}\int_{t_{i-1}}^{t}(t-s)^{-\mu}s^{\mu+\gamma-1}\big(K(t,s)u_i(s)-I_{N_i+1,i}^{\lambda,L}\big(K(t,s)U_i(s)\big)\big)ds\right)\right|^2t^{\lambda-1}dt.$$

Let $\omega_i^{\alpha,\beta}(s):=(t_i^\lambda-s)^\alpha(s-t_{i-1}^\lambda)^\beta$ for $s\in\hat{\sigma}_i$. By (2.4) and the property of Jacobi-Gauss quadrature, one has

$$\begin{aligned}\Upsilon_i^1&=\lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda}\big|\hat{I}_{N_i,i}^{\alpha,\beta}\big(a_\gamma(\rho(s))u_i(\rho(s))-a_\gamma(\rho(s))U_i(\rho(s))\big)\big|^2ds\\
&\le\lambda^{-1}h_{i,\lambda}^{-\alpha-\beta}\int_{t_{i-1}^\lambda}^{t_i^\lambda}\big|\hat{I}_{N_i,i}^{\alpha,\beta}\big(a_\gamma(\rho(s))u_i(\rho(s))-a_\gamma(\rho(s))U_i(\rho(s))\big)\big|^2\omega_i^{\alpha,\beta}(s)ds\\
&=\lambda^{-1}h_{i,\lambda}^{-\alpha-\beta}\sum_{k=0}^{N_i}\Big(a_\gamma(\rho(\xi_{i,k}^{\alpha,\beta}))u_i(\rho(\xi_{i,k}^{\alpha,\beta}))-a_\gamma(\rho(\xi_{i,k}^{\alpha,\beta}))U_i(\rho(\xi_{i,k}^{\alpha,\beta}))\Big)^2\omega_{i,k}^{\alpha,\beta}.\end{aligned}$$

According to

$$\sum_{k=0}^{N_i}\omega_{i,k}^{\alpha,\beta}=\left(\frac{h_{i,\lambda}}{2}\right)^{1+\alpha+\beta}\sum_{k=0}^{N_i}\theta_{i,k}^{\alpha,\beta}=\left(\frac{h_{i,\lambda}}{2}\right)^{1+\alpha+\beta}B(1+\alpha,1+\beta),\tag{4.5}$$

where $B(\cdot,\cdot)$ is the Beta function, we get

$$\Upsilon_i^1\le c\lambda^{-1}h_{i,\lambda}\big\|e(\rho(s))\big\|_{L^\infty(\hat{\sigma}_i)}^2.$$

Lemma 3.3 then yields

$$\begin{aligned}\Upsilon_i^1&\le c\lambda^{-1}\left(\int_{t_{i-1}^\lambda}^{t_i^\lambda}\big|e(\rho(s))\big|^2ds+h_{i,\lambda}^2\int_{t_{i-1}^\lambda}^{t_i^\lambda}\big|\partial_se(\rho(s))\big|^2ds\right)\\
&\le c\int_{t_{i-1}}^{t_i}|e(t)|^2t^{\lambda-1}dt+c\lambda^{-2}T^{2-2\lambda}h_{i,\lambda}^2\int_{t_{i-1}}^{t_i}|e'(t)|^2t^{\lambda-1}dt.\end{aligned}\tag{4.6}$$

Next, we estimate the term Υ_i^2. Let $\Psi(t,s):=F(t,s)+H(t,s)$. By (2.4), one gets

$$\begin{aligned}\Upsilon_i^2&=\int_{t_{i-1}}^{t_i}\left|I_{N_i,i}^{\lambda,\alpha,\beta}t^{-\gamma}\int_0^t(t-s)^{-\mu}s^{\mu+\gamma-1}\Psi(t,s)ds\right|^2t^{\lambda-1}dt\\
&=\lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda}\left|\hat{I}_{N_i,i}^{\alpha,\beta}\rho(\tau)^{-\gamma}\int_0^{\rho(\tau)}\big(\rho(\tau)-s\big)^{-\mu}s^{\mu+\gamma-1}\Psi\big(\rho(\tau),s\big)ds\right|^2d\tau\\
&\le\lambda^{-1}h_{i,\lambda}^{-\alpha-\beta}\int_{t_{i-1}^\lambda}^{t_i^\lambda}\left|\hat{I}_{N_i,i}^{\alpha,\beta}\rho(\tau)^{-\gamma}\int_0^{\rho(\tau)}\big(\rho(\tau)-s\big)^{-\mu}s^{\mu+\gamma-1}\Psi\big(\rho(\tau),s\big)ds\right|^2\omega_i^{\alpha,\beta}(\tau)d\tau.\end{aligned}$$

The property of Jacobi-Gauss quadrature and (4.5) yield that

$$\begin{aligned}\Upsilon_i^2&\le\lambda^{-1}h_{i,\lambda}^{-\alpha-\beta}\sum_{k=0}^{N_i}\left(\big(t_{i,k}^{\alpha,\beta}\big)^{-\gamma}\int_0^{t_{i,k}^{\alpha,\beta}}\big(t_{i,k}^{\alpha,\beta}-s\big)^{-\mu}s^{\mu+\gamma-1}\Psi\big(t_{i,k}^{\alpha,\beta},s\big)ds\right)^2\omega_{i,k}^{\alpha,\beta}\\
&\le\lambda^{-1}h_{i,\lambda}^{-\alpha-\beta}\sum_{k=0}^{N_i}\Big(\max_{s\in[0,t_i]}\big|\Psi\big(t_{i,k}^{\alpha,\beta},s\big)\big|\Big)^2\left(\big(t_{i,k}^{\alpha,\beta}\big)^{-\gamma}\int_0^{t_{i,k}^{\alpha,\beta}}\big(t_{i,k}^{\alpha,\beta}-s\big)^{-\mu}s^{\mu+\gamma-1}ds\right)^2\omega_{i,k}^{\alpha,\beta}\\
&\le c\lambda^{-1}h_{i,\lambda}\max_{0\le k\le N_i}\Big(\max_{s\in[0,t_i]}\big|\Psi(t_{i,k}^{\alpha,\beta},s)\big|\Big)^2.\end{aligned}$$

The triangle inequality gives

$$\Big(\max_{s\in[0,t_i]}\big|\Psi\big(t_{i,k}^{\alpha,\beta},s\big)\big|\Big)^2 \le 2\max_{s\in[0,t_i]}\Big(\big|F\big(t_{i,k}^{\alpha,\beta},s\big)\big|^2+\big|H\big(t_{i,k}^{\alpha,\beta},s\big)\big|^2\Big).$$

Then by Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}\Upsilon_i^2 \le{}& ch_{i,\lambda}\max_{1\le j\le i} h_{j,\lambda}^{2m_j+1}(N_j+1)^{-2m_j}\big\|\partial_s^{m_j+1}\big(K\big(t,s^{\frac{1}{\lambda}}\big)u_j\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(\sigma_i;L^2(\hat\sigma_j))}\\ &+cT^{2-2\lambda}h_{i,\lambda}\max_{1\le j\le i}\Big(h_{j,\lambda}^{-1}\int_{t_{j-1}}^{t_j}|e(t)|^2t^{\lambda-1}dt+h_{j,\lambda}\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{\lambda-1}dt\Big). \qquad (4.7)\end{aligned}$$

Combining (4.4), (4.6) and (4.7), we can get the desired result. □

Using Lemma 4.1, we can derive the following error bound for the collocation solution under a weighted H^1-norm.

Theorem 4.1. *Assume that $u\in H^1_{0,\lambda-1}(I), u(t^{1/\lambda})|_{\hat\sigma_i}\in H^{m_i+1}(\hat\sigma_i)$ and $v(t^{1/\lambda})|_{\hat\sigma_i}\in H^{m_i}(\hat\sigma_i)$ with $1\le m_i\le N_i+1$ for $i=1,\dots,M$ and $a_\gamma(t)\in C(I), K(t,s^{1/\lambda})\in C^{m+1}(I\times\hat I)$ with $m=\max_{1\le j\le M}m_j$. The collocation points are chosen as in (2.3) with $\alpha,\beta\le 0$. Then there exists a constant c such that*

$$\begin{aligned}\|e\|^2_{H^1_{0,\lambda-1}(I)}\le{}&\exp(cT^{4-3\lambda})\sum_{i=1}^M h_{i,\lambda}^{2m_i}N_i^{-2m_i}\\ &\times\Big(\big|v\big(s^{\frac{1}{\lambda}}\big)\big|^2_{H^{m_i}(\hat\sigma_i)}+h_{i,\lambda}\big\|\partial_s^{m_i+1}\big(K\big(t,s^{\frac{1}{\lambda}}\big)u_i\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(I;L^2(\hat\sigma_i))}\Big).\end{aligned}$$

Proof. By (2.4) and Lemma 3.2, one gets

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}\big|v_i(t)-I_{N_i,i}^{\lambda,\alpha,\beta}v_i(t)\big|^2t^{\lambda-1}dt\\ &=\lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda}\big|v\big(\rho(s)\big)-\hat I_{N_i,i}^{\alpha,\beta}v\big(\rho(s)\big)\big|^2ds\\ &\le c\lambda^{-1}h_{i,\lambda}^{2m_i}N_i^{-2m_i}\big\|\partial_s^{m_i}v\big(s^{\frac{1}{\lambda}}\big)\big\|^2_{L^2(\hat\sigma_i)}. \qquad (4.8)\end{aligned}$$

Combining (4.1), Lemma 4.1 and (4.8) yields

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}|e'(t)|^2\,t^{\lambda-1}dt\\ &\le c\tilde Q_i+cT^{2-2\lambda}h_{i,\lambda}\max_{1\le j\le i}\Big(h_{j,\lambda}^{-1}\int_{t_{j-1}}^{t_j}|e(t)|^2t^{\lambda-1}dt+h_{j,\lambda}\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{\lambda-1}dt\Big), \qquad (4.9)\end{aligned}$$

where

$$\tilde Q_i:=Q_i+h_{i,\lambda}^{2m_i}N_i^{-2m_i}\big\|\partial_s^{m_i}v\big(s^{\frac{1}{\lambda}}\big)\big\|^2_{L^2(\hat\sigma_i)}.$$

For sufficiently small $h_{i,\lambda}$, (4.9) gives

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}|e'(t)|^2t^{\lambda-1}dt\\ &\le c\tilde Q_i+cT^{2-2\lambda}\int_{t_{i-1}}^{t_i}|e(t)|^2t^{\lambda-1}dt\\ &\quad+cT^{2-2\lambda}h_{i,\lambda}\max_{1\le j\le i-1}\Big(h_{j,\lambda}^{-1}\int_{t_{j-1}}^{t_j}|e(t)|^2t^{\lambda-1}dt+h_{j,\lambda}\int_{t_{j-1}}^{t_j}|e'(t)|^2t^{\lambda-1}dt\Big). \qquad (4.10)\end{aligned}$$

It is easy to get that for $j = 1, \ldots, i$,

$$\begin{aligned} e^2(t_j) - e^2(t_{j-1}) &= 2\int_{t_{j-1}}^{t_j} e(t)e'(t)dt \\ &\leq 2T^{1-\lambda}\Big(\int_{t_{j-1}}^{t_j} |e(t)|^2 t^{\lambda-1}dt + \int_{t_{j-1}}^{t_j} |e'(t)|^2 t^{\lambda-1}dt\Big). \end{aligned}$$

Since $e(0) = 0$, one obtains

$$\begin{aligned} e^2(t_{i-1}) &= \sum_{j=1}^{i-1} \big(e^2(t_j) - e^2(t_{j-1})\big) \\ &\leq 2T^{1-\lambda}\sum_{j=1}^{i-1}\Big(\int_{t_{j-1}}^{t_j} |e(t)|^2 t^{\lambda-1}dt + \int_{t_{j-1}}^{t_j} |e'(t)|^2 t^{\lambda-1}dt\Big). \end{aligned} \tag{4.11}$$

By the variable transformation $\tau = s^{1/\lambda}$ and $t = x^{1/\lambda}$, we derive

$$\begin{aligned} \int_{t_{i-1}}^{t_i} |e(t)|^2 t^{\lambda-1}dt &= \int_{t_{i-1}}^{t_i} \Big(\int_{t_{i-1}}^{t} e'(\tau)d\tau + e(t_{i-1})\Big)^2 t^{\lambda-1}dt \\ &= \lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda} \Big(\int_{t_{i-1}^\lambda}^{x} \partial_s e\big(s^{\frac{1}{\lambda}}\big)ds + e(t_{i-1})\Big)^2 dx. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality and (4.11) that

$$\begin{aligned} \int_{t_{i-1}}^{t_i} |e(t)|^2 t^{\lambda-1}dt &\leq 2\lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda} \Big(\int_{t_{i-1}^\lambda}^{x} \partial_s e\big(s^{\frac{1}{\lambda}}\big)ds\Big)^2 dx + 2\lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda} e^2(t_{i-1})dx \\ &\leq 2\lambda^{-1}h_{i,\lambda}^2\int_{t_{i-1}^\lambda}^{t_i^\lambda} \big|\partial_s e\big(s^{\frac{1}{\lambda}}\big)\big|^2 ds + 2\lambda^{-1}h_{i,\lambda}e^2(t_{i-1}) \\ &\leq 2\lambda^{-2}T^{2-2\lambda}h_{i,\lambda}^2\int_{t_{i-1}}^{t_i} |e'(t)|^2 t^{\lambda-1}dt \\ &\quad + 4\lambda^{-1}T^{1-\lambda}h_{i,\lambda}\sum_{j=1}^{i-1}\Big(\int_{t_{j-1}}^{t_j} |e(t)|^2 t^{\lambda-1}dt + \int_{t_{j-1}}^{t_j} |e'(t)|^2 t^{\lambda-1}dt\Big). \end{aligned} \tag{4.12}$$

Let

$$R_i := \int_{t_{i-1}}^{t_i} |e(t)|^2 t^{\lambda-1}dt + \int_{t_{i-1}}^{t_i} |e'(t)|^2 t^{\lambda-1}dt.$$

Then substituting (4.12) into (4.10) yeilds

$$\begin{aligned} &\int_{t_{i-1}}^{t_i} |e'(t)|^2 t^{\lambda-1}dt \\ &\leq c\tilde{Q}_i + cT^{4-4\lambda}h_{i,\lambda}^2\int_{t_{i-1}}^{t_i} |e'(t)|^2 t^{\lambda-1}dt + cT^{3-3\lambda}h_{i,\lambda}\sum_{j=1}^{i-1} R_j \\ &\quad + cT^{2-2\lambda}h_{i,\lambda}\Bigg(cT^{1-\lambda}\sum_{k=1}^{i-1} R_k + cT^{2-2\lambda}\sum_{j=1}^{i-1} h_{j,\lambda}\int_{t_{j-1}}^{t_j} |e'(t)|^2 t^{\lambda-1}dt\Bigg). \end{aligned} \tag{4.13}$$

By (4.12) and (4.13), we can deduce that, for sufficiently small $h_{i,\lambda}$,

$$R_i \le c\tilde{Q}_i + cT^{4-4\lambda} h_{i,\lambda} \sum_{j=1}^{i-1} R_j.$$

Taking $\epsilon_i = h_{i,\lambda}^{-1} R_i$, and applying the Gronwall inequality (see, for example, [24, Lemma 3.3]), one can obtain

$$R_i \le c\tilde{Q}_i + \exp(cT^{4-3\lambda}) h_{i,\lambda} \sum_{j=1}^{i-1} \tilde{Q}_i. \tag{4.14}$$

Then substituting Q_i into (4.14) gives

$$\begin{aligned}
\|e\|^2_{H^1_{0,\lambda-1}(I)} &= \sum_{i=1}^{M} R_i \le c\sum_{i=1}^{M} \tilde{Q}_i + \exp(cT^{4-3\lambda}) \sum_{i=1}^{M} h_{i,\lambda} \sum_{j=1}^{i-1} \tilde{Q}_i \le \exp(cT^{4-3\lambda}) \sum_{i=1}^{M} \tilde{Q}_i \\
&\le \exp(cT^{4-3\lambda}) \sum_{i=1}^{M} \Big(h_{i,\lambda}^{2m_i} N_i^{-2m_i} \big|v\big(s^{\frac{1}{\lambda}}\big)\big|^2_{H^{m_i}(\hat{\sigma}_i)} + h_{i,\lambda} \max_{1\le j\le i} h_{j,\lambda}^{2m_j+1} (N_j+1)^{-2m_j} \\
&\qquad \times \big\|\partial_s^{m_j+1}\big(K\big(t, s^{\frac{1}{\lambda}}\big) u_j\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(\sigma_i; L^2(\hat{\sigma}_j))} \Big) \\
&\le \exp(cT^{4-3\lambda}) \sum_{i=1}^{M} h_{i,\lambda}^{2m_i} N_i^{-2m_i} \\
&\qquad \times \Big(\big|v\big(s^{\frac{1}{\lambda}}\big)\big|^2_{H^{m_i}(\hat{\sigma}_i)} + h_{i,\lambda} \big\|\partial_s^{m_i+1}\big(K\big(t, s^{\frac{1}{\lambda}}\big) u_i\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(I; L^2(\hat{\sigma}_i))} \Big),
\end{aligned}$$

which is the desired result. □

Remark 4.1. Note that Theorem 4.1 holds for arbitrary meshes. If we consider the following graded mesh with grading exponent q ($q \ge 1$):

$$\mathcal{T}_M = \left\{ t_i = T\left(\frac{i}{M}\right)^q,\ i = 0, 1, \ldots, M \right\}, \tag{4.15}$$

one can obtain by the mean value theorem that

$$h_\lambda \le \begin{cases} T^\lambda M^{-q\lambda}, & \text{if} \quad 1 \le q \le 1/\lambda, \\ T^\lambda q\lambda M^{-1}, & \text{if} \quad q > 1/\lambda, \end{cases}$$

where

$$h_\lambda = \max_{1\le i\le M} h_{i,\lambda} = \max_{1\le i\le M} \big(t_i^\lambda - t_{i-1}^\lambda\big).$$

In this case, we further have the following error estimate:

$$\begin{aligned}
&\|e\|^2_{H^1_{0,\lambda-1}(I)} \\
&\le \exp(cT^{4-3\lambda}) T^\lambda \sum_{i=1}^{M} M^{-2\min\{q\lambda m_i, m_i\}} N_i^{-2m_i} \\
&\qquad \times \Big(\big|v\big(s^{\frac{1}{\lambda}}\big)\big|^2_{H^{m_i}(\hat{\sigma}_i)} + h_{i,\lambda} \big\|\partial_s^{m_i+1}\big(K\big(t, s^{\frac{1}{\lambda}}\big) u_i\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(I; L^2(\hat{\sigma}_i))} \Big).
\end{aligned}$$

Remark 4.2. From the definition of the $H^1_{0,\lambda-1}$-norm it follows that

$$\|e\|^2_{H^1(I)} \le T^{1-\lambda}\|e\|^2_{H^1_{0,\lambda-1}(I)}.$$

Using Lemma 3.3, we can derive similar error estimates under the L^2- and L^∞-norms. For example, if the assumptions in Theorem 4.1 are satisfied and if the graded mesh (4.15) is employed, then we have

$$\begin{aligned}&\max_{t\in I}|e(t)|^2\\ &\le 2(1+T^2)\exp(cT^{4-3\lambda})\sum_{i=1}^{M} M^{-2\min\{q\lambda m_i, m_i\}}N_i^{-2m_i}\\ &\qquad \times\Big(\big|v\big(s^{\frac{1}{\lambda}}\big)\big|^2_{H^{m_i}(\hat\sigma_i)} + h_{i,\lambda}\big\|\partial_s^{m_i+1}\big(K\big(t,s^{\frac{1}{\lambda}}\big)u_i\big(s^{\frac{1}{\lambda}}\big)\big)\big\|^2_{L^\infty(I;L^2(\hat\sigma_i))}\Big).\end{aligned}$$

Remark 4.3. The choice of the fractional exponent λ plays a crucial role in the effectiveness of our method. More precisely, one can observe from Theorem 4.1 that the order of convergence of the method depends on the regularity of $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$. So, λ should be selected such that $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ are smooth enough. When the structural properties of the solution $u(t)$ are known, we can select optimal λ accordingly. When the singularity of $u(t)$ is unknown and $g_\gamma(t)$ has a weak singularity, a simple but practical strategy is to take $\lambda = 1/r$ with r being a moderately large integer. On the one hand, the weak singularity of $g_\gamma(t)$ implies that $u'(t)$ possibly has a similar weak singularity. On the other hand, the regularity of $u(t^r)$ and $u'(t^r)$ is always better than the regularity of $u(t)$ and $u'(t)$ respectively. Hence, r could be selected in such a way that $g_\gamma(t^r)$ is smooth enough.

5. Numerical Results

In this section, we consider the following numerical example:

$$\begin{cases} t^\gamma u'(t) = g(t) + t^{\frac{5}{3}}u(t) + \displaystyle\int_0^t \frac{\sqrt{3}}{3\pi}(t-s)^{-\mu}s^{\gamma+\mu-1}e^s u(s)ds, \quad t\in[0,1],\\ u(0)=0, \end{cases} \tag{5.1}$$

where $g(t)$ is a given function. By selecting different $g(t)$, we test the performance of the proposed method in different situations.

In the numerical tests, we set $\alpha = \beta = -1/2$, take uniform mode $N_i = N$ and employ graded meshes with grading exponent q ($q \ge 1$) defined in (4.15). We will test the p-version convergence of the method by increasing N for fixed subinterval number M. To show the h-version convergence of the method, a fixed N is used and mesh sizes will be refined.

In the following, we denote by $E_{M,N}$ the $H^1_{0,\lambda-1}$-norm of the errors, namely

$$E_{M,N} = \|e\|_{H^1_{0,\lambda-1}(I)}.$$

Let r represent the h-version convergence order, computed by $r = \log_2(E_{M,N}/E_{2M,N})$.

Example 5.1. Consider VIDE (5.1) with

$$g(t) = t^{1+\mu}e^{-t}\big((1+\mu)t^{\gamma-1} - t^\gamma - t^{\frac{5}{3}}\big) - \frac{\sqrt{3}}{3\pi}B(1-\mu, 2\mu+\gamma+1)t^{1+\mu+\gamma}.$$

The exact solution of this problem is $u(t) = t^{1+\mu}e^{-t}$. Note that $u(t)$ exhibits weak singularity at the initial point for $\mu \in (0,1)$.

In this example, we set $\mu = 1/2$ and $\gamma = 1$. Taking $\lambda = \mu = 1/2$, one can see that, by transformation $t = s^{1/\lambda}$, $u(s^{1/\lambda})$ and $u'(s^{1/\lambda})$ are analytic although $u(t)$ and $u'(t)$ are weakly singular at $t = 0$. In Fig. 5.1, the convergence rates of p-version of the method are shown for fixed $M = 1$ and $M = 8$. When $\lambda = 1$, namely polynomial collocation scheme is applied, the convergence rates are algebraic in both cases of $M = 1$ and $M = 8$. For the proposed fractional collocation method with $\lambda = 1/2$, the convergence results are given in Figs. 5.1(b) and 5.1(c). On both uniform mesh and graded mesh with $q = 2$, the exponential convergence is achieved.

The errors and h-version convergence of the method are listed in Tables 5.1 and 5.2. One can see from Table 5.1 that the fractional collocation method ($\lambda = 1/2$) can achieve high order convergence on uniform mesh ($q = 1$), while the polynomial collocation method ($\lambda = 1$) has an order barrier. Table 5.2 also shows that, when $\lambda = 1/2$, the optimal convergence rates can be obtained if $q = 1/\lambda$ is taken. The numerical results are compatible with the theorem analysis.

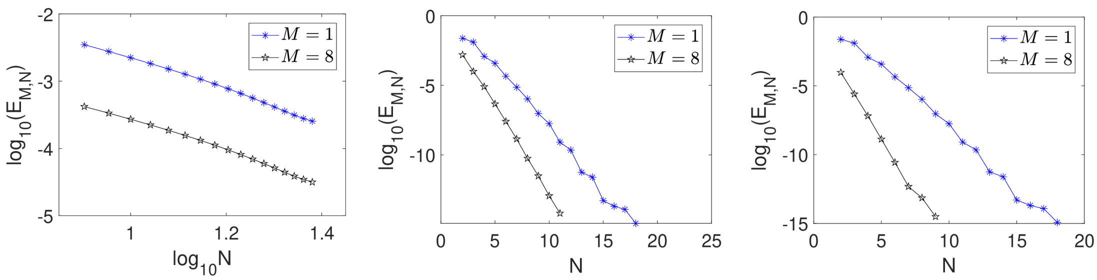

Fig. 5.1. $H^1_{0,\lambda-1}$-norm errors and p-version convergence for Example 5.1.

Table 5.1: $H^1_{0,\lambda-1}$-norm errors and h-version convergence rates on uniform mesh for Example 5.1.

M	$\lambda = 1$ ($q = 1$)				$\lambda = 1/2$ ($q = 1$)			
	$N = 2$		$N = 4$		$N = 2$		$N = 4$	
	$E_{M,N}$	r	$E_{M,N}$	r	$E_{M,N}$	r	$E_{M,N}$	r
64	5.04E-04	1.01	1.76E-04	1.00	5.09E-05	1.71	3.50E-08	2.70
128	2.51E-04	1.00	8.79E-05	1.00	1.54E-05	1.73	5.31E-09	2.72
256	1.25E-04	1.00	4.40E-05	1.00	4.60E-06	1.74	7.98E-10	2.73

Table 5.2: $H^1_{0,\lambda-1}$-norm errors and h-version convergence rates on graded mesh for Example 5.1.

M	$\lambda = 1/2$ ($q = 2$)					
	$N = 1$		$N = 2$		$N = 3$	
	$E_{M,N}$	r	$E_{M,N}$	r	$E_{M,N}$	r
64	4.86E-05	2.00	1.84E-07	3.00	6.23E-09	4.00
128	1.22E-05	2.00	2.30E-08	3.00	3.89E-10	4.00
256	3.04E-06	2.00	2.87E-09	3.00	2.43E-12	4.00

Example 5.2. Consider VIDE (5.1) with

$$g(t) = t^{1+\nu_1}e^{-t}\big((1+\nu_1)t^{\gamma-1} - t^{\gamma} - t^{\frac{5}{3}}\big) - +t^{1+\nu_2}e^{-t}\big((1+\nu_2)t^{\gamma-1} - t^{\gamma} - t^{\frac{5}{3}}\big)$$
$$- \frac{\sqrt{3}}{3\pi}\left(B(1-\nu_1, 2\nu_1+\gamma+1)t^{1+\nu_1+\gamma} + B(1-\nu_2, 2\nu_2+\gamma+1)t^{1+\nu_2+\gamma}\right).$$

The exact solution of this problem is $u(t) = (t^{1+\nu_1} + t^{1+\nu_2})e^{-t}$.

One can see that we cannot guarantee that $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ are always analytic for general ν_1 and ν_2. Using this example, we test the performance of our method in the case that $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ are not smooth enough.

In this example, we set $\mu = 1/2$, $\gamma = 1$, $\nu_1 = 1/2$, $\nu_2 = \sqrt{5}$. The fractional parameter λ is chosen as $\lambda = 1$ and $\lambda = 1/2$, respectively. The p-version convergence results with fixed $M = 1$ and $M = 8$ are shown in Fig. 5.2, which show that fractional collocation method performs better than polynomial collocation method for this problem. In Tables 5.3 and 5.4, we list the h-version convergence results on uniform mesh and graded mesh. One can see that the fractional collocation method can still have high order of convergence.

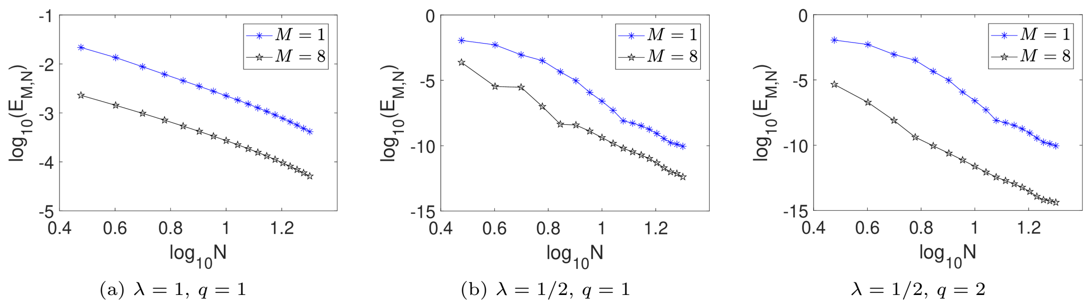

Fig. 5.2. $H^1_{0,\lambda-1}$-norm errors and p-version convergence for Example 5.2.

Table 5.3: $H^1_{0,\lambda-1}$-norm errors and h-version convergence rates on uniform mesh for Example 5.2.

M	$\lambda = 1$ $(q = 1)$				$\lambda = 1/2$ $(q = 1)$			
	$N = 2$		$N = 4$		$N = 2$		$N = 4$	
	$E_{M,N}$	r	$E_{M,N}$	r	$E_{M,N}$	r	$E_{M,N}$	r
64	5.04E-04	1.00	1.76E-04	1.00	4.34E-05	1.53	8.51E-08	2.30
128	2.51E-04	1.00	8.79E-05	1.00	1.40E-05	1.63	1.57E-08	2.44
256	1.25E-04	1.00	4.40E-05	1.00	4.36E-06	1.68	2.79E-09	2.50

Table 5.4: $H^1_{0,\lambda-1}$-norm errors and h-version convergence rates on graded mesh for Example 5.2.

M	$\lambda = 1/2$ $(q = 2)$					
	$N = 1$		$N = 2$		$N = 3$	
	$E_{M,N}$	r	$E_{M,N}$	r	$E_{M,N}$	r
64	2.57E-05	2.00	2.01E-07	3.00	1.12E-09	4.00
128	6.42E-06	2.00	2.51E-08	3.00	6.99E-11	4.00
256	1.60E-06	2.00	3.14E-09	3.00	4.37E-12	4.00

6. Concluding Remarks

In this paper, we considered a fractional collocation method for solving Volterra integral-differential equations with noncompact operators and nonsmooth solutions. We proved the solvability of the numerical scheme, and obtained *hp*-error estimates under the $H^1_{0,\lambda-1}$-norm. The theoretical results showed that, by choosing appropriate fractional exponent λ, both the *p*-version and the *h*-version of the method can attain high order convergence even for nonsmooth solution. The numerical experiments showed the effectiveness of the method and confirmed the theoretical analysis.

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