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SIMULTANEOUSLY IMAGING AN INHOMOGENEOUS CONDUCTIVE MEDIUM AND VARIOUS IMPENETRABLE OBSTACLES*

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Abstract

Consider the inverse scattering of time-harmonic acoustic waves by a mixed-type scatterer consisting of an inhomogeneous penetrable medium with a conductive transmission condition and various impenetrable obstacles with different kinds of boundary conditions. Based on the establishment of the well-posedness result of the direct problem, we intend to develop a modified factorization method to simultaneously reconstruct both the support of the inhomogeneous conductive medium and the shape and location of various impenetrable obstacles by means of the far-field data for all incident plane waves at a fixed wave number. Numerical examples are carried out to illustrate the feasibility and effectiveness of the proposed inversion algorithms.

Mathematics subject classification: 35R30, 35Q60, 35P25, 78A46.
Key words: Inverse acoustic scattering, Modified factorization method, Numerical reconstruction, Inhomogeneous medium.

1. Introduction

In this paper, we study the inverse problem of reconstructing a mixed-type scatterer from the far-field measurements produced by all the incident plane waves at a fixed wave number. The scatterer is supposed to be the union of an inhomogeneous medium with the conductive transmission condition and different kinds of impenetrable obstacles. This problem occurs in lots of application areas such as radar and sonar, medical imaging and non-destructing testing, etc. Precisely, let an open bounded obstacle D_1 denote the inhomogeneous penetrable medium with a C^2-smooth boundary ∂D_1 and an open bounded obstacle D_2 denote the impenetrable obstacle with a C^2-smooth boundary ∂D_2. Denote by $D_0 := \mathbb{R}^n \backslash (\overline{D}_1 \cup \overline{D}_2)$ (where $n = 2, 3$, for convenience, we will consider the case when $n = 3$) which is connected. We further assume that $D_1 \cap D_2 = \emptyset$ (See Fig. 1.1 for the geometric configuration of the mixed scattering problem).

Suppose that D_1 is filled with an inhomogeneous material characterized by $n(x)$, which is known as the refractive index satisfying that $n(x) \in L^\infty(\mathbb{R}^3)$ with $\mathrm{Re}[n(x)] < 1$ and $\mathrm{Im}[n(x)] \geq c_0 > 0$ with a positive constant c_0, whereas the exterior part D_0 is filled with a homogeneous material with the refractive index $n(x) = 1$. For simplicity, we only consider the case when an impedance boundary condition is imposed on ∂D_2. The same results can be similarly extended to the other cases, e.g. the Dirichlet or the Neumann boundary condition on ∂D_2. Consider the incident wave field $u^i = e^{ikx\cdot d}$ with the wave number $k > 0$ and the incident

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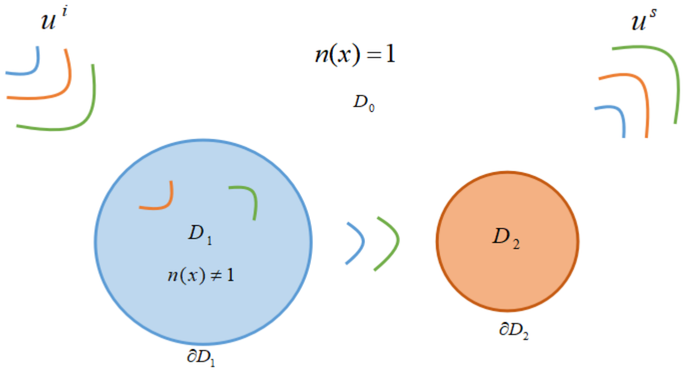


Fig. 1.1. Graphical representation of the mixed scattering problem.

direction $d \in \mathbb{S}^2$. Then scattering of time-harmonic acoustic waves by the mixed-type scatterer can be modeled by the following Helmholtz equation with a conductive transmission boundary condition on ∂D_1 and an impedance boundary condition on ∂D_2:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D_0, \\ \Delta v + k^2 n(x) v = 0 & \text{in } D_1, \\ u - v = 0 & \text{on } \partial D_1, \\ \dfrac{\partial u}{\partial \nu} - \dfrac{\partial v}{\partial \nu} + \mu u = 0 & \text{on } \partial D_1, \\ \dfrac{\partial u}{\partial \nu} + i\lambda u = 0 & \text{on } \partial D_2. \end{cases} \tag{1.1}$$

Here ν is the unit normal on ∂D_1 directed into $\mathbb{R}^3\backslash\overline{D}_1$, and on ∂D_2 directed into $\mathbb{R}^3\backslash\overline{D}_2$, respectively, and μ is the constant conductivity parameter satisfying that $\mathrm{Re}(\mu) < 0, \mathrm{Im}(\mu) \geq \mu_0 > 0, \lambda > 0$ is a positive constant, and $u = u^i + u^s$ denotes the total field in D_0 and $v = u^i + v^s$ denotes the total field in D_1 with the incident wave $u^i = e^{ikx\cdot d}$ and the scattered fields u^s and v^s, respectively. Moreover, the scattered field u^s satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial |x|} - iku^s = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \ \to \ \infty. \tag{1.2}$$

It is well-known that the scattered field u^s has the asymptotic behavior [6]

$$u^s(x) = \frac{e^{ik|x|}}{4\pi|x|} u_\infty(\widehat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \ \to \ \infty, \tag{1.3}$$

uniformly for all $\widehat{x} = x/|x|$, where u_∞ is known as the far-field pattern of u^s, which is an analytic function defined on $\mathbb{S}^2$.

The well-posedness of the scattering problem (1.1)-(1.2) can be established by applying the variational method (see also [15, 23]). In the current paper, we are interested in the inverse problem of simultaneously reconstructing the shape and location of the inhomogeneous penetrable medium D_1 and the impenetrable obstacle D_2 from a knowledge of the far-field pattern u_∞ for all incident plane waves at a fixed frequency by using a modified factorization method. The factorization method was first introduced by Kirsch [9] for the Dirichlet scattering problem. We also refer the readers to the monographs [3, 12] for a comprehensive account on the inverse scattering by obstacles with different kinds of boundary conditions. Kirsch *et al.* [10,11] also extended the method to the inhomogeneous medium scattering problems or to the layered cavity scattering problems [16,21]. Recently, the factorization method has been applied to the inverse problem of reconstructing an inhomogeneous medium with unknown buried objects inside [20,25] or recovering an impenetrable buried obstacle from an inhomogeneous background medium [5,14,26]. There are also some related numerical results for the inverse scattering of time-harmonic acoustic plane waves by mixed scatterers. In [22] the classical factorization method of [9] has been justified in reconstructing a mixed scatterer which is the union of a sound-soft impenetrable obstacle and an imperfect crack. A mixed inverse scattering problem of acoustic waves by a union of an impenetrable sound-soft obstacle and an inhomogeneous penetrable medium was studied in [13] by using the factorization method. For the special case when an impenetrable sound-soft obstacle is buried in an inhomogeneous medium, the numerical analysis of the factorization method for the recovery of the inhomogeneous medium can be found in [24]. For the case when $D_2 = \emptyset$, the validity of the classical factorization method proposed in [9] was justified in [2], which was later extended to the anisotropic medium scattering case in [1]. However, the mathematical theory and numerical method developed in [1,2] can not be applied to solve our inverse problem due to the fact that the factorization of the far-field operator is only compact. To overcome this difficulty, we shall develop a modified factorization method for our inverse problem. In fact, we are trying to construct a sequence of perturbation operators F_m of the far-field operator F in an appropriate way such that F_m is independent of the refractive index $n(x)$ of the inhomogeneous medium and the boundary conditions imposed on the impenetrable obstacle D_2. It is expected that the perturbation operators F_m can satisfy the range identity in [12, Theorem 2.15] for each $m \in \mathbb{N}_+$. Then the far-field operator F can be viewed as a sufficiently small perturbation of a perturbation operator F_{m_0} for some sufficient large $m_0 \in \mathbb{N}_+$. This further means that the noisy operator F^δ is also a small perturbation of $F^\delta_{m_0}$. Consequently, the inhomogeneous medium D_1 and the impenetrable obstacle D_2 can be numerically reconstructed by using the spectral data of F and F^δ.

Some other qualitative methods such as the linear sampling method or the reciprocity gap functional method have been developed for the inverse scattering associated with the inhomogeneous background [4,17,18]. We remark that the factorization method could give a rigorous characterization of the support of the target, which implies that it is the most rigorously justified technique within the class of qualitative methods in inverse scattering. So we would like to derive a modified factorization method as an analytical as well as a numerical tool for solving our inverse problem. Many other non-iterative techniques for inverse medium scattering problems are also developed, including point source methods [19] and the iteration method [8,27].

The remaining part of this paper is organized as follows. In Section 2, we provide the well-posedness result of the direct scattering problem (1.1)-(1.2) and some properties of the data-to-pattern operator. Section 3 is devoted to the justification of a modified factorization method for simultaneously recovering the inhomogeneous conductive medium and the shape and location of the impenetrable obstacle. Numerical examples are provided to illustrate the efficiency of the developed inversion algorithms in Section 4.

2. Properties of the Data-to-pattern Operator G

In this section we provide some important properties for the data-to-pattern operator G defined in the following (2.3). We begin with the statement of the well-posedness result of the scattering problem (1.1)-(1.2) (for a proof we refer the reader to [15, 23]). Noting that the incident field u^i satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$, thus the scattering field denoted by $(u, v) := (u^s, v^s)$ satisfies the following boundary value problem:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D_0, \\ \Delta v + k^2 n(x) v = -qf & \text{in } D_1, \\ u - v = 0 & \text{on } \partial D_1, \\ \dfrac{\partial u}{\partial \nu} - \dfrac{\partial v}{\partial \nu} + \mu u = -g & \text{on } \partial D_1, \\ \dfrac{\partial u}{\partial \nu} + i\lambda u = -h & \text{on } \partial D_2, \\ \lim\limits_{r\to\infty} r\Big(\dfrac{\partial u}{\partial r} - iku\Big) = 0, & r = |x|, \end{cases} \tag{2.1}$$

where $f = u^i$ in $D_1, g = \mu u^i$ on $\partial D_1, h = \partial u^i/\partial \nu + i\lambda u^i$ on ∂D_2 and $q := k^2[n(x) - 1]$ in D_1.

We now state the well-posedness results for problem (2.1).

Theorem 2.1. *For any $f \in L^2(D_1), g \in H^{-1/2}(\partial D_1)$ and $h \in H^{-1/2}(\partial D_2)$, there exists a unique solution $(u, v) \in H^1(B_R\backslash(\overline{D}_1 \cup \overline{D}_2)) \times H^1(D_1)$ to problem (2.1) satisfying that*

$$\|u\|_{H^1(B_R\backslash(\overline{D}_1\cup\overline{D}_2))} + \|v\|_{H^1(D_1)} \leq C\left(\|f\|_{L^2(D_1)} + \|g\|_{H^{-\frac{1}{2}}(\partial D_1)} + \|h\|_{H^{-\frac{1}{2}}(\partial D_2)}\right), \tag{2.2}$$

where C is a positive constant depending on R. Here B_R is a large ball with the radius R large enough such that $\overline{D}_1 \cup \overline{D}_2 \subset B_R$.

Based on Theorem 2.1, we introduce the data-to-pattern operator $G : Y \to L^2(\mathbb{S}^2)$ by

$$G(f, g, h)^T = u_\infty, \tag{2.3}$$

where

$$Y := L^2(D_1) \times H^{-\frac{1}{2}}(\partial D_1) \times H^{-\frac{1}{2}}(\partial D_2),$$

and u_∞ is the far-field pattern of the solution u to the problem (2.1) with the given data $(f, g, h)^T \in Y$. For the solution operator G, we have following lemma.

Lemma 2.1. *G is compact and has dense range in $L^2(\mathbb{S}^2)$.*

Proof. It is obvious that the compactness of the operator G can easily derived from the interior regularity results of elliptic equations [7]. In order to prove the denseness of the range of G in $L^2(\mathbb{S}^2)$, it suffices to prove that the L^2-adjoint operator G^* of G is injective.

Let (u, v) be a solution of the problem (1.1)-(1.2) corresponding to the incident field

$$u^i(y) = \int_{\mathbb{S}^2} e^{-ikd\cdot y}\overline{\varphi(d)}ds(d), \quad y \in \mathbb{R}^3, \quad \varphi \in L^2(\mathbb{S}^2). \tag{2.4}$$

Assume that (w, p) is a solution of the problem (2.1) with the data $(f, g, h)^T$. It then follows from the Green's Representation theorem that

$$\begin{aligned} w_\infty(d) = &\int_{\partial D_1} \left[w(y)\frac{\partial e^{-ikd\cdot y}}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y)e^{-ikd\cdot y}\right]ds(y) \\ &+ \int_{\partial D_2} \left[w(y)\frac{\partial e^{-ikd\cdot y}}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y)e^{-ikd\cdot y}\right]ds(y). \end{aligned}$$

Hence, it is deduced from the definition of the operator G that for $\varphi \in L^2(\mathbb{S}^2)$, we have

$$
\begin{aligned}
\left\langle G(f,g,h)^T, \varphi \right\rangle_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} w_\infty(d) \cdot \overline{\varphi(d)} ds(d) \\
&= \int_{\partial D_1} \left[w(y) \frac{\partial u^i}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y) u^i(y) \right] ds(y) \\
&\quad + \int_{\partial D_2} \left[w(y) \frac{\partial u^i}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y) u^i(y) \right] ds(y). \qquad (2.5)
\end{aligned}
$$

Then by using the transmission boundary conditions on ∂D_1 and the impedance boundary condition on ∂D_2 and the fact that both u^s and w satisfy the Sommerfeld radiation condition, we have

$$
\begin{aligned}
\left\langle G(f,g,h)^T, \varphi \right\rangle_{L^2(\mathbb{S}^2)} &= \int_{\partial D_1} \left[w(y) \frac{\partial u}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y) u(y) \right] ds(y) \\
&\quad + \int_{\partial D_2} \left[w(y) \frac{\partial u}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y) u(y) \right] ds(y) \\
&= \int_{D_1} q v f dy + \int_{\partial D_1} g u ds + \int_{\partial D_2} h u ds \\
&= \left\langle (f,g,h)^T, G^* \varphi \right\rangle_{L^2(\mathbb{S}^2)}.
\end{aligned}
$$

Therefore, the adjoint operator G^* can be characterized as

$$
G^* \varphi = (\overline{qv}|_{D_1}, \overline{u}|_{\partial D_1}, \overline{u}|_{\partial D_2})^T. \qquad (2.6)
$$

Let $G^* \varphi = 0$, which leads to that $v = 0$ in D_1 and $u = 0$ on $\partial D_1 \cup \partial D_2$. This together with the transmission conditions on ∂D_1 further gives that $u = v = 0$ on ∂D_1 and $u = \partial u / \partial \nu = 0$ on ∂D_1. It then follows from the Holmgren's uniqueness theorem that $u = u^i + u^s = 0$ in $\mathbb{R}^3 \backslash \overline{D}_2$. Since u^i does not satisfy the radiation condition, one thus obtains that $u^i = 0$ in $\mathbb{R}^3 \backslash \overline{D}_2$. This allows us to employ [6, Theorem 3.19] to deduce that $\varphi = 0$, which shows the injectivity of the operator G^*. This completes the proof of the lemma. □

Theorem 2.2. *For $z \in \mathbb{R}^3$, define $\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z}$ for $\widehat{x} \in \mathbb{S}^2$. Then we have*

$$
z \in (D_1 \cup D_2) \iff \phi_z(\widehat{x}) \in \mathcal{R}(G),
$$

where $\mathcal{R}(G)$ denotes the range of G.

Proof. Let us first assume that $z \in (D_1 \cup D_2)$. Then we can choose a small ball $B_\epsilon(z)$ with center at z and the radius $\epsilon > 0$ satisfying that $\overline{B_\epsilon(z)} \subseteq (D_1 \cup D_2)$. Choose a cut-off function $\chi \in C^\infty(\mathbb{R}^3)$ with $\chi(t) = 1$ for $|t| \geq \epsilon$ and $\chi(t) = 0$ for $|t| \leq \epsilon/2$ and define

$$
w_z(x) = \chi(|x - z|) \Phi(x, z), \quad x \in \mathbb{R}^3.
$$

It is easily seen that $w_z(x) \in C^\infty(\mathbb{R}^3)$, which satisfies that $w_z = \Phi(\cdot, z)$ for $|x - z| \geq \epsilon$. A direct computation yields

$$
\Delta w_z + k^2 n w_z = \Phi \Delta \chi + \chi \Delta \Phi + 2 \nabla \chi \nabla \Phi + k^2 n \chi \Phi =: -q f^{(0)} \quad \text{in } D_1,
$$

and

$$
\mu w_z|_{\partial D_1} =: -g^{(0)}, \quad \left(\frac{\partial w_z}{\partial \nu} + i \lambda w_z \right)|_{\partial D_2} =: -h^{(0)}.
$$

It can be checked that $f^{(0)} \in L^2(D_1), g^{(0)} \in H^{-1/2}(\partial D_1)$ and $h^{(0)} \in H^{-1/2}(\partial D_2)$. Clearly, w_z is a solution of (2.1) with the data $(f^{(0)}, g^{(0)}, h^{(0)})$. Thus, $G(f^{(0)}, g^{(0)}, h^{(0)})^T = w_z^\infty = \phi_z$, that is $\phi_z \in \mathcal{R}(G)$.

Now let $z \notin (D_1 \cup D_2)$. We assume that there exists $(\tilde{f}, \tilde{g}, \tilde{h})^T \in Y$ such that $G(\tilde{f}, \tilde{g}, \tilde{h})^T = \phi_z$ and we let $\tilde{w}$ be a solution to the problem (2.1) with the data $(\tilde{f}, \tilde{g}, \tilde{h})$. Thus one has $\tilde{w}_\infty = G(\tilde{f}, \tilde{g}, \tilde{h})^T = \phi_z$. With the aid of Rellich's lemma and the unique continuation principle, we immediately have that $\tilde{w}(x) = \Phi(x, z)$ for $x \in \mathbb{R}^3 \backslash (\overline{D}_1 \cup \overline{D}_2 \cup \{z\})$. However, $\|\tilde{w}\|_{H^1(B(z))} < +\infty$ and $\|\Phi(\cdot, z)\|_{H^1(B(z))}$ tends to $+\infty$, where $B(z)$ is a sufficiently small ball centered at z. This leads to a contradiction. The theorem is thus proved. □

3. A Modified Factorization Method for the Simultaneous Reconstruction

In this section we focus on the simultaneous reconstruction of the location and shape of the inhomogeneous media and the impenetrable obstacle. We begin with introducing the far-field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by

$$(Fg)(\widehat{x}) = \int_{\mathbb{S}^2} u_\infty(\widehat{x}; d) g(d) ds(d), \quad g \in L^2(\mathbb{S}^2), \tag{3.1}$$

where u_∞ is the far-field pattern of the scattered field u of the problem (2.1) associated with the incident wave $u^i = e^{ikx\cdot d}$. Obviously, Fg is the far-field pattern corresponding to the incident field of the Herglotz wave function

$$v_g(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \quad x \in \mathbb{R}^3. \tag{3.2}$$

Define the incident operator $H : L^2(\mathbb{S}^2) \to Y$ by $H = (H_1, H_2, H_3)^T$ with

$$H_1 g(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \qquad x \in D_1, \tag{3.3}$$

$$H_2 g(x) = \mu \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \qquad x \in \partial D_1, \tag{3.4}$$

$$H_3 g(x) = \int_{\mathbb{S}^2} \left(\frac{\partial}{\partial \nu} + i\lambda \right) e^{ikx\cdot d} g(d) ds(d), \quad x \in \partial D_2. \tag{3.5}$$

It then follows from the superposition principle and the definition of the operator G that $F = GH$. In order to derive the factorization of the far-field operator F, we next introduce the operators $\mathbf{V}_{D_1D_1}, V_{D_1\partial D_i}, \tilde{S}_{\partial D_j D_1}, S_{\partial D_i \partial D_j}, \tilde{K}_{\partial D_j D_1}, K_{\partial D_i \partial D_j}, K'_{\partial D_i \partial D_j}, T_{\partial D_i \partial D_j}$, $i, j = 1, 2$, defined by

$$(\mathbf{V}_{D_1D_1}\varphi)(x) = \int_{D_1} \Phi(x, y)\varphi(y) dy, \qquad x \in D_1,$$

$$(V_{D_1\partial D_i}\varphi)(x) = \int_{D_1} \Phi(x, y)\varphi(y) dy, \qquad x \in \partial D_i,$$

$$(\tilde{S}_{\partial D_j D_1}\varphi)(x) = \int_{\partial D_j} \Phi(x, y)\varphi(y) ds(y), \quad x \in D_1,$$

$$(S_{\partial D_i \partial D_j}\varphi)(x) = \int_{\partial D_i} \Phi(x, y)\varphi(y) ds(y), \quad x \in \partial D_j,$$

$$\begin{aligned}
\big(\tilde{K}_{\partial D_j D_1}\varphi\big)(x) &= \int_{\partial D_j} \frac{\partial \Phi(x,y)}{\partial \nu_y}\varphi(y)ds(y), \qquad x \in D_1,\\
(K_{\partial D_i \partial D_j}\varphi)(x) &= \int_{\partial D_i} \frac{\partial \Phi(x,y)}{\partial \nu_y}\varphi(y)ds(y), \qquad x \in \partial D_j,\\
\big(K'_{\partial D_i \partial D_j}\varphi\big)(x) &= \frac{\partial}{\partial \nu_x}\int_{\partial D_i} \Phi(x,y)\varphi(y)ds(y), \quad x \in \partial D_j,\\
(T_{\partial D_i \partial D_j}\varphi)(x) &= \frac{\partial}{\partial \nu_x}\int_{\partial D_i} \frac{\partial \Phi(x,y)}{\partial \nu_y}\varphi(y)ds(y), \quad x \in \partial D_j.
\end{aligned}$$

By the boundedness of the trace operator, we deduce that the operators

$$\begin{array}{ll}
\mathbf{V}_{D_1D_1} : L^2(D_1) \;\to\; H^2(D_1), & V_{D_1\partial D_i} : L^2(D_1) \;\to\; H^{\frac{3}{2}}(\partial D_i),\\
\tilde{S}_{\partial D_j D_1} : H^{-\frac{1}{2}}(\partial D_j) \;\to\; H^1(D_1), & S_{\partial D_i \partial D_j} : H^{-\frac{1}{2}}(\partial D_i) \;\to\; H^{\frac{1}{2}}(\partial D_j),\\
\tilde{K}_{\partial D_j D_1} : H^{\frac{1}{2}}(\partial D_j) \to H^1(D_1), & K_{\partial D_i \partial D_j} : H^{\frac{1}{2}}(\partial D_i) \;\to\; H^{\frac{1}{2}}(\partial D_j),\\
K'_{\partial D_i \partial D_j} : H^{-\frac{1}{2}}(\partial D_i) \;\to\; H^{-\frac{1}{2}}(\partial D_j), & T_{\partial D_i \partial D_j} : H^{\frac{1}{2}}(\partial D_i) \;\to\; H^{-\frac{1}{2}}(\partial D_j)
\end{array}$$

are all bounded. Based on these operators, we have the following factorization theorem.

Theorem 3.1. *F has the following factorization form:*

$$F = GM^*G^*, \tag{3.6}$$

where $M : Y^* \to Y$ is defined by

$$M=\begin{pmatrix} q^{-1}I-\boldsymbol{V}_{D_1D_1} & -\overline{\mu}\tilde{S}_{\partial D_1 D_1} & -\tilde{K}_{\partial D_2 D_1}+i\lambda\tilde{S}_{\partial D_2 D_1}\\ -\mu V_{D_1\partial D_1} & -|\mu|^2 S_{\partial D_1\partial D_1}+\overline{\mu}I & -\mu K_{\partial D_2\partial D_1}+i\lambda\mu S_{\partial D_2\partial D_1}\\ -\dfrac{\partial V_{D_1\partial D_2}}{\partial\nu}-i\lambda V_{D_1\partial D_2} & -\overline{\mu}K'_{\partial D_1\partial D_2}-i\lambda\overline{\mu}S_{\partial D_1\partial D_2} & A_{33} \end{pmatrix} \tag{3.7}$$

with $A_{33} := -T_{\partial D_2\partial D_2} + i\lambda K'_{\partial D_2\partial D_2} - i\lambda K_{\partial D_2\partial D_2} - \lambda^2 S_{\partial D_2\partial D_2} - i\lambda I$.

Proof. By the definition of the incident operator H, one can deduce that the adjoint incident operator $H^* : Y^* \to L^2(\mathbb{S}^2)$ has the form

$$\begin{aligned}
(H^*\varphi)(d) = &\int_{D_1} e^{-iky\cdot d}\varphi_1(y)dy + \overline{\mu}\int_{\partial D_1} e^{-iky\cdot d}\varphi_2(y)ds(y)\\
&+ \int_{\partial D_2}\left(\frac{\partial}{\partial\nu} - i\lambda\right) e^{-iky\cdot d}\varphi_3(y)ds(y),
\end{aligned} \tag{3.8}$$

which is the far-field pattern of the function W defined by

$$\begin{aligned}
W(x) = &\int_{D_1} \Phi(x,y)\varphi_1(y)dy + \overline{\mu}\int_{\partial D_1} \Phi(x,y)\varphi_2(y)ds(y)\\
&+ \int_{\partial D_2}\left(\frac{\partial}{\partial\nu} - i\lambda\right)\Phi(x,y)\varphi_3(y)ds(y), \quad x \in \mathbb{R}^3\backslash\overline{D}_2.
\end{aligned} \tag{3.9}$$

It is easily found that W solves problem (2.1) with the following data:

$$\begin{aligned}
f &= q^{-1}\varphi_1 - \mathbf{V}_{D_1D_1}\varphi_1 - \overline{\mu}\tilde{S}_{\partial D_1 D_1}\varphi_2 - \tilde{K}_{\partial D_2 D_1}\varphi_3 + i\lambda\tilde{S}_{\partial D_2 D_1}\varphi_3,\\
g &= -\mu V_{D_1\partial D_1}\varphi_1 - |\mu|^2 S_{\partial D_1\partial D_1}\varphi_2 + \overline{\mu}\varphi_2 - \mu K_{\partial D_2\partial D_1}\varphi_3 + i\lambda\mu S_{\partial D_2\partial D_1}\varphi_3,\\
h &= -\frac{\partial V_{D_1\partial D_2}}{\partial\nu}\varphi_1 - i\lambda V_{D_1\partial D_2}\varphi_1 - \overline{\mu}K'_{\partial D_1\partial D_2}\varphi_2 - i\lambda\overline{\mu}S_{\partial D_1\partial D_2}\varphi_2 + A_{33}\varphi_3,
\end{aligned}$$

where A_{33} is defined above. Therefore,

$$H^*\varphi = W_\infty = G(f,g,h)^T = GM\varphi.$$

Thus, $H = M^*G^*$. Recalling $F = GH$ yields that $F = GM^*G^*$. This completes the proof of the theorem. $\square$

In what follows, we first decompose the middle operator M into $M = M_1 + M_2$ as follows:

$$\begin{aligned} M =& \begin{pmatrix} q^{-1}I & 0 & 0 \\ 0 & -|\mu|^2 S^i_{\partial D_1 \partial D_1} & 0 \\ 0 & 0 & -T^i_{\partial D_2\partial D_2} \end{pmatrix} \\ &+ \begin{pmatrix} -\mathbf{V}_{D_1D_1} & -\overline{\mu}\tilde{S}_{\partial D_1 D_1} & -\tilde{K}_{\partial D_2 D_1} + i\lambda \tilde{S}_{\partial D_2 D_1} \\ -\mu V_{D_1\partial D_1} & -|\mu|^2(S_{\partial D_1\partial D_1} - S^i_{\partial D_1\partial D_1}) + \overline{\mu} I & -\mu K_{\partial D_2\partial D_1} + i\lambda\mu S_{\partial D_2 \partial D_1} \\ -\dfrac{\partial V_{D_1\partial D_2}}{\partial \nu} - i\lambda V_{D_1\partial D_2} & -\overline{\mu}K'_{\partial D_1\partial D_2} - i\lambda\overline{\mu}S_{\partial D_1 \partial D_2} & A^1_{33} \end{pmatrix} \\ =&: M_1 + M_2, \end{aligned} \tag{3.10}$$

where

$$A^1_{33} := -\big(T_{\partial D_2\partial D_2} - T^i_{\partial D_2\partial D_2}\big) + i\lambda K'_{\partial D_2\partial D_2} - i\lambda K_{\partial D_2\partial D_2} - \lambda^2 S_{\partial D_2\partial D_2} - i\lambda I,$$

$S^i_{\partial D_1\partial D_1}$ and $T^i_{\partial D_2\partial D_2}$ are the single-layer and the derivative of the double-layer boundary operators corresponding to the wave number $k = i$, respectively.

Then by direct calculations we have the following theorem.

Theorem 3.2. *Suppose that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_2. Then the operator M defined in Theorem* 3.1 *is invertible and $M^{-1} = M_1^{-1} + M_3$, where*

$$M_1^{-1} = \begin{pmatrix} qI & 0 & 0 \\ 0 & -|\mu|^2 S^{i,-1}_{\partial D_1\partial D_1} & 0 \\ 0 & 0 & -T^{i,-1}_{\partial D_2\partial D_2} \end{pmatrix}, \tag{3.11}$$

and the operator $M_3 = -M_1^{-1}M_2M^{-1}$ is compact.

Proof. Obviously, M can be decomposed into (3.10). Then we can derive that M_1 is invertible on Y. The compactness of M_2 follows from the compact embedding theorem and the compactness of the ingredient operators in M_2. This ensures that $M = M_1 + M_2$ is a Fredholm-type operator. Now we let $M\varphi = 0$ for $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in Y^*$. Based on Theorem 3.1, it can be concluded that $w(x)$ defined by (3.9) is the solution to problem (2.1) with the boundary data $(f,g,h) = (0,0,0)^T$. Then the uniqueness of problem (2.1) leads to that $W(x) = 0$ in $\mathbb{R}^3\backslash\overline{D_2}$. Since $\Delta W + k^2 W = -\varphi_1$ in D_1, we have that $\varphi_1 = 0$. In addition, $\varphi_2 = 0$ can be derived by using the jump relations of the derivative of the single layer boundary operator defined on ∂D_1. Moreover, it can be verified that $W(x)$=0 in D_2 due to the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_2, whereas $\varphi_3 = 0$ follows again from the jump relations of the derivative of the single layer boundary operator on ∂D_2. This proves the fact that the operator M is invertible, and a direct calculation yields that $M^{-1} = M_1^{-1} - M_1^{-1}M_2M^{-1} := M_1 + M_3$, where M_3 is compact. This ends the proof of the theorem. $\square$

From Theorem 3.2 we can easily observe that the middle operator M in the factorization of the far-field operator F can not be decomposed into a coercive part for the case when $\mathrm{Re}[n(x)] < 1$ since

$$\langle S^i\varphi, \varphi\rangle_{\partial D_l} \geq C_l\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial D_l)}, \quad -\langle T^i\varphi, \varphi\rangle_{\partial D_l} \geq C_l\|\varphi\|^2_{H^{\frac{1}{2}}(\partial D_l)}, \quad l = 1, 2.$$

Hence, the classical factorization method proposed by Kirsch [12] can not be applied directly. In order to derive a suitable factorization of the far-field operator F, we first rewrite it in the form

$$F = H^* M^{-1} H. \tag{3.12}$$

Then we intend to introduce a series of perturbation operators F_m of F in the sense that $\lim_{m\to\infty} \|F_m - F\|_{L^2(\mathbb{S}^2)} = 0$. It will be shown that for any $m \in \mathbb{N}$, F_m has a suitable factorization satisfying the range identity [12, Theorem 2.15]. Therefore, the mixed-type scatterer can be recovered approximately from a knowledge of the far-field data F. Before going further, we can easily obtain the fact that $\mathcal{R}(H^*) = \mathcal{R}(G)$ since $H^* = GM$ and M is invertible. This together with Theorem 2.2 yields the following result.

Theorem 3.3. *It holds that*

$$z \in (D_1 \cup D_2) \iff \phi_z(\widehat{x}) \in \mathcal{R}(H^*). \tag{3.13}$$

To derive a suitable modified factorization of the far-field operator, we introduce the following auxiliary operators:

$$\begin{aligned} \widetilde{H}_{D_1} : L^2(\mathbb{S}^2) &\to H^{\frac{1}{2}}(\partial\Omega_1), \\ \widetilde{H}_{D_2} : L^2(\mathbb{S}^2) &\to H^{\frac{1}{2}}(\partial\Omega_2), \end{aligned}$$

defined by

$$\big(\widetilde{H}_{D_1} g\big)(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \quad x \in \partial\Omega_1, \tag{3.14}$$

$$\big(\widetilde{H}_{D_2} g\big)(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \quad x \in \partial\Omega_2, \tag{3.15}$$

Clearly, $\widetilde{H}_{D_l}$, $l = 1, 2$, is bounded and well-defined. Here the open and bounded domains Ω_l with C^2-boundaries $\partial\Omega_l$, $l = 1, 2$, satisfy that $\overline{D}_1 \subset \Omega_1$, $\overline{D}_2 \subset \Omega_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$, see Fig. 3.1. Hence, we can define the perturbation operators

$$F_m^{D_1} = F - \rho_m^{(3)} \widetilde{H}^*_{D_2} S^{-1}_{\partial\Omega_2} \widetilde{H}_{D_2}, \tag{3.16}$$

$$F_m^{D_2} = F + \big(\rho_m^{(1)} + \rho_m^{(2)}\big) \widetilde{H}^*_{D_1} S^{-1}_{\partial\Omega_1} \widetilde{H}_{D_1}, \tag{3.17}$$

where $S_{\partial\Omega_l}$, $l = 1, 2$, are the single-layer operators on $\partial\Omega_l$ with respect to the wave number $k = i$, and $\rho_m^{(\widetilde{l})}$, $\widetilde{l} = 1, 2, 3$, are positive numbers satisfying that $\rho_m^{(\widetilde{l})} \to 0$ as $m \to \infty$. This ensures that

$$\big\|F_m^{D_l} - F\big\|_{L^2(\mathbb{S}^2)} \;\to\; 0 \quad \text{as} \quad m \;\to\; \infty, \quad l = 1, 2.$$

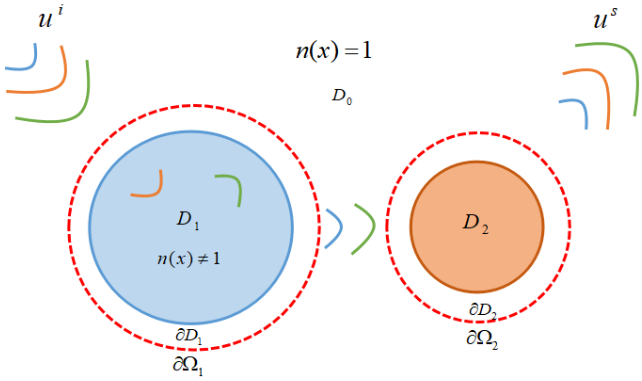


Fig. 3.1. Graphical representation of domains.

Theorem 3.4. *The operators $F_m^{D_1}$ and $F_m^{D_2}$ have the following factorizations:*

$$F_m^{D_1} = \widetilde{\boldsymbol{H}}^*_{D_1} M_{D_1} \widetilde{\boldsymbol{H}}_{D_1}, \quad \widetilde{\boldsymbol{H}}_{D_1} = \left(H_1, H_2, \widetilde{H}_{D_2}\right)^T, \tag{3.18}$$

$$F_m^{D_2} = \widetilde{\boldsymbol{H}}^*_{D_2} M_{D_2} \widetilde{\boldsymbol{H}}_{D_2}, \quad \widetilde{\boldsymbol{H}}_{D_2} = \left(\widetilde{H}_{D_1}, \widetilde{H}_{D_1}, H_3\right)^T \tag{3.19}$$

with the middle operators

$$M_{D_1} : L^2(D_1) \times H^{-\frac{1}{2}}(\partial D_1) \times H^{\frac{1}{2}}(\partial\Omega_2) \;\to\; L^2(D_1) \times H^{\frac{1}{2}}(\partial D_1) \times H^{-\frac{1}{2}}(\partial\Omega_2),$$
$$M_{D_2} : H^{\frac{1}{2}}(\partial\Omega_1) \times H^{\frac{1}{2}}(\partial\Omega_1) \times H^{-\frac{1}{2}}(\partial D_2) \;\to\; H^{-\frac{1}{2}}(\partial\Omega_1) \times H^{-\frac{1}{2}}(\partial\Omega_1) \times H^{\frac{1}{2}}(\partial D_2).$$

Here M_{D_1} and M_{D_2} are given by

$$M_{D_1} = \begin{pmatrix} qI & 0 & 0 \\ 0 & -|\mu|^2 S^{i,-1}_{\partial D_1 \partial D_1} & 0 \\ 0 & 0 & -\rho_m^{(3)} S^{-1}_{\partial\Omega_2} \end{pmatrix} + M^2_{D_1}, \tag{3.20}$$

$$M_{D_2} = \begin{pmatrix} \rho_m^{(1)} S^{-1}_{\partial\Omega_1} & 0 & 0 \\ 0 & \rho_m^{(2)} S^{-1}_{\partial\Omega_1} & 0 \\ 0 & 0 & -T^{i,-1}_{\partial D_2 \partial D_2} \end{pmatrix} + M^2_{D_2} \tag{3.21}$$

with the compact operators $M^2_{D_1}$ and $M^2_{D_2}$, and

$$\mathrm{Re}(-M_{D_1}) = C_{D_1} + Q_{D_1}, \quad \mathrm{Re}(M_{D_2}) = C_{D_2} + Q_{D_2}$$

with the positive coercive operators C_{D_1}, C_{D_2} and the compact operators Q_{D_1}, Q_{D_2}.

Proof. First, we define a compact operator $L_3 : H^{1/2}(\partial\Omega_2) \to H^{-1/2}(\partial D_2)$ by

$$L_3 h_3 = \left(\frac{\partial w_3}{\partial \nu} + i\lambda w_3 \right)\Big|_{\partial D_2},$$

where w_3 satisfies the following boundary value problem:

$$\begin{cases} \Delta w_3 + k^2 w_3 = 0 & \text{in } \Omega_2, \\ w_3 = h_3 & \text{on } \partial\Omega_2 \end{cases}$$

with $h_3 \in H^{1/2}(\partial\Omega_2)$. This leads to that $H_3 = L_3\widetilde{H}_{D_2}$, which allows us to obtain

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & L_3 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ \widetilde{H}_{D_2} \end{pmatrix} =: \boldsymbol{L}_{D_1}\widetilde{\boldsymbol{H}}_{D_1}. \tag{3.22}$$

Therefore, with the aid of (3.12), (3.22) and Theorem 3.2, we derive that

$$\begin{aligned}
F_m^{D_1} &= F - \rho_m^{(3)}\widetilde{H}_{D_2}^* S_{\partial\Omega_2}^{-1}\widetilde{H}_{D_2} = \widetilde{\boldsymbol{H}}_{D_1}^*\big[\boldsymbol{L}_{D_1}^* M^{-1}\boldsymbol{L}_{D_1} + J_m^{(1)}\big]\widetilde{\boldsymbol{H}}_{D_1} \\
&= \widetilde{\boldsymbol{H}}_{D_1}^*\left\{\begin{pmatrix} qI & 0 & 0 \\ 0 & -|\mu|^2 S_{\partial D_1\partial D_1}^{i,-1} & 0 \\ 0 & 0 & -\rho_m^{(3)} S_{\partial\Omega_2}^{-1} \end{pmatrix}\right. \\
&\qquad\left. + \left[\boldsymbol{L}_{D_1}^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -T_{\partial D_2\partial D_2}^{i,-1} \end{pmatrix} \boldsymbol{L}_{D_1} + \boldsymbol{L}_{D_1}^* M_3 \boldsymbol{L}_{D_1}\right]\right\}\widetilde{\boldsymbol{H}}_{D_1} \\
&=: \widetilde{\boldsymbol{H}}_{D_1}^*\big(M_{D_1}^1 + M_{D_1}^2\big)\widetilde{\boldsymbol{H}}_{D_1} =: \widetilde{\boldsymbol{H}}_{D_1}^* M_{D_1}\widetilde{\boldsymbol{H}}_{D_1},
\end{aligned}$$

where $J_m^{(1)}$ is defined as

$$J_m^{(1)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho_m^{(3)} S_{\partial\Omega_2}^{-1} \end{pmatrix},$$

whence (3.20) follows.

Second, we define the compact operators

$$\begin{aligned}
L_1 : H^{\frac{1}{2}}(\partial\Omega_1) &\to L^2(D_1) \quad \text{by} \quad L_1 h = w|_{D_1}, \\
L_2 : H^{\frac{1}{2}}(\partial\Omega_1) &\to H^{\frac{1}{2}}(\partial D_1) \quad \text{by} \quad L_2 h = \mu w|_{\partial D_1},
\end{aligned}$$

where w satisfies the following boundary value problem:

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } \Omega_1, \\ w = h & \text{on } \partial\Omega_1 \end{cases}$$

with $h \in H^{1/2}(\partial\Omega_1)$.

It is easily seen that $H_1 = L_1\widetilde{H}_{D_1}$ and $H_2 = L_2\widetilde{H}_{D_1}$, and

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{H}_{D_1} \\ \widetilde{H}_{D_1} \\ H_3 \end{pmatrix} =: \boldsymbol{L}_{D_2}\widetilde{\boldsymbol{H}}_{D_2}.$$

Therefore, we conclude that

$$\begin{aligned}
F_m^{D_2} &= F + \rho_m^{(1)}\widetilde{H}_{D_1}^* S_{\partial\Omega_1}^{-1}\widetilde{H}_{D_1} + \rho_m^{(2)}\widetilde{H}_{D_1}^* S_{\partial\Omega_1}^{-1}\widetilde{H}_{D_1} \\
&= \widetilde{\boldsymbol{H}}_{D_2}^*\big[\boldsymbol{L}_{D_2}^* M^{-1}\boldsymbol{L}_{D_2} + J_m^{(2)}\big]\widetilde{\boldsymbol{H}}_{D_2} \\
&= \widetilde{\boldsymbol{H}}_{D_2}^*\left\{\begin{pmatrix} \rho_m^{(1)} S_{\partial\Omega_1}^{-1} & 0 & 0 \\ 0 & \rho_m^{(2)} S_{\partial\Omega_1}^{-1} & 0 \\ 0 & 0 & -T_{\partial D_2\partial D_2}^{i,-1} \end{pmatrix}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left[\boldsymbol{L}_{D_2}^* \begin{pmatrix} qI & 0 & 0 \\ 0 & -|\mu|^2 S_{\partial D_1 \partial D_1}^{i,-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{L}_{D_2} + \boldsymbol{L}_{D_2}^* M_3 \boldsymbol{L}_{D_2}\right]\Bigg\} \widetilde{\boldsymbol{H}}_{D_2} \\
=:\ &\widetilde{\boldsymbol{H}}_{D_2}^* (M_{D_2}^1 + M_{D_2}^2) \widetilde{\boldsymbol{H}}_{D_2} =: \widetilde{\boldsymbol{H}}_{D_2}^* M_{D_2} \widetilde{\boldsymbol{H}}_{D_2},
\end{aligned}
$$

where $J_m^{(2)}$ is defined by

$$
J_m^{(2)} := \begin{pmatrix} \rho_m^{(1)} S_{\partial\Omega_1}^{-1} & 0 & 0 \\ 0 & \rho_m^{(2)} S_{\partial\Omega_1}^{-1} & \\ 0 & 0 & 0 \end{pmatrix},
$$

whence (3.21) follows.

The decomposition of $\mathrm{Re}(-M_{D_1})$ and $\mathrm{Re}(M_{D_2})$ follows directly from (3.20), (3.21), the coercive properties of $S_{\partial D_1 \partial D_1}^i, -T_{\partial D_2 \partial D_2}^i, S_{\partial\Omega_1}^{-1}, S_{\partial\Omega_2}^{-1}$ and the fact that $\mathrm{Re}(q) < 0$. The proof of the theorem is now completed. $\square$

Theorem 3.5. *$\widetilde{\boldsymbol{H}}_{D_1}^*$ and $\widetilde{\boldsymbol{H}}_{D_2}^*$ are compact and have dense range in $L^2(\mathbb{S}^2)$.*

Proof. It is obvious that $\widetilde{\boldsymbol{H}}_{D_1}^*$ and $\widetilde{\boldsymbol{H}}_{D_2}^*$ are compact since they have continuous kernels. Moreover, based on Theorem 3.4, we know that $H^* = \widetilde{\boldsymbol{H}}_{D_1}^* \boldsymbol{L}_{D_1}^* = \widetilde{\boldsymbol{H}}_{D_2}^* \boldsymbol{L}_{D_2}^*$, this indicates that $\mathcal{R}(H^*) \subset \mathcal{R}(\widetilde{\boldsymbol{H}}_{D_1}^*)$ and $\mathcal{R}(H^*) \subset \mathcal{R}(\widetilde{\boldsymbol{H}}_{D_2}^*)$. In addition, since $H^* = GM$ and M is known to be invertible, one has $G = H^* M^{-1}$, which implies that $\mathcal{R}(G) \subset \mathbb{R}(H^*)$. So, we derive that $\mathcal{R}(G) \subset \mathcal{R}(\widetilde{\boldsymbol{H}}_{D_1}^*)$ and $\mathcal{R}(G) \subset \mathcal{R}(\widetilde{\boldsymbol{H}}_{D_2}^*)$. Then by Lemma 2.1, we conclude that $\widetilde{\boldsymbol{H}}_{D_1}^*$ and $\widetilde{\boldsymbol{H}}_{D_2}^*$ have dense range in $L^2(\mathbb{S}^2)$. The proof of the theorem is complete. $\square$

Theorem 3.6. *It holds that*

(i) $\mathrm{Im}\langle M_{D_1}\varphi, \varphi\rangle \geq 0$ *for all* $\varphi \in L^2(D_1) \times H^{-1/2}(\partial D_1) \times H^{1/2}(\partial\Omega_2)$, *and* $\mathrm{Im}\langle M_{D_2}\varphi, \varphi\rangle \geq 0$ *for all* $\varphi \in H^{1/2}(\partial\Omega_1) \times H^{1/2}(\partial\Omega_1) \times H^{-1/2}(\partial D_2)$.

(ii) $\mathrm{Im}\langle M_{D_1}\varphi, \varphi\rangle > 0$ *for all* $\varphi \in \overline{\mathcal{R}(\widetilde{\boldsymbol{H}}_{D_1})}$ *with* $\varphi \neq 0$, *and* $\mathrm{Im}\langle M_{D_2}\varphi, \varphi\rangle > 0$ *for all* $\varphi \in \overline{\mathcal{R}(\widetilde{\boldsymbol{H}}_{D_2})}$ *with* $\varphi \neq 0$.

Proof. (i) For any $\varphi \in L^2(D_1) \times H^{-1/2}(\partial D_1) \times H^{1/2}(\partial\Omega_2)$, define $\psi = (M^{-1})^* \boldsymbol{L}_{D_1}\varphi$, we have

$$
\begin{aligned}
\mathrm{Im}\langle M_{D_1}\varphi, \varphi\rangle &= \mathrm{Im}\big\langle \boldsymbol{L}_{D_1}^* M^{-1} \boldsymbol{L}_{D_1}\varphi, \varphi\big\rangle = \mathrm{Im}\big\langle M^{-1}\boldsymbol{L}_{D_1}\varphi, \boldsymbol{L}_{D_1}\varphi\big\rangle \\
&= \mathrm{Im}\big\langle \boldsymbol{L}_{D_1}\varphi, (M^{-1})^* \boldsymbol{L}_{D_1}\varphi\big\rangle = \mathrm{Im}\langle M^*\psi, \psi\rangle = \mathrm{Im}\langle \psi, M\psi\rangle.
\end{aligned}
$$

To prove $\mathrm{Im}\langle M_{D_1}\varphi, \varphi\rangle \geq 0$, we first prove that $\mathrm{Im}\langle M\psi, \psi\rangle \leq 0$. Define a function $W(x)$ by (3.9) with φ replaced by $\psi = (\psi_1, \psi_2, \psi_3)^T \in Y$. Then by similar arguments as that in Theorem 3.1, we obtain that

$$
\begin{aligned}
\langle M\psi, \psi\rangle &= \big(q^{-1}\psi_1, \psi_1\big)_{D_1} - (W_-, \psi_1)_{D_1} + \overline{\mu}\langle \psi_2, \psi_2\rangle_{\partial D_1} - \mu\langle W_+, \psi_2\rangle_{\partial D_1} \\
&\quad - \left\langle \frac{\partial W_+}{\partial \nu}, \psi_3 \right\rangle_{\partial D_2} - i\lambda\langle W_+, \psi_3\rangle_{\partial D_2} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

It is easily verified that

$$\begin{aligned}\mathrm{Im}(I_1) &= \int_{D_1} \mathrm{Im}(q^{-1})|\psi_1|^2 dx,\\ \mathrm{Im}(I_3) &= \mathrm{Im}(\overline{\mu}) \int_{\partial D_1} |\psi_2|^2 ds.\end{aligned}$$

Clearly, $\mathrm{Im}(I_1) = 0$ if $\mathrm{Im}[n(x)] = 0$, $\mathrm{Im}(I_1) \le 0$ if $\mathrm{Im}[n(x)] \ge c_0 > 0$ and $\mathrm{Im}(I_3) \le 0$ since $\mathrm{Im}(\mu) \ge \mu_0 > 0$. Applying Green's theorem, the jump relations, the transmission conditions on ∂D_1 and the asymptotic relationships yields that

$$\begin{aligned}
I_2 &= -(W_-, \psi_1)_{D_1} = \int_{D_1} W_-\big(\Delta\overline{W}_- + k^2\overline{W}_-\big)dx\\
&= \left\langle W_-, \frac{\partial W_-}{\partial \nu}\right\rangle_{\partial D_1} - \int_{D_1} \big(|\nabla W|^2 - k^2|W|^2\big)dx,\\
I_4 &= -\mu\langle W, \psi_2\rangle_{\partial D_1} = \int_{\partial D_1} W_+\frac{\partial \overline{W}_+}{\partial \nu}ds - \int_{\partial D_1} W_-\frac{\partial \overline{W}_-}{\partial \nu}ds\\
&= \int_{\partial B_R} W\frac{\partial \overline{W}}{\partial \nu}ds - \left\langle W_+, \frac{\partial W_+}{\partial \nu}\right\rangle_{\partial D_2} - \left\langle W_-, \frac{\partial W_-}{\partial \nu}\right\rangle_{\partial D_1}\\
&\quad - \int_{B_R\backslash(\overline{D}_1\cup\overline{D}_2)} \big(|\nabla W|^2 - k^2|W|^2\big)dx,\\
I_5 &= -\left\langle \frac{\partial W_+}{\partial \nu}, \psi_3\right\rangle_{\partial D_2} = \int_{\partial D_2} \frac{\partial W_+}{\partial \nu}\overline{W}_- ds - \int_{\partial D_2} \frac{\partial W_+}{\partial \nu}\overline{W}_+ ds\\
&= \int_{D_2} \big(|\nabla W|^2 - k^2|W|^2\big)dx + \langle i\lambda\psi_3, W_+\rangle_{\partial D_2} - \langle i\lambda\psi_3, \psi_3\rangle_{\partial D_2} - \left\langle \frac{\partial W_+}{\partial \nu}, W_+\right\rangle_{\partial D_2},\\
I_6 &= \langle W_+, i\lambda\psi_3\rangle_{\partial D_2}.
\end{aligned}$$

Therefore, we have

$$\mathrm{Im}(I_2 + I_4 + I_5 + I_6) = \mathrm{Im}\bigg(\int_{\partial B_R} W\frac{\partial \overline{W}}{\partial \nu}ds\bigg) - \lambda\int_{\partial D_2} |\psi_3|^2 ds.$$

Combining the above analysis leads to that

$$\begin{aligned}
\mathrm{Im}\langle M\psi, \psi\rangle &= \mathrm{Im}\left(\int_{\partial B_R} W\frac{\partial \overline{W}}{\partial \nu}ds\right) + \mathrm{Im}(q^{-1})\int_{D_1} |\psi_1|^2 dx\\
&\quad + \mathrm{Im}(\overline{\mu})\int_{\partial D_1} |\psi_2|^2 ds - \lambda\int_{\partial D_2} |\psi_3|^2 ds\\
&= -k\lim_{R\to\infty}\int_{\partial B_R} |W|^2 ds + \mathrm{Im}(q^{-1})\int_{D_1} |\psi_1|^2 dx\\
&\quad + \mathrm{Im}(\overline{\mu})\int_{\partial D_1} |\psi_2|^2 ds - \lambda\int_{\partial D_2} |\psi_3|^2 ds\\
&= -\frac{k}{|\gamma|^2}\int_{\mathbb{S}^2} |W_\infty|^2 ds + \mathrm{Im}(q^{-1})\int_{D_1} |\psi_1|^2 dx\\
&\quad + \mathrm{Im}(\overline{\mu})\int_{\partial D_1} |\psi_2|^2 ds - \lambda\int_{\partial D_2} |\psi_3|^2 ds \le 0,
\end{aligned} \tag{3.23}$$

where $\gamma = e^{ikR}/4\pi R$. Hence, we have

$$\operatorname{Im}\langle M_{D_1}\varphi, \varphi\rangle \geq 0. \tag{3.24}$$

Similarly, it can be deduced that

$$\operatorname{Im}\langle M_{D_2}\varphi, \varphi\rangle \geq 0. \tag{3.25}$$

(ii) Recalling $H^* = GM$ implies $G = H^*M^{-1}$, which further gives $G^* = (M^{-1})^*H$. This in combination with the fact that $H = \boldsymbol{L}_{D_1}\widetilde{\boldsymbol{H}}_{D_1}$ yields that $G^* = (M^{-1})^*\boldsymbol{L}_{D_1}\widetilde{\boldsymbol{H}}_{D_1}$. For any $\varphi \in \overline{\mathcal{R}(\widetilde{\boldsymbol{H}}_{D_1})}$, one has that

$$\psi = (M^{-1})^*\boldsymbol{L}_{D_1}\varphi \in \overline{\mathcal{R}(G^*)}.$$

Therefore, to obtain strictly positive property of the operator $\operatorname{Im}(M_{D_1})$, it is sufficient to prove that

$$\operatorname{Im}\langle \psi, M\psi\rangle > 0 \quad \text{for all} \quad \psi \in \overline{\mathcal{R}(G^*)} \quad \text{with} \quad \psi \neq 0. \tag{3.26}$$

Let $\operatorname{Im}\langle \psi, M\psi\rangle = 0$ for some $\psi \in \overline{\mathcal{R}(G^*)}$. Since $\operatorname{Im}[n(x)] \geq c_0 > 0, \operatorname{Im}(\mu) \geq \mu_0 > 0$ and $\lambda > 0$, it then follows from (3.23) that $\psi = (\psi_1, \psi_2, \psi_3) = 0$. The strictly positive property of the operator $\operatorname{Im}(M_{D_2})$ can be similarly obtained. The theorem is thus completely proved. □

Theorem 3.7. *For $z \in \mathbb{R}^3$, define $\phi_z \in L^2(\mathbb{S}^2)$ by $\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z}$, $\widehat{x} \in \mathbb{S}^2$. Assume that* $\operatorname{Re}[n(x)] < 1$. *We have the following results:*

(i) For any point $z \notin \overline{\Omega}_2$, then $z \in D_1 \iff \phi_z \in \mathcal{R}(\widetilde{\boldsymbol{H}}^*_{D_1})$.

(ii) For any point $z \notin \overline{\Omega}_1$, then $z \in D_2 \iff \phi_z \in \mathcal{R}(\widetilde{\boldsymbol{H}}^*_{D_2})$.

Proof. (i) Let $z \in D_1$, it is found from Theorem 2.2 that $\phi_z \in \mathcal{R}(H^*)$. Since $H^* = \widetilde{\boldsymbol{H}}^*_{D_1}\boldsymbol{L}^*_{D_1}$, we have $\mathcal{R}(H^*) \subseteq \mathcal{R}(\widetilde{\boldsymbol{H}}^*_{D_1})$. Thus, $\phi_z \in \mathcal{R}(\widetilde{\boldsymbol{H}}^*_{D_1})$.

Now assume $z \notin D_1$ and let $\phi_z \in \mathcal{R}(\widetilde{\boldsymbol{H}}^*_{D_1})$, which means that there exists φ^z such that $\widetilde{\boldsymbol{H}}^*_{D_1}\varphi^z = \phi_z$. Then by Rellich's Lemma and the unique continuation principle, we derive

$$\int_{D_1} \Phi(x,y)\varphi_1^z dy + \overline{\mu}\int_{\partial D_1} \Phi(x,y)\varphi_2^z ds + \int_{\partial\Omega_2}\left(\frac{\partial}{\partial\nu} - i\lambda\right)\Phi(x,y)\varphi_3^z ds(y) = \Phi(x,z)$$

for $x \in \mathbb{R}^3\backslash(\overline{D}_1 \cup \overline{\Omega}_2 \cup \{z\})$. However, the left hand is continuous at $x = z (z \notin \overline{\Omega}_2$ and $z \notin D_1)$ but the right hand is singular at $x = z$, which yields a contradiction.

(ii) By applying the similar arguments as that in the proof of (i), one can derive that

$$\int_{\partial\Omega_1} \Phi(x,y)\varphi_1^z dy + \overline{\mu}\int_{\partial\Omega_1} \Phi(x,y)\varphi_2^z ds + \int_{\partial D_2}\left(\frac{\partial}{\partial\nu} - i\lambda\right)\Phi(x,y)\varphi_3^z ds(y) = \Phi(x,z)$$

for $x \in \mathbb{R}^3\backslash(\overline{\Omega}_1 \cup \overline{D}_2 \cup \{z\})$. Similarly, we arrive at a singularity contradiction. This ends the proof the theorem. □

Finally, Theorems 3.3-3.7 in conjunction with the range identity [12, Theorem 2.15] and Picard's range criterion implies the main result of this section.

Theorem 3.8. *For $z \in \mathbb{R}^3$, define $\phi_z \in L^2(\mathbb{S}^2)$ by $\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z}$, $\widehat{x} \in \mathbb{S}^2$. Assume that* $\operatorname{Re}[n(x)] < 1$. *Then* *(i) For any point $z \notin \overline{\Omega}_2$, we have that*

$$z \in D_1 \iff \phi_z \in \mathcal{R}\Big(\big(F^{D_1}_{m,\#}\big)^{\frac{1}{2}}\Big)$$
$$\iff W^{D_1}_m(z) = \left[\sum_{j=1}^{\infty} \frac{\big|\big(\phi_z, \varphi_j^{(m)}\big)_{L^2(\mathbb{S}^2)}\big|^2}{\lambda_j^{(m)}}\right]^{-1} > 0$$

with $m \in \mathbb{N}$, where $\{\lambda_j^{(m)}, \varphi_j^{(m)}\}$ is the eigensystem of the self-adjoint operator

$$F^{D_1}_{m,\#} := \big|\mathrm{Re}\big(F^{D_1}_m\big)\big| + \big|\mathrm{Im}\big(F^{D_1}_m\big)\big|.$$

(ii) For any point $z \notin \overline{\Omega}_1$, we have that

$$z \in D_2 \iff \phi_z \in \mathcal{R}\Big(\big(F^{D_2}_{m,\#}\big)^{\frac{1}{2}}\Big)$$
$$\iff W^{D_2}_m(z) = \left[\sum_{j=1}^{\infty} \frac{\big|\big(\phi_z, \varphi_j^{(m)}\big)_{L^2(\mathbb{S}^2)}\big|^2}{\lambda_j^{(m)}}\right]^{-1} > 0,$$

with $m \in \mathbb{N}$, where $\{\lambda_j^{(m)}, \varphi_j^{(m)}\}$ is the eigensystem of the self-adjoint operator

$$F^{D_2}_{m,\#} := \big|\mathrm{Re}\big(F^{D_2}_m\big)\big| + \big|\mathrm{Im}\big(F^{D_2}_m\big)\big|.$$

4. Numerical Examples

In this section we give some numerical examples of the digital simultaneous imaging of the inhomogeneous medium D_1 and the obstacle D_2 in $\mathbb{R}^2$ for verifying the validity and applicability of the developed factorization method for our inverse problem. We use the limited incident directions $d = d_j \in \mathbb{S}$, which are equidistantly distributed on the unit circle $\mathbb{S}$, and the limited observation directions $\widehat{x} = \widehat{x}_j \in \mathbb{S}$ with $j = 1, 2, \ldots, m$. Moreover, the finite data is used to discretize the far-field operator. Therefore, one can obtain the matrix

$$F_m = \big(u_\infty(\widehat{x}_p, d_p)\big) \in \mathbb{C}^{m\times m}$$

represented by the measurement data. So, the indicator function $W_m(z)$ is defined by the finite sum

$$W_m(z) = \left[\sum_{p=1}^{m} \frac{1}{\lambda_p}\left|\sum_{q=1}^{m} \phi_{z,q}\overline{\varphi_{p,q}}\right|^2\right]^{-1} \quad \text{for} \quad z \in \mathbb{R}^2, \tag{4.1}$$

where $\{\lambda_p; \varphi_p\}_{p=1}^m$ is the characteristic system of the matrix

$$F_{m,\#} = |\mathrm{Re}(F_m)| + |\mathrm{Im}(F_m)|,$$

and $\{\phi_{z,q}\}_{q=1}^m$ is the discretization of the test function ϕ_z and $\{\varphi_{p,q}\}_{q=1}^m$ is the discretization of the eigenvector φ_p.

In each example, we will also show the results of the reconstruction with partially noisy data. In fact, we have added artificial noise to make the results more realistic. We choose a noise level δ and a noise matrix X and define the following noisy far-field operator:

$$F^\delta_m := F_m + \delta\frac{X}{\|X\|_2}\|F_m\|_2, \quad \big(F^\delta_m\big)_\# := \big|\mathrm{Re}\big(F^\delta_m\big)\big| + \big|\mathrm{Im}\big(F^\delta_m\big)\big|. \tag{4.2}$$

Accordingly, the indicator function $W_m(z)$ can be directly calculated from the characteristic system of the perturbation matrix $(F_m^\delta)_\#$ for simultaneously reconstruct the shape and location of the medium D_1 and the impenetrable obstacle D_2.

In all numerical examples, we focus on determining the boundary and position of both the inhomogeneous medium D_1 and the impenetrable obstacle D_2 in the two-dimensional case. For a more concise representation, we take $k_1^2 = k^2 n(x)$ to indicate that the material in the medium D_1 is homogeneous, which is different from the background medium in the $\mathbb{R}^2 \backslash (\overline{D}_1 \cup \overline{D}_2)$. The numerical shape expressions of all tested curves are given in the Table 4.1. In all numerical examples, we set the same fixed parameters: $k = 5, k_1 = 2+8i, \mu = -1+7i, \lambda = 8$ and $M = 64$.

Table 4.1: Parametrization of the Graph.

Graph type	**Parametrization**
Circle shaped	$x(t) = R(\cos t, \sin t),\ t \in [0, 2\pi],\ R > 0$
Ellipse shaped	$x(t) = (5\cos t, 4\sin t),\ t \in [0, 2\pi]$
Apple shaped	$x(t) = [(0.5 + 0.4\cos t + 0.1\sin 2t)/(1 + 0.7\cos t)](\cos t, \sin t),\ t \in [0, 2\pi]$
Rounded triangle	$x(t) = (2 + 0.3\cos 3t)(\cos t, \sin t),\ t \in [0, 2\pi]$
Peanut shaped	$x(t) = \sqrt{\cos^2 t + 0.25\sin^2 t}(\cos t, \sin t),\ t \in [0, 2\pi]$

Example 4.1. In this example, we consider the simultaneous reconstruction of a penetrable circle shaped medium and an impenetrable ellipse shaped obstacle from the far-field data without noise, with 2% noise and with 5% noise respectively. See Fig. 4.1.

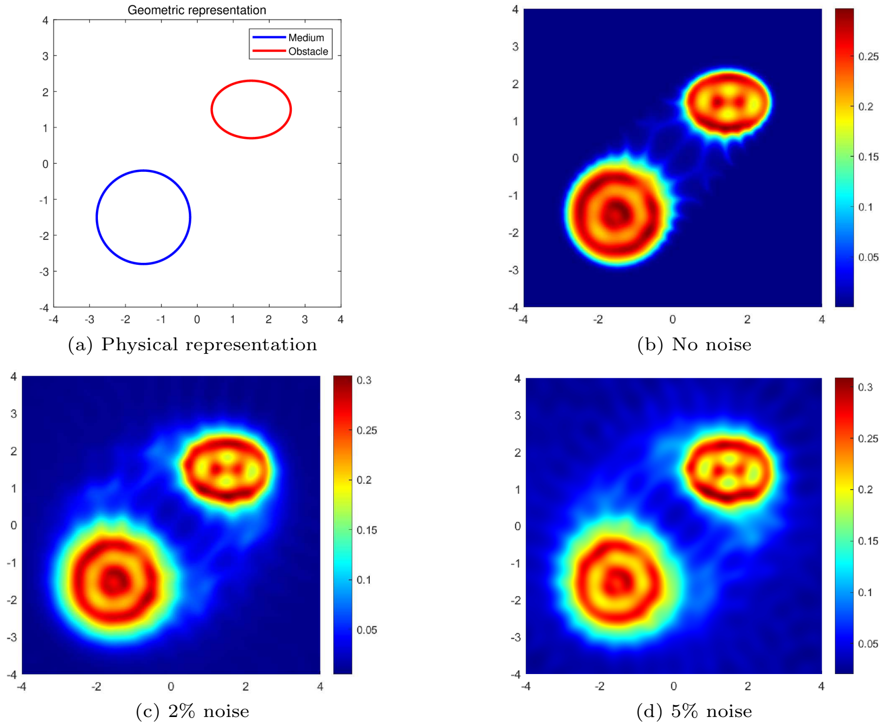
Example 4.2. In this example, we consider the simultaneous reconstruction of a penetrable circle shaped medium and an impenetrable peanut shaped obstacle from the far-field data without noise, with 2% noise and with 5% noise respectively. See Fig. 4.2.

Example 4.3. In this example, we consider the simultaneous reconstruction of a penetrable circle shaped medium and an impenetrable rounded triangle obstacle from the far-field data without noise, with 2% noise and with 5% noise respectively. See Fig. 4.3.

Example 4.4. We now consider the simultaneous reconstruction of a penetrable circle shaped medium and an impenetrable apple shaped obstacle from the far-field data without noise, with 2% noise and with 5% noise respectively. See Fig. 4.4.

Example 4.5. Finally, we consider the simultaneous reconstruction of a penetrable peanut shaped medium and an impenetrable apple shaped obstacle from the far-field data without noise, with 2% noise and with 5% noise respectively. See Fig. 4.5.

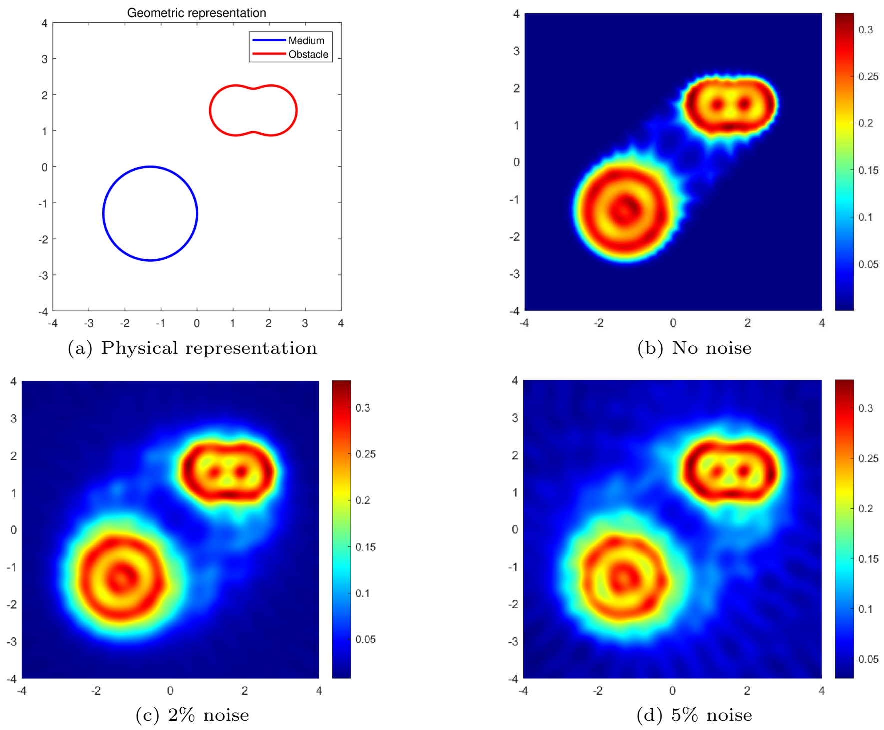
As can be seen from the above five examples and other cases that have been carried out but not given here that the shapes and positions of the mixed type scatterer of a penetrable medium D_1 and an impenetrable obstacle D_2 can be numerically reconstructed by means of the spectral data of the far-field operator. This proves the validity and applicability of the modified factorization method proposed in the current paper. In future, we plan to extend our results to more challenging problems of the electromagnetic scattering and the other related scattering problems.



(a) Physical representation (b) No noise

(c) 2% noise (d) 5% noise

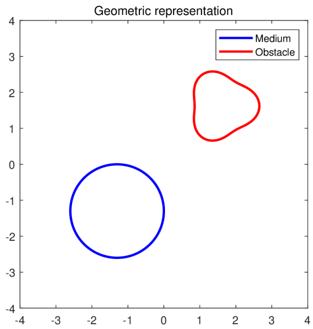
Fig. 4.1. Reconstruction of circle shaped and ellipse shaped.



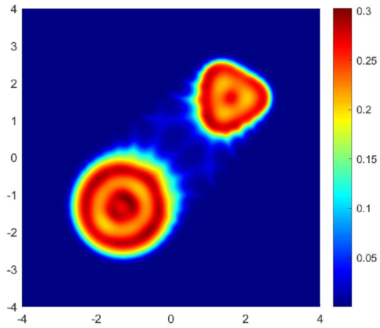
(a) Physical representation (b) No noise

(c) 2% noise (d) 5% noise

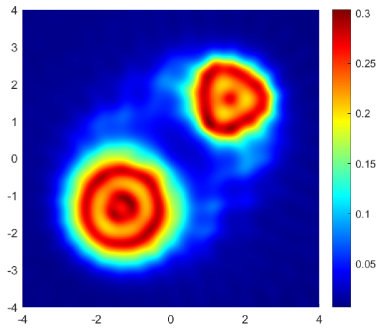
Fig. 4.2. Reconstruction of circle shaped and peanut shaped.



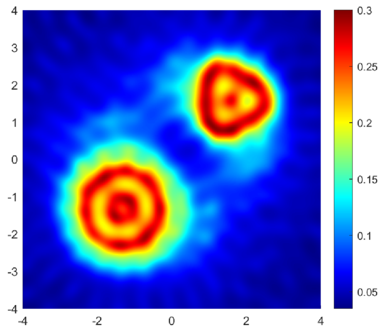
(a) Physical representation



(b) No noise

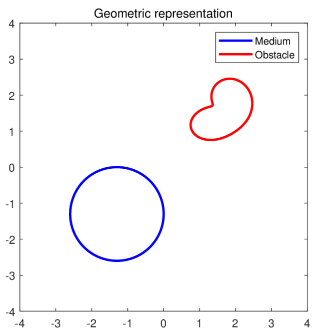


(c) 2% noise

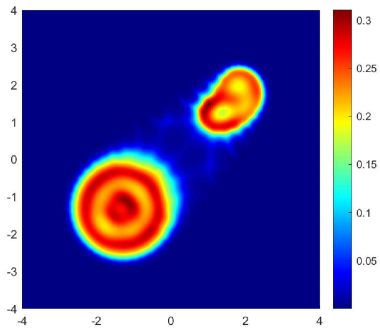


(d) 5% noise

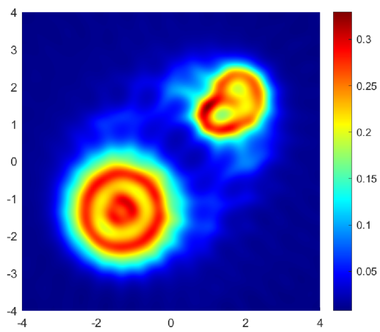
Fig. 4.3. Reconstruction of circle shaped and rounded triangle.



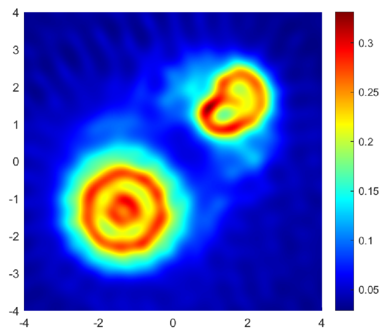
(a) Physical representation



(b) No noise

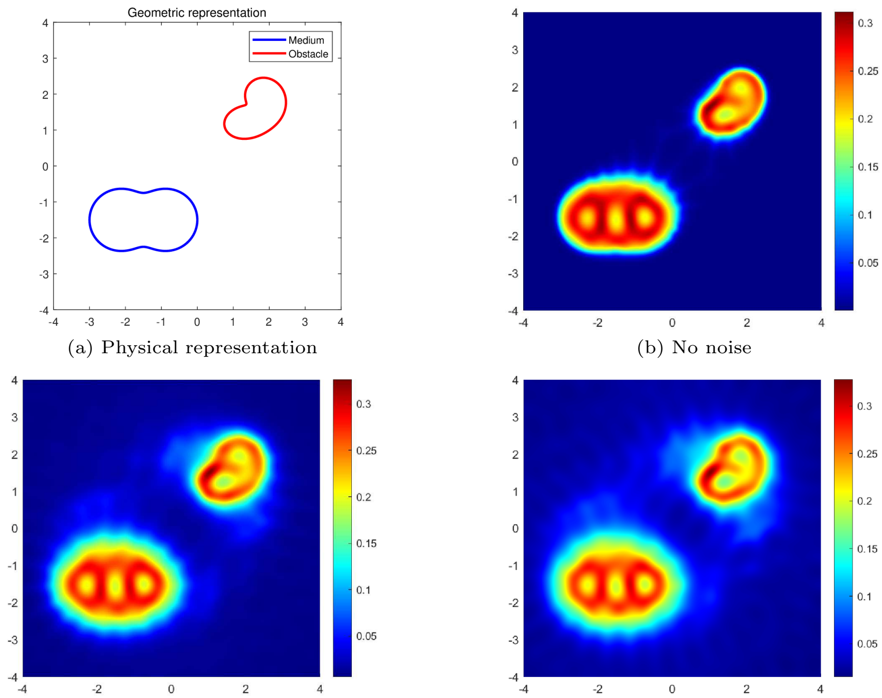


(c) 2% noise



(d) 5% noise

Fig. 4.4. Reconstruction of circle shaped and apple shaped.



(a) Physical representation (b) No noise

(c) 2% noise (d) 5% noise

Fig. 4.5. Reconstruction of peanut shaped and a apple shaped.

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