SOME DOMAIN DECOMPOSITION AND ITERATIVE REFINEMENT ALGORITHMS FOR ELLIPTIC FINITE ELEMENT PROBLEMS

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Abstract

In this contribution, we report on some results recently obtained in joint work with Maksymilian Dryja. We first study an additive variant of Schwarz' alternating algorithm and establish that a fast method of this kind can be devised which is optimal in the sense that the number of conjugate gradient iterations required, to reach a certain tolerance, is independent of the mesh size as well as the number of subregions. An interesting feature of this algorithm is that all the subproblems can be solved at the same time. The method is therefore quite promising for parallel computers. Using a similar mathematical framework, we then consider the solution of elliptic finite element problems on composite meshes. Such problems can be built up systematically by introducing a basic finite element approximation on the entire region and then repeatedly selecting subregions, and subregions of subregions, where the finite element model is further refined. We consider conjugate gradient algorithms where, in each iteration, problems on the subregions representing finite element models prior to further refinement are solved. This makes it possible to use solvers for problems with uniform or relatively uniform mesh sizes, while the composite mesh can be strongly graded. We remark that this work is technically quite closely related to our previous work on iterative substructuring methods, which are domain decomposition algorithms using non-overlapping subregions.

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1. Introduction

In this paper, we report on results obtained in joint work with Maksymilian Dryja of the University of Warsaw, Poland. Our work began with the study of a recent paper by P.-L. Lions in which a variational framework is introduced for the classical, multiplicative (sequential) Schwarz' method; see Schwarz [17] and Lions [12]. As pointed out by Lions, the Variational framework is far from new; cf. [18], [2], [1] and other references given in Lions [12].

In the second section of this paper, we introduce a framework similar to Lions' and also an additive (parallel) version of the Schwarz algorithm. While Lions considered the continuous problems, we work consistently with conforming finite element approximations; cf. [6].

In view of the interest in Parallel computers with many processors, we are principally interested in the case of many subregions. Our main result is that we can design additive methods of this kind which have rates of convergence which are independent of the number of subregions and the number of unknowns. No proofs are given in this paper; cf. Dryja and Widlund [8] for details. We note that the technical tools used there are similar to those of our previous work on so-called iterative substructuring methods, which are domain decomposition methods with nonoverlapping subregions; cf. e.g. [7] and [19] and the articles sited in those papers.

In the final section, we discuss optimal iterative refinement methods. This work begun with the discovery that the so called FAC and AFAC methods, which have been studied by McCormick and his Coworkers [13], [14], [15] and [11], are structured quite similarly to the classical Schwarz procedure and its additive variant, respectively. In our view, the central theoretical issue of the iterative refinement algorithms is the design and study of algorithms for which the rate of convergence is independent of the number of subproblems, i.e. the number of refinement levels, as well as the mesh size. After introducing certain finite element models on composite meshes, which can systematically be built up inside a framework of conforming finite elements, we describe several algorithms, for which we have established optimality in this sense; cf. [21].

Let us also note that we have discovered that the very interesting so called hierarchical basis method, introduced recently by Yserentant [22], can be derived and understood inside our general framework. We plan to return to this topic elsewhere.

Since the theme of this seminar is boundary intégral equations and boundary element methods, a remark on a connection between this main topic and domain decomposition algorithms might be in order. In our early work on domain decomposition algorthms for regions divided into two nonoverlapping subdomains, we