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ON THE CONVERGENCE OF THE RELAXATION METHODS FOR POSITIVE DEFINITE LINEAR SYSTEMS[*1)]

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Abstract

We establish the convergence theories of the symmetric relaxation methods for the system of linear equations with symmetric positive definite coefficient matrix, and more generally, those of the unsymmetric relaxation methods for the system of linear equations with positive definite matrix.

Key words: System of linear equations, Relaxation method, Convergence theory, Positive definite matrix.

1. Introduction

The classical iterative methods, such as the Jacobi method, the Gauss-Seidel method and the SOR method, as well as their symmetrized variants, play an important role for solving the large sparse system of linear equations

$$Ax = b, \tag{1.1a}$$

where

$$\begin{aligned} &A = (a_{mj}) \in L(R^n) \quad \text{is a given nonsingular matrix;} \\ &x = (x_1, x_2, \cdots, x_n)^T \in R^n \quad \text{is the unknown vector; and} \\ &b = (b_1, b_2, \cdots, b_n)^T \in R^n \quad \text{is a given vector.} \end{aligned} \tag{1.1b}$$

In accordance with the basic extrapolation principle of the linear iterative method, Hadjidimos[1] further proposed a class of accelerated overrelaxation (AOR) method for solving the linear system (1.1) in 1978. This method includes two arbitrary parameters, and their suitable choices not only can naturally recover the Jacobi, the Gauss-Seidel and the SOR methods, etc., but also can considerably improve the convergence property of this AOR method. After many authors' extensive and deepened researches, the convergence theories of the afore-mentioned relaxation methods have been established in a more complete manner when the coefficient matrix of the linear system (1.1) is

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an L-matrix, an M-matrix, an H-matrix, and a symmetric positive definite matrix, respectively. For details one can refer to [1]–[7] and references therein.

Based on Hadjidimos' work[1], many researchers have designed the symmetrized and, more generally, the unsymmetrized versions of the AOR method, called as the SAOR method and the UAOR method, respectively, and discussed in detail the convergence properties of these methods under the conditions that the coefficient matrix of the linear system (1.1) is either an L-matrix, or an M-matrix, or an H-matrix. For more details one can see [4] and references therein. These studies not only afford efficient algorithm choices for the linear system (1.1), but also establish systematical convergence theories for the relaxation methods.

However, to our knowledge, except for the symmetric positive definite matrix with property–A, there is no convergence result about either the SAOR method or the UAOR method for general (symmetric) positive definite matrix class. The difficulty seems to be that the commutativity as in the SSOR method does not still hold in these methods. In this paper, we will emphatically establish the convergence theory of the SAOR method for the symmetric positive definite matrix class, or more generally, that of the UAOR method for the positive definite matrix class.

2. Reviews of the Relaxation Methods

More generally, from now on, we will turn to consider the system of linear equations (1.1) which has the following partitioned form:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}, \tag{2.1a}$$

where

$$A_{i,j} \in L(R^{n_j}, R^{n_i}), \quad x_i, b_i \in R^{n_i}, \quad i,j = 1,2,\cdots,N \tag{2.1b}$$

and n_i $(i = 1,2,\cdots,N)$ are positive integers satisfying

$$n_1 + n_2 + \cdots + n_N = n. \tag{2.1c}$$

Also, we will stipulate that $A_{i,i}$ $(i = 1,2,\cdots,N)$ are nonsingular matrices.

If we take

$$\begin{aligned} A_D &= \text{diag}\,(A_{1,1}, A_{2,2}, \cdots, A_{N,N}), \\ A_L &= \begin{pmatrix} 0 & & & & & \\ -A_{2,1} & 0 & & & & \\ \vdots & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & & \\ -A_{N-1,1} & \cdots & \cdots & -A_{N-1,N-2} & 0 & \\ -A_{N,1} & \cdots & \cdots & -A_{N,N-2} & -A_{N,N-1} & 0 \end{pmatrix}, \end{aligned}$$