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A COMPACT UPWIND SECOND ORDER SCHEME FOR THE EIKONAL EQUATION*

J.-D. Benamou
INRIA, INRIA B.P. 105, 78153 Le Chesnay Cedex, France
Email: jean-david.benamou@inria.fr
Songting Luo
Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
Email: luos@math.msu.edu
Hongkai Zhao
Department of Mathematics, University of California, Irvine, CA 92697, USA
Email: zhao@math.uci.edu

Abstract

We present a compact upwind second order scheme for computing the viscosity solution of the Eikonal equation. This new scheme is based on:

1. the numerical observation that classical first order monotone upwind schemes for the Eikonal equation yield numerical upwind gradient which is also first order accurate up to singularities;
2. a remark that partial information on the second derivatives of the solution is known and given in the structure of the Eikonal equation and can be used to reduce the size of the stencil.

We implement the second order scheme as a correction to the well known sweeping method but it should be applicable to any first order monotone upwind scheme. Care is needed to choose the appropriate stencils to avoid instabilities. Numerical examples are presented.

Mathematics subject classification: 35L60, 65N06, 65N12, 65N15
Key words: Eikonal equation, Upwind scheme, Hamilton-Jacobi, Viscosity Solution, Sweeping method.

1. Introduction

The Eikonal equation:

$$\begin{cases} |\nabla\phi(x)| = n(x), & x \in \Omega/\Gamma \\ \phi(x) = \phi_0(x), & x \in \Gamma \end{cases} \tag{1.1}$$

is a special class of the Hamilton-Jacobi equations. It has wide applications in geometric optics, computer vision, optimal control and etc. This boundary value problem (1.1) is a first order hyperbolic partial differential equation (PDE). The classical method of characteristics can be applied to solve the problem. Solutions remain smooth until the characteristics cross and the fronts (level sets of the solution) intersect. Crandall and Lions [3] introduced the concept of the viscosity solutions for the Hamilton-Jacobi equations and a unique global weak solution can be defined in that sense. A weak solution remains smooth locally with the singularity in the gradient along some sub-manifold of codimension 1, 2 or 3 (in 3D).

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It therefore makes sense to design high order schemes which have to remain high order away from the singularities called kinks. High order schemes are of particular importance in the high frequency wave propagation where the Eikonal equation is coupled to a transport equation through its gradient [17, 26].

Different first order numerical schemes have been developed to compute the viscosity solutions. There are two types of approaches to compute the viscosity solution of the Eikonal equation. One approach is to transform it to a time-dependent problem. For example, Osher [15] provided a natural link between the time-dependent and the static problems by using level-set ideas. Semi-Lagrangian schemes [7, 8] are obtained with the dynamical programming principle under the optimal control framework. Another approach is to treat the problem as a stationary problem and directly solve it with efficient numerical algorithms such as Dijkstral type of fast marching method (FMM) [6, 9, 21–23, 29] and iterative fast sweeping method [2, 5, 11, 12, 18–20, 27, 28, 31, 32].

However both approaches rely, in theory and in practice, on the idea of "upwind" or "causality". An efficient ordering of the application of the stencil on the grid must follow the traveltime or the level set propagation. First order upwind schemes can do it monotonically (the iterations converge monotonically to the solution) and the convergence is proven using the viscosity theory [1, 4, 25]. Even though the gradient may be singular, the method remains stable because the characteristics defining the upwind directions enter the kinks (exactly as in the case of the shocks for the hyperbolic conservation laws).

On the other hand, second order schemes cannot be monotone [14] and in that case the viscosity theory to prove the convergence is inoperative. The popular high order ENO and WENO methods [10, 16, 24] use adaptive stencils (actually different stencils) to capture the smoothest possible approximation of the second derivatives and therefore avoid, in theory, the possible singularity of the solution. These ENO and WENO type of discretizations have been incorporated into fast sweeping method in [30]. Recently second order discontinuous Galerkin method has been developed for fast sweeping method for the Eikonal equation [13]. Also a second order fast marching method was proposed in [22]. In this approach both the solution and its gradient at accepted points, which are computed and stored during previous updates, are used to provide high order approximation of directional derivatives at a considered point during the marching process. The discretization is based on direction by direction approximation and accurate ∇u are needed near the boundary to start with the fast marching method.

Our approach is different and is based on the following two observations. The first observation is a superconvergence phenomena for first order monotone upwind methods. More precisely, the upwind numerical gradient of the solution obtained by these methods seems to remain first order accurate up to singularity, i.e., away from kinks. Apparently this phenomena has not been observed and studied in the literature. We substantiate this observation by a detailed numerical study and we are currently working on a proof. The second observation is that one only needs second derivative approximation tangential to the front, i.e., the curvature estimation, to achieve second order local truncation error by using the PDE and a decomposition of the Taylor expansion. This results in a compact and upwind second order stencil. For example, the stencil is 4 points in 2D (3 points is needed for the first order upwind scheme), which is more compact than direction by direction second order approximation. Moreover, if the second order accurate stencil can be placed upwind then we can avoid the singularity in the gradient and remain second order accurate. Our method can be regarded as an efficient one pass deferred correction to any first order monotone upwind method. It works in this way: after