Journal of Computational Mathematics
Vol.36, No.2, 2018, 276–309.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1701-m2016-0525

EXPONENTIAL INTEGRATORS FOR STOCHASTIC SCHRÖDINGER EQUATIONS DRIVEN BY ITÔ NOISE*

Rikard Anton and David Cohen
Department of Mathematics and Mathematical Statistics, Umeå University, SE–901 87 Umeå, Sweden
Department of Mathematics, University of Innsbruck, A–6020 Innsbruck, Austria
Email: rikard.anton@umu.se, david.cohen@umu.se, david.cohen@uibk.ac.at

Abstract

We study an explicit exponential scheme for the time discretisation of stochastic Schrödinger Equations Driven by additive or Multiplicative Itô Noise. The numerical scheme is shown to converge with strong order 1 if the noise is additive and with strong order 1/2 for multiplicative noise. In addition, if the noise is additive, we show that the exact solutions of the linear stochastic Schrödinger equations satisfy trace formulas for the expected mass, energy, and momentum (i. e., linear drifts in these quantities). Furthermore, we inspect the behaviour of the numerical solutions with respect to these trace formulas. Several numerical simulations are presented and confirm our theoretical results.

Mathematics subject classification: 35Q55, 60H15, 65C20, 65C30, 65C50, 65J08.
Key words: Stochastic partial differential equations, Stochastic Schrödinger equations, Numerical methods, Geometric numerical integration, Stochastic exponential integrators, Strong convergence, Trace formulas.

1. Introduction

We consider temporal discretisations of nonlinear stochastic Schrödinger Equations Driven by Itô Noise

$$\begin{aligned} &\mathrm{i}\mathrm{d}u = \Delta u \,\mathrm{d}t + F(x,u)\,\mathrm{d}t + G(u)\,\mathrm{d}W \qquad &&\text{in } \mathbb{R}^d \times (0,\infty), \\ &u(\cdot,0) = u_0 &&\text{in } \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$, and $\mathrm{i} = \sqrt{-1}$. The product between G and $\mathrm{d}W$ is of Itô type, and further details on F and G and on the dimension d will be specified later. The stochastic process $\{W(t)\}_{t\geq 0}$ is a square integrable complex-valued Q-Wiener process with respect to a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. The regularity of the covariance operator Q will be specified later in the text. The initial value u_0 is an $\mathcal{F}_0$-measurable complex-valued function, which will be further specified below.

The Schrödinger equation is widely used within physics and takes several different forms depending on the situation. It is used in hydrodynamics, nonlinear optics and plasma physics to only mention a few areas. In certain physical situations it may be appropriate to incorporate some kind of randomness into the equation. One possibility is to add a driving random force to then obtain an equation of the form (1.1). See for example [7] and references therein for further details.

* Received January 26, 2016 / Revised version received November 8, 2016 / Accepted January 22, 2017 / Published online February 5, 2018 /

Stochastic Schrödinger equations have received much attention from a more theoretical point of view during the last decades. Connected to the present article and without being exhaustive, we mention the works [5, 6, 8] on Itô problems and [5, 7, 8, 31, 32] for the Stratonovich setting.

It is seldom possible to solve stochastic partial differential equations exactly, and efficient numerical schemes are therefore needed. For the time integration of the above stochastic Schrödinger equations, we will consider stochastic exponential integrators. These numerical methods are explicit and easy to implement, furthermore they offer good geometric properties. Exponential integrators are widely used and studied nowadays as witnessed by the recent review [19] for the time integration of deterministic problems. Applications of such schemes to the deterministic (nonlinear) Schrödinger equation can be found in, for example, [3, 10, 11, 18, 27–29] and references therein. These numerical methods were recently investigated for stochastic parabolic partial differential equations in, for example, [20, 23, 24] and for the stochastic wave equations in [1, 12, 13, 26].

We now review previous works on temporal discretisations of stochastic Schrödinger equations. In [4] a Crank-Nicolson scheme is studied for the equation with nonlinearity $F(u)$. First order of convergence is obtained in the case of additive noise, and with multiplicative Itô noise the convergence rate is one half. Observe that this numerical scheme is implicit. A stochastic Schrödinger equation with Stratonovich noise is considered in [15], where, again, a Crank-Nicolson scheme is studied for the equation with nonlinearity $F(x,u)=\lambda|u|^{2\sigma}u$, with $\lambda=\pm 1$ and $\sigma>0$. The authors prove convergence to the exact solution and mass preservation of the scheme. Further, in [21] a mass-preserving splitting scheme for Eq. (1.1) with $F(x,u)=V(x)u$ and $G(u)=u$ is considered. The noise is of Stratonovich type and first order convergence is obtained. In [22], $V(x)$ is replaced by $|u|^2$ and first order convergence is again obtained. Still in the Stratonovich setting, [33] derives multi-symplectic schemes for stochastic Schrödinger equations. We finally mention [2, 16], in which thorough numerical simulations are presented for both additive noise and multiplicative Stratonovich noise.

In the present work we show that

- the exponential integrator applied to the linear stochastic Schrödinger equation without potential and with additive noise converges strongly with order 1 and satisfies exact trace formulas for the mass, the energy, and for the momentum;
- the exponential integrator applied to the linear stochastic Schrödinger equation with a multiplicative potential of the form $V(x)u$ and additive noise converges with strong order 1, but has a small error in the trace formulas for the mass and energy;
- the exponential integrator applied to stochastic Schrödinger Equations Driven by Multiplicative Itô Noise strongly converges with order $1/2$.

We begin the exposition by introducing some notations and useful results that we will use in our proofs. After that we will follow a similar approach as in [4]. That is, we will begin by analysing the numerical method applied to the linear Schrödinger equation with additive noise in Section 3. Then we study stochastic Schrödinger equations with a multiplicative potential in Section 4 and finally we consider the stochastic Schrödinger equation with a multiplicative potential and multiplicative noise in Section 5. For each of the above problems, we analyse the speed of convergence of the exponential methods (in the strong sense) and for additive problems we show some trace formulas (such results could be interpreted as weak error estimates). Various numerical experiments accompany the presentation and illustrate the main properties of these exponential methods when applied to stochastic Schrödinger Equations Driven by Itô Noise.