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Robust and Globally Divergence-Free Weak Galerkin Methods for Oseen Equations

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Abstract. In this paper, a robust and globally divergence-free weak Galerkin finite element method of Oseen equations is proposed and analyzed. We use the $\mathbf{P}_k/P_{k-1}$ discontinuous finite element combination for the approximation of velocity and pressure, and piecewise $\mathbf{P}_k/P_k$ for the numerical traces of velocity and pressure. This method not only yields globally divergence-free velocity approximations, but is also robust in the sense that a priori error estimates are uniform with respect to the coefficients of Oseen equations, providing the exact solutions are sufficiently smooth. Finally, numerical examples are given to confirm our theoretical results.

AMS subject classifications: 65N30, 65M12, 65M70

Key words: Oseen equations, weak Galerkin finite element, divergence-free, robust.

1 Introduction

Oseen equations are linear equations, which are proved to be an auxiliary problem in many numerical methods for solving the Navier-Stokes equations. Both the implementation and the analysis of numerical schemes for Oseen equations are important for the Navier-Stokes equations.

It is well-known that Galerkin mixed method for Oseen equations requires the pair of finite element spaces for the velocity and pressure to satisfy an inf-sup stability condition (see [6, 24]) which excludes the use of low-order elements and equal-order elements. To circumvent this difficulty, many stabilization techniques have been developed to get stabilized finite element methods. Some recent examples include pressure projection methods [4,11], Galerkin least-square methods [3,12], and local projection stabilized methods [2,23], etc.

As an extension of the standard finite elements, the discontinuous Galerkin (DG) method has become increasingly popular due to its attractive features like local conservation of physical quantities and flexibility in meshing. However, an inconvenient feature

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of the DG method is that it may require the penalization parameter to be sufficiently large for stability. This inconvenience was avoided by local discontinuous Galerkin (LDG) methods [8,9,18] and hybridizable discontinuous Galerkin (HDG) methods [19–21]. Oseen equations have been numerically analyzed in [16, 17] using LDG method. This method yields optimal priori estimates for the error in the velocity and pressure when using polynomials of degree k for the velocity and $k-1$ for the pressure. We also mention HDG schemes were introduced in [10] which gives a priori error analysis. As far as the authors know, these methods mentioned in [10, 16, 17] are not robust with respected to the fluid viscosity coefficient.

In [26, 27] the weak Galerkin (WG) methods were proposed and analyzed to solve elliptic problem. The WG methods replace the classical operators (e.g. gradient, divergence, and curl) by the weakly defined operators according to integration by parts. These methods have a lot of attractive computational features: more general finite element partitions of arbitrary polygons or polyhedra with certain shape regularity, parameter free, etc. In [25] the WG method was proposed for Oseen equations with constant coefficients. We refer to [13] for a class of robust globally divergence-free WG methods for Stokes equations. The methods not only yield globally divergence-free velocity solutions, but also have uniform error estimates with respect to the Reynolds number. In [15] a robust WG method for convection-diffusion-reaction equations was proposed and analyzed on conforming or nonconforming polygon or polyhedral meshes. The method is robust in the sense that the derived priori error estimates is uniform with respect to the coefficients for sufficient smooth true solutions.

Inspired by [13,15,25], this paper is concerned with a robust WG method for Oseen equations. In the method the convection fields $\mathbf{b}$ is not assumed to be divergence-free and the reaction coefficient σ is a function. We use the $\mathbf{P}_k/P_{k-1}$ discontinuous finite element combination for the approximation of velocity and pressure, and piecewise $\mathbf{P}_k/P_k$ for the numerical traces of velocity and pressure. We shall show that the method can yield globally divergence-free velocity approximations and the error estimates are uniform with respect to $\mathbf{b}$,σ, and the fluid viscosity coefficient ε, provided the exact solutions are sufficiently smooth. So, our method can be used to the Navier-Stokes equations with high Reynolds number.

This paper is organized as follows. In Section 2, we introduce some notation for the Oseen equations and Sobolev spaces. In Section 3, the fundamental definitions and WG finite element scheme for the Oseen problem are developed. We discuss stability and give the error estimation in Sections 4 and 5 respectively. Numerical examples demonstrate our theoretical results in Section 6.

Throughout this paper, we will use $a \lesssim b$ $(a \gtrsim b)$ to denote $a \leq Cb$ $(a \geq Cb)$, where C is a positive constant independent of mesh h, h_K, h_E and ε. Denote by

$$\langle \mathbf{u},\mathbf{v}\rangle_{\partial\mathcal{K}_h} := \sum_{K\in\mathcal{K}_h} \langle \mathbf{u},\mathbf{v}\rangle_{0,\partial K}, \quad \|\mathbf{u}\|^2_{0,\partial\mathcal{K}_h} := \sum_{K\in\mathcal{K}_h} \|\mathbf{u}\|^2_{0,\partial K}.$$