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A Crank-Nicolson Finite Difference Scheme for the (2+1)D Saturable Nonlinear Schrödinger Equation with Generalized Damping

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Abstract. In this study, we implement a Crank-Nicolson finite difference scheme to discretize the (2+1)D saturable nonlinear Schrödinger equation with general damping. We show the existence and uniqueness of the discrete solution. The boundedness of the discrete mass and energy is established. The error between the exact and discrete solutions is shown to converge at a second-order rate in both time and space, according to the L^2 and H^1 discrete norms. Moreover, we show that the proposed scheme preserves the mass conservation and energy conservation for the (2+1)D saturable nonlinear Schrödinger equation without damping. Numerical simulations are conducted to validate these convergence properties and the conservation laws.

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Key words: Crank-Nicolson finite difference, (2+1)D Nonlinear Schrödinger equation, 2D soliton, nonlinear damping.

1 Introduction

In this paper, we propose a finite difference scheme in space and Crank-Nicolson in time for the (2+1)D saturable nonlinear Schrödinger (NLS) equation with generalized damping:

$$\begin{cases} i\dfrac{\partial u}{\partial t}+\Delta u+\lambda\dfrac{u|u|^2}{1+|u|^2}+i\epsilon u g(|u|^2)=0, & \forall (x,y)\in\Omega, \quad t\in(0,T), \\ u(x,y,0)=u_0(x,y), & \forall (x,y)\in\Omega, \\ u(x,y,t)=0, & \forall (x,y)\in\partial\Omega, \quad t\in[0,T], \end{cases} \tag{1.1}$$

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where $\Omega=(a,b)\times(c,d)$, λ is a real constant, and ε is the nonlinear damping coefficient, $\varepsilon\geq 0$. In Eq. (1.1), the function

$$g:[0,\infty)\mapsto[0,\infty)$$

with $g\in C^2([0,\infty))$, satisfies the following properties:

- For all $s\geq 0$, $g(s)\geq 0$, and g is an increasing function.
- There exist constants $C_{1,g}$, $C_{2,g}$, $C_{3,g}$ and exponents $q_1,q_2,q_3\geq 0$ such that g satisfies the following:

$$g(s)\leq C_{1,g}|s|^{q_1}, \qquad \forall s\geq 0, \tag{1.2a}$$
$$|g(s)-g(t)|\leq C_{2,g}|s-t|(|s|^{q_2}+|t|^{q_2}), \qquad \forall s,t\geq 0, \tag{1.2b}$$
$$|g'(s)-g'(t)|\leq C_{3,g}|s-t|(s^{q_3}+t^{q_3}), \qquad \forall s,t\geq 0. \tag{1.2c}$$

In applications in optics and Bose-Einstein condensates (BEC), the function g can be used in form of $g(s)=s^\sigma$ with $s\geq 0$ and $\sigma\geq 2$ or $\sigma=0,1$. When $\sigma=1$, the damping represents a cubic loss, which arises in optics due to two-photon absorption. In general, when $\sigma=m$, where $m\geq 1$, the term $i\varepsilon u g(|u|^2)$ represents the generalized nonlinear loss that occurs in optics due to $(m+1)$-photon absorption. In Bose-Einstein condensates, three-body inelastic recombinations can lead to a quintic nonlinear damping term [9].

We assume that the exact solution of Eq. (1.1) satisfies:

$$u\in C^2([0,T],W^{3,\infty}(\Omega))\cap C([0,T],W^{5,\infty}(\Omega)\cap H_0^1(\Omega)). \tag{1.3}$$

Due to the wide range applications of NLS-type equations in optics and plasma physics, many numerical schemes for NLS-type equations have been proposed. These include the finite difference scheme, the Ablowitz-Ladik scheme, the pseudo-spectral split-step approach, and the relaxation finite difference method [16, 19, 21, 23], among others. For 1D or 2D space, the most common approximation method are the split-step Fourier method [4, 17, 19, 20, 22] and the finite difference method, see, e.g., [1, 18, 23] and finite element method in [8,10,13]. Additionally, the $(n$+1)D cubic NLS $(n=2,3)$ and its particular version, the Gross-Pitaevskii equation, with a potential in Bose-Einstein condensates, has been thoroughly examined in several studies (e.g., [2, 3, 5–7, 11]). A finite element scheme and a Crank Nicolson approximation are used to discretize for a general class of NLS equations in [10], the article shows conservative mass and energy and the optimal L^∞-error (H^1-error) estimates. In [5, 6], the time-splitting sine-spectral method (TSSP) was implemented to solve the damped $(n+1)$D NLS equation $(n=2,3)$ with an external potential and with a power law nonlinearity. It has been showed that the TSSP scheme is efficient for high regularity solutions. However, unlike the finite difference scheme, the TSSP does not hold the energy conservation. Recently, in [3], the authors established the optimal convergence rate through the study of finite difference methods for