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Solutions of Fractional Partial Differential Equations of Quantum Mechanics

S. D. Purohit*

Department of Basic Sciences (Mathematics), College of Technology and Engineering, M.P. University of Agriculture and Technology, Udaipur-313001, India

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Abstract. The aim of this article is to investigate the solutions of generalized fractional partial differential equations involving Hilfer time fractional derivative and the space fractional generalized Laplace operators, occurring in quantum mechanics. The solutions of these equations are obtained by employing the joint Laplace and Fourier transforms, in terms of the Fox's H-function. Several special cases as solutions of one dimensional non-homogeneous fractional equations occurring in the quantum mechanics are presented. The results given earlier by Saxena et al. [Fract. Calc. Appl. Anal., 13(2) (2010), pp. 177–190] and Purohit and Kalla [J. Phys. A Math. Theor., 44 (4) (2011), 045202] follow as special cases of our findings.

AMS subject classifications: 26A33, 44A10, 33C60, 35J10

Key words: Fractional Schrödinger equation, Laplace transform, Fourier transform, Hilfer fractional derivative, Fox's H-function and Quantum mechanics.

1 Introduction

The partial differential equations of fractional order have been successfully used for modeling some relevant physical processes, therefore, a large amount of research in the solutions of these equations has been published in the literature. Debnath [3] has discussed solutions of the various type of partial differential equations occurring in the fluid mechanics. Nikolova and Boyadjiev [17] found solution of the time-space fractional diffusion equations by means of the fractional generalization of Fourier transform and the classical Laplace transform. Solution of generalized diffusion equation containing two space-fractional derivatives have been recently analyzed by Pagnini and Mainardi [19]. Solutions of fractional reaction-diffusion equations are investigated in a number of recent papers by Saxena et al. [23–25] and Haubold et al. [6].

*Corresponding author.
Email: sunil_a_purohit@yahoo.com (S. D. Purohit)

Laskin [9–11] constructed space fractional quantum mechanics and formulated the fractional Schrödinger equation by generalizing the Feynman path integrals from Brownian-like to Lévy-like quantum mechanical paths. The Schrödinger equation thus obtained containing the space and time fractional derivatives. Several authors including Naber [16] and Saxena et al. [26,27] studied various aspects of the Schrödinger equations in terms of the fractional derivatives as dimensionless objects. For some physical applications of fractional Schrödinger equation, one can refer the work of Guo and Xu [5].

The Grunwald-Letnikov method is employed by Scherer et al. [28, 29] to solve fractional differential equations numerically.

In a recent paper, Purohit and Kalla [21] have investigated solutions of generalized fractional partial differential equations by employing the joint Laplace and Fourier transforms. Several special cases in terms of the solutions of one dimensional non-homogeneous fractional equations occurring in the fluid and quantum mechanics (diffusion, wave and Schrödinger equations) are also presented in the same paper.

The object of this paper is to investigate solutions of generalized fractional partial differential equations involving Hilfer time-fractional derivative and the space-fractional generalized Laplace operators by employing the joint Laplace and Fourier transforms. Several special cases in terms of the solutions of one dimensional non-homogeneous fractional equations occurring in the quantum mechanics are presented. It is to be noted that the problem considered here (involving Hilfer time-fractional derivative) is different than those considered by Saxena et al. [26] and the authors [21], where Caputo time-fractional derivative and Liouville space-fractional derivative were employed. Additionally, the problem considered here is more general than the problem considered by Saxena et al. [27]. Hilfer fractional derivative has advantage that it generalizes the Riemann-Liouville and Caputo type fractional derivative operators, therefore, several authors called this a general operator. The results given earlier by Saxena et al. [27] and Purohit and Kalla [21] follow as special cases of our findings.

In order to obtain the solutions of generalized fractional partial differential equations, the definitions and notations of the well-known operators are described below:

The Laplace transform of a function $U=U(x,t)$ (which is supposed to be continuous or sectionally continuous, and of exponential order as $t\to\infty$) with respect to the variable t is defined by

$$\mathcal{L}\{U(x,t)\}=\widehat{U}(x,s)=\int_0^\infty e^{-st}U(x,t)dt, \quad (t>0, \quad x\in\mathbb{R}), \tag{1.1}$$

where $\Re(s)>0$, and the inverse Laplace transform of $\widehat{U}(x,s)$ with respect to s is given by

$$\mathcal{L}^{-1}\{\widehat{U}(x,s)\}=U(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\widehat{U}(x,s)ds, \tag{1.2}$$

where γ being a fixed real quantity.