Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **16**, No. 6, pp. 1381-1409

DOI: 10.4208/aamm.OA-2023-0040
December 2024

Unconditional Stability and Fourth-Order Convergence of a Two-Step Time Split Explicit/Implicit Scheme for Two-Dimensional Nonlinear Unsteady Convection-Diffusion-Reaction Equation

Eric Ngondiep[1,2,*] and Ali H. Tedjani[1]

[1] *Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud, Islamic University (IMSIU), 90950 Riyadh 11632, Saudi Arabia*
[2] *Hydrological Research Center, Institute for Geological and Mining Research, 4110 Yaounde-Cameroon*

Received 31 January 2023; Accepted (in revised version) 6 June 2023

Abstract. This paper deals with an efficient two-step time split explicit/implicit scheme applied to a two-dimensional nonlinear unsteady convection-diffusion-reaction equation. The computational cost of the new algorithm at each time level is equivalent to solving a pentadiagonal matrix equation with strictly dominant diagonal elements. Such a bandwidth matrix can be easily inverted using the Gaussian Decomposition and the corresponding linear system should be solved by the back substitution method. The proposed approach is unconditionally stable, temporal second-order accuracy and fourth-order convergence in space. These results suggest that the developed technique is faster and more efficient than a large class of numerical methods studied in the literature for the considered initial-boundary value problem. Numerical experiments are carried out to confirm the theoretical analysis and to demonstrate the performance of the constructed numerical scheme.

AMS subject classifications: 65M12, 65M06

Key words: 2D nonlinear unsteady convection-diffusion-reaction equation, explicit method, implicit scheme, two-step time split fourth-order explicit/implicit approach, unconditional stability, error estimates.

1 Introduction

A broad range of applications arising in the field of mathematical biology, flow problems or heat transfer, chemical theory, convection heat transport problems, fluid dynamics,

*Corresponding author.
Emails: ericngondiep@gmail.com, engondiep@imamu.edu.sa (E. Ngondiep)

water pollution problems, nuclear reactions, bio-modeling and simulation of oil extraction from under-ground reservoirs and semiconductor device simulation are modeled by nonlinear ordinary and partial differential equations (ODEs/PDEs) with nonlinear terms [1–14]. Analytical solutions of such time-dependent PDEs are available for only simple cases. Specifically, the two-dimensional transport (convection-diffusion-reaction) equations have been used to describe a wide set of physical phenomena such as: thermal pollution in river systems, heat transfer in draining film, spread of contaminants coastal seas and estuaries, dispersion of pollutants in rivers and streams, flow in porous media, etc. [15–19]. When the diffusion coefficient is too small compared to a constant convection speed vector, the analytical solution of the transport problem may suffer from boundary and interior layers which can cause spurious instabilities due to the rapid transition of the solution over a very small space of the domain [20]. In addition, the flow behaviors usually contain steep gradients that require special treatment of numerical methods. Due to the complexity of many of these problems, the nonlinear nonstationary convection-diffusion-reaction equations are usually solved numerically. A large class of classical schemes are faced to strong oscillations or yield excessive numerical dispersion [21, 22]. To overcome these drawbacks, a broad range of stabilized numerical techniques have been developed in an approximate solution of such complex equations. For more details, the readers can consult the works discussed in [23–34] and references therein.

In this paper, we consider the following two-dimensional nonlinear unsteady convection-diffusion-reaction equation defined in [35] as

$$\begin{aligned}&\frac{\partial v}{\partial t}-a_1(x)\frac{\partial^2 v}{\partial x^2}-a_2(y)\frac{\partial^2 v}{\partial y^2}-\nabla\cdot\left[\left(\widehat{b}_1(x,y),\widehat{b}_2(x,y)\right)v\right]-(\widehat{c}_1(x)+\widehat{c}_2(y))v\\&=g(t,x,y,v)\quad\text{for}\ \ (x,y,t)\in\Omega\times(0,T],\end{aligned}\tag{1.1}$$

with initial condition

$$v(x,y,0)=\rho(x,y)\quad\text{for}\ \ (x,y)\in\Omega\cup\partial\Omega,\tag{1.2}$$

and boundary condition

$$v(x,y,t)=f(x,y,t)\quad\text{on}\ \ \partial\Omega\times[0,T].\tag{1.3}$$

Here: $a_1(x)>0$, $a_2(y)>0$, are diffusion coefficients, $(\widehat{b}_1(x,y),\widehat{b}_2(x,y))$ represents the convection speed vector, : $\widehat{c}_1(x)$ and $\widehat{c}_2(y)$, denote the reaction coefficients, $g(t,x,y,v)$ is the nonlinear source term while ρ and f designate the initial and boundary conditions, respectively. For the sake of stability analysis and error estimates, we assume that the nonlinear source term $g(t,x,y,v)$ is locally Lipschitz with respect to its fourth variable v. An efficient numerical approach should be able to simulate a transport phenomenon accurately while being capable to suppress the numerical instabilities arising during the spatial discretization. Both spatial second-order central difference schemes and first-order upwing difference formulations fail to approach the exact solution of Eq. (1.1), unless a