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A Well-Conditioned Spectral Integration Method for High-Order Differential Equations with Variable Coefficients

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Abstract. A well-conditioned spectral integration (SI) method is introduced, developed and applied to nth-order differential equations with variable coefficients and general boundary conditions. The approach is based on integral reformulation techniques which lead to almost banded linear matrices, and the main system to be solved is further banded by utilizing a Schur complement approach. Numerical experiments indicate the spectral integration method can solve high order equations efficiently, oscillatory problems accurately and is adaptable to large systems. Applications in Korteweg-de Vries (KdV) type and Kawahara equations are carried out to illustrate the proposed method is effective to complicated mathematical models.

AMS subject classifications: 65N35, 65L10, 33C45

Key words: Spectral integration methods, KdV equation, Kawahara equation.

1 Introduction

Spectral methods are extensively used in applied mathematics, scientific computing and engineering for solving differential equations [1–4]. Classical techniques, for instance, the spectral collocation [2, 5] and coefficients methods [6, 7] are well known for their remarkable accuracy. The former can be applied to differential equations with variable coefficients and nonlinear terms, several approaches of the latter can construct well-conditioned systems. However, the spectral collocation method will lead to dense and ill-conditioned matrices with a condition number growing like $\mathcal{O}(N^{2n})$ (n is the order of the differential equation and N is the number of collocation points).

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Within the spectral coefficients methods, a large body of literature has been contributed to improve the conditioning. Classical spectral methods of Galerkin type use compact combinations of orthogonal polynomials as trial/test functions and force the residual to zero [1]. Shen proposed Legendre-Galerkin approaches for solving problems with constant coefficients [1, 6, 8]. The resulting matrices of spectral-Galerkin methods are sparse and well-conditioned. These approaches are very efficient, but there are difficulties associated with them when automatically imposing the nonhomogeneous boundary conditions. Besides, spectral-Galerkin methods are only adaptable to equations with polynomial coefficients. Olver and Townsend developed an ultraspherical spectral (US) method in [7] for linear ordinary differential equations (ODEs) with variable coefficients. US method is adaptable to problems with general boundary conditions and leads to almost banded matrices. However, the condition number of the resulting matrices will still be large when the boundary conditions contain unknown solution's derivatives (see Table 1). To improve the shortcoming of techniques above, we develop a method in this paper which can solve high order linear ODEs with variable coefficients and construct well-conditioned discrete systems for general linearly independent boundary conditions.

Our work is based on the integral reformulation techniques which can date back to Clenshaw 's work [9]. In general, there are essentially two categories of integral reformulation techniques.

The first is to treat the coefficients of solution's highest order derivatives as unknowns, and further obtain the approximation of unknown function by the integral relationships between the coefficients of basis functions [10–15]. Greengard proposed a boundary value problem solver in [10, 12], this solver is efficient and accurate for second-order problems with constant coefficients, but it is not adaptable to high order equation and variable coefficients. Hariharan developed an ultraspherical and spectral integration methods in [14]. They combined the integration constants with the relationship of unknown function and its kth-order derivatives' Chebyshev coefficients to discretize the original equation. However, there will be n integration constants for nth order equation, which is complicated to construct discrete system for high-order equation.

For constant coefficients, the second is to integrate original equation [13, 16]. Du presented a Chebyshev spectral method in [13], which construct the discrete system by the integral operator and the multiplication operator in [7]. This method is effective and can preserve the well-conditioning property. Nevertheless, this kind of approach requires to reformulate the equation into an integral equation. To improve the issues of the integral reformulation techniques, the aim of our paper is to avoid integrating the original equation and handling the integration constants. Specifically, we generalize the spectral integration method [10] to the nth-order linear ordinary differential equations with variable coefficients and general boundary conditions. The main contributions of this work are as follows:

1. Banded and well-conditioned: As the classical spectral integration method [10] is only adaptable to second-order equation with constant coefficients, we propose to