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A Truncated-Type Explicit Numerical Method for the Stochastic Allen-Cahn Equation

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Abstract. The stochastic Allen-Cahn equations, as a typical example of nonlinear stochastic partial differential equations, play an important role in phase theory. In this paper, we investigate the rate of convergence in the pth mean for a truncated-type explicit Euler time-stepping method applied to the stochastic Allen-Cahn equations as well as using the spectral Galerkin approximation in spatial discretization. Finally, a numerical example is given to confirm the strong convergence order.

AMS subject classifications: 60H35, 60H15

Key words: Stochastic Allen-Cahn equations, truncated Euler–Maruyama, spectral Galerkin method, strong convergence.

1 Introduction

Over the past decades, there are plenty of research articles about stochastic partial differential equations (SPDEs). Since it is difficult to obtain the analytical solution to a SPDE, various numerical methods have been introduced for the SPDEs with drift and diffusion coefficients satisfying the global Lipschitz or linear growth condition [1, 14, 18, 19, 25, 29]. When numerically solving a SPDE, spatial discretizations are usually achieved with finite element, finite difference, and spectral Galerkin methods; see, for example, [20,21,26,37]. The temporally discretization is usually carried out by Euler-type methods [3–5,23,25,36].

As a typical example of SPDEs, stochastic Allen-Cahn equations play an important role in the phase theory and phase transition, and this class of equations have received increasing attention in the last few years [2, 8, 23–26]. However, once the nonlinear term of the coefficient of a stochastic differential equation grows super-linearly, the standard Euler-Maruyama (EM) method is known to diverge in the strong sense [17]. Thus, the implicit scheme [9, 30], modified EM method [5, 10], splitting scheme [6, 7], and other

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numerical methods [11, 28] have been applied to the stochastic Allen-Cahn equation. In a very recent work, Wang [36] proved a tamed EM method for temporal discretization combined with the spectral Galerkin method in space can achieve strong order 1/2. On the other side, truncated-type scheme is another efficient tool to approximate a stochastic differential equation with superlinear coefficients. There are several kinds of truncated methods designed for solving stochastic differential equation [4, 15, 16, 27, 32, 33]. In this paper, to our best knowledge, we apply, for the first time, this type of truncated scheme for the temporal discretization to construct fully discrete numerical estimators for the stochastic Allen-Cahn equation.

The rest of this paper is organized as follows. The next section presents assumptions and notations that are used throughout this paper. Section 3 is devoted to the analysis of the strong convergence rate for the full space-time discretization method to solve the underlying stochastic Allen-Cahn equation. Numerical experiments are illustrated in Section 4 to verify the theoretical findings.

2 Notations and assumptions

In this paper, we are interested in stochastic Allen-Cahn equations with additive space-time white noise, and it is described by

$$\begin{cases} \mathrm{d}X(t)+AX(t)\mathrm{d}t=F(X(t))\mathrm{d}t+\mathrm{d}W(t), & t\in(0,T], \\ X(0)=X_0. \end{cases} \tag{2.1}$$

Here the linear operator A, deterministic mappings F and I-cylindrical Wiener process $W(t,\cdot)$ that will be specified precisely later. Next, we give some notations and assumptions.

Throughout this article, unless otherwise specified, we use the following notation. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}_0$-null sets). We list some of the notation used throughout the paper as follows.

Symbol	Meaning
$(\mathbb{H},\langle\cdot,\cdot\rangle,\|\cdot\|)$	a real separable Hilbert space with $\|\cdot\|=\langle\cdot,\cdot\rangle^{\frac{1}{2}}$.
$\mathcal{L}(\mathbb{H})$	the space of bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ endowed with the usual operator norm $\|\cdot\|_{\mathcal{L}(\mathbb{H})}$.
$\mathcal{L}_2:=\mathcal{L}_2(\mathbb{H})\subset\mathcal{L}(\mathbb{H})$	the subspace consisting of all Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{H}$.
$L^\gamma(D):=L^\gamma(D;\mathbb{R})$, $\gamma\geq 1$	a Banach space consisting of γ-times integrable functions.
$V:=C(D,\mathbb{R})$	a Banach space of continuous functions with usual norms.
$\dot{\mathbb{H}}^\gamma:=\mathrm{dom}(A^{\frac{\gamma}{2}})$	the Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle_\gamma:=\langle A^{\frac{\gamma}{2}}\cdot,A^{\frac{\gamma}{2}}\cdot\rangle$ and norm $\|\cdot\|_\gamma=\langle\cdot,\cdot\rangle_\gamma^{\frac{1}{2}}$.
$-A$	the Laplacian with homogeneous Dirichlet boundary conditions, defined by $-Au=\Lambda u$.
λ_i, $e_i(x)$	eigenvalues and eigenvectors, here we let $\lambda_i=\pi^2 i^2$, $i\in\mathbb{N}$ and an orthonormal basis $\{e_i(x)=\sqrt{2}\sin(i\pi x),x\in(0,1)\}_{i\in\mathbb{N}}$, such that $Ae_i=\lambda_i e_i$.

Throughout this paper, let C stand for a generic positive constant independent of discretization parameters and its values may change between occurrences.

Assumption 2.1 (Linear operator A)**.** Let $D:=(0,1)$ and $\mathbb{H}=L^2(D;\mathbb{R})$ be a real separable Hilbert space, equipped with usual product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|=\langle\cdot,\cdot\rangle^{\frac{1}{2}}$. Let $-A:\ \mathrm{dom}(A)\subset\mathbb{H}\to\mathbb{H}$ be the Laplacian with homogeneous Dirichlet boundary conditions ($X(0,x)=X_0(x)$, $x\in D$, $X(t,0)=X(t,1)=0$, $t\in(0,T]$), defined by $-Au=\Lambda u$, $u\in\mathrm{dom}(A):=\mathbb{H}^2\cap\mathbb{H}_0^1$.

In the paper, we will utilize the semigroup approach in [34] to define a mild solution of (2.1). We list some properties of the analytic semigroup. Noting that $-A$ generates an analytic semigroup $E(t)=e^{-tA}$, $t\geq 0$ on $\mathbb{H}$ and we define the fractional powers of A, i.e., A^γ, $\gamma\in\mathbb{R}$. It is well-known that (see Lemma 3.22 in [31])

$$\|A^\gamma E(t)\|_{\mathcal{L}(H)}\leq Ct^{-\gamma},\qquad t>0,\quad \gamma\geq 0, \tag{2.2a}$$
$$\|A^{-\rho}(I-E(t))\|_{\mathcal{L}(H)}\leq Ct^{\rho},\qquad t>0,\quad \rho\in[0,1], \tag{2.2b}$$

and that (see Lemma 6.3 in [34])

$$\|A^{-\alpha}\|_{\mathcal{L}(H)}<\infty\quad \text{for any }\ 0<\alpha<1. \tag{2.3}$$

Assumption 2.2 (Nonlinearity)**.** Let $F:\ L^6(D;\mathbb{R})\to\mathbb{H}$ be a deterministic mapping defined by $F(v)(x)=f(v(x)):=v(x)-v^3(x), v\in L^6(D;\mathbb{R})$.

It is easy to find a constant $L\in(0,\infty)$ such that

$$\langle u-v,F(u)-F(v)\rangle\leq L\|u-v\|^2,\qquad u,v\in V, \tag{2.4a}$$
$$\|F(u)-F(v)\|\leq L(1+\|u\|^2+\|v\|^2)\|u-v\|,\qquad u,v\in V. \tag{2.4b}$$

The property in (2.4) immediately implies

$$\langle u,F(u)\rangle\leq C(1+\|u\|^2),\qquad u\in V. \tag{2.5a}$$
$$\|F(u)\|\leq C(1+\|u\|^2)\|u\|,\qquad u\in V. \tag{2.5b}$$

Assumption 2.3 (Noise process)**.** Let $\{W(t)\}_{t\in[0,T]}$ be an I-cylindrical Wiener process (see [12]) represented by a formal series,

$$W(t):=\sum_{n=1}^{\infty}\beta_n(t)e_n,\quad t\in[0,T], \tag{2.6}$$

where $\{\beta_n(t)\}_{n\in\mathbb{N}}$, $t\in[0,T]$ is a sequence of independent real-valued standard Brownian motions.

Assumption 2.4 (Initial value)**.** Let the initial data $X_0:\ \Omega\to H$, be an $\mathcal{F}_0/\mathcal{B}(H)$-measurable random variable satisfying, for sufficiently large positive number $p\in\mathbb{N}$ and $\beta\in(0,\frac{1}{2})$,

$$\mathbb{E}[\|X_0\|_V^p]+\mathbb{E}[\|A^{\frac{\beta}{2}}X_0\|^p]\leq C. \tag{2.7}$$

3 Spatial-temporal full discretization

In this paper, we study the accuracy of our spatio-temporal discretization scheme applied to SPDE (2.1). The results about the well-posedness, uniqueness of the mild solution, and the regularity results of SPDE (2.1) can be found in [13]. Meanwhile, we discrete the spatial by the spectral Galerkin methods, had obtained the convergence rate in [36]. To completeness of the paper, we list the results as the lemma in the appendix. Next, we give our time discretization method, the truncated EM scheme.

Firstly, we define the truncated EM numerical solutions, we choose a strictly increasing continuous function $\mu:\mathbb{R}_+\to\mathbb{R}_+$ such that $\mu(w)\to\infty$ as $w\to\infty$ and

$$\sup_{|x|\leq w}|F^N(x)|\leq\mu(w),\quad \forall w\geq 1. \tag{3.1}$$

Denote by μ^{-1} the inverse function of μ and we see that $\mu^{-1}:[\mu(0),\infty)\to\mathbb{R}_+$ is a strictly increasing continuous function. We also choose a strictly decreasing function $h:(0,1]\to(0,+\infty)$ and a constant $\bar{h}\geq 1$ such that for any $\beta\in[0,\frac{1}{2})$,

$$\lim_{\Delta\to 0}h(\Delta)=\infty \quad \text{and} \quad \Delta^{\frac{\beta}{2}}h(\Delta)\leq\bar{h},\quad \forall\Delta\in(0,1]. \tag{3.2}$$

For a given step size $\Delta\in(0,\Delta^*]$, let us define a mapping π_Δ from $\mathbb{R}^n$ to the closed ball $\{x\in\mathbb{R}^n:|x|\leq\mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x)=(|x|\wedge\mu^{-1}(h(\Delta)))\frac{x}{|x|},$$

where we set $x/|x|=0$ when $x=0$. That is, π_Δ will map x to itself when $|x|\leq\mu^{-1}(h(\Delta))$ and to $\mu^{-1}(h(\Delta))x/|x|$ when $|x|>\mu^{-1}(h(\Delta))$. We then define the truncated functions

$$F^N_\Delta(x)=F^N(\pi_\Delta(x))$$

for any $x\in\mathbb{R}^n$. It is easy to see that

$$|F^N_\Delta(x)|\leq\mu(\mu^{-1}(h(\Delta)))=h(\Delta). \tag{3.3}$$

For $M\in\mathbb{N}$ we construct a uniform mesh on $[0,T]$ with $\Delta=\frac{T}{M}$ being the time stepsize, and propose a spatio-temporal full discretization as,

$$\begin{aligned}Y^{M,N}_{t_{m+1}}=&E_N(t_{m+1}-t_m)Y^{M,N}_{t_m}+A_N^{-1}\big(I-E_N(\Delta)\big)F^N_\Delta(Y^{M,N}_{t_m})\\&+\int_{t_m}^{t_{m+1}}E_N(t_{m+1}-s)P^N dW(s)\end{aligned} \tag{3.4}$$

for $m=0,1,\cdots,M-1$ and $Y_0^{M,N}=P^N X_0$. Equivalently, the full discretization (3.4) is written by

$$\begin{aligned} Y_{t_{m+1}}^{M,N} =& E_N(t_{m+1}-t_m)Y_{t_m}^{M,N}+\int_{t_m}^{t_{m+1}} E_N(t_{m+1}-s)F_\Delta^N(Y_{t_m}^{M,N})ds \\ &+\int_{t_m}^{t_{m+1}} E_N(t_{m+1}-s)P^N dW(s), \end{aligned} \tag{3.5}$$

and the continuous version of (3.5) as

$$Y_t^{M,N}=E_N(t)Y_0^{M,N}+\int_0^t E_N(t-s)F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})ds+\int_0^t E_N(t-s)P^N dW(s), \tag{3.6}$$

where $\lfloor t\rfloor := t_i$ for $t\in[t_i,t_{i+1})$ $i\in\{0,1,\cdots,M-1\}$.

3.1 A priori moment bounds of the approximations

This subsection aims to obtain a priori estimates of the full discrete approximation.

Lemma 3.1. *Let Assumptions* 2.1-2.4 *hold. Then the approximation process* $Y_t^{M,N}$ *produced by* (3.6) *obeys*

$$\sup_{0\le t\le T}\mathbb{E}\|Y_t^{M,N}\|^p\le C. \tag{3.7}$$

Proof. To simply write, we set

$$\mathcal{O}_t^N=P^N\mathcal{O}_t=\int_0^t E_N(t-s)P^N dW(s).$$

Next we introduce a process $Z_t^{M,N}$ given by

$$\begin{aligned} Z_t^{M,N}&=Y_t^{M,N}-\mathcal{O}_t^N \\ &=E_N(t)Z_0^{M,N}+\int_0^t E_N(t-s)F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)ds, \end{aligned} \tag{3.8}$$

which satisfies

$$\frac{d}{dt}Z_t^{M,N}=-A_N Z_t^{M,N}+F_\Delta^N(Z_{\lfloor t\rfloor}^{M,N}+\mathcal{O}_{\lfloor t\rfloor}^N). \tag{3.9}$$

According to Lemma A.1, in order to prove the (3.7), we only need to obtain the boundness of the $Z_t^{M,N}$ i.e.,

$$\sup_{0\le u\le T}\mathbb{E}\|Z_u^{M,N}\|^p\le C.$$

Thus

$$\begin{aligned}\mathbb{E}\|Z_t^{M,N}\|^p =&\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\langle Z_s^{M,N},-A_N Z_s^{M,N}+F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\rangle ds\\ \leq&\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\langle Z_s^{M,N},F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\rangle ds\\ =&\underbrace{\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\langle Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N},F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\rangle ds}_{J_1}\\ &+\underbrace{\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\langle Z_{\lfloor s\rfloor}^{M,N},F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\rangle ds}_{J_2}. \end{aligned}\tag{3.10}$$

Because of J_1 is difficult to tackled, we first to deal with the J_2. By the Lemma 5.8 in [22], when the $F(x)=c_1x-c_2x^3$ $(c_1,c_2>0)$, we have

$$\langle x,F^N(x+y)\rangle=\langle x,F(x+y)\rangle\leq K[\|x\|^2+1+\|y\|^K],\tag{3.11}$$

where $K=\max\{6,c_1+|c_1|^2+|c_2|^2\}$. According to Lemma A.3, we have following inequality

$$\langle x,F_\Delta^N(x+y)\rangle\leq K_1[\|x\|^2+1+\|y\|^{K_1}].$$

Therefore

$$\begin{aligned}J_2=&\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\langle Z_{\lfloor s\rfloor}^{M,N},F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\rangle ds\\ \leq&\mathbb{E}\int_0^t pK_1\|Z_s^{M,N}\|^{p-2}\Big[\|Z_{\lfloor s\rfloor}^{M,N}\|^2+1+\|\mathcal{O}_{\lfloor s\rfloor}^N\|^{K_1}\Big]ds\\ \leq&\mathbb{E}\int_0^t C\|Z_s^{M,N}\|^{p-2}\Big[\|Z_{\lfloor s\rfloor}^{M,N}\|^2+1\Big]ds\\ \leq&C\int_0^t\sup_{0\leq u\leq s}\mathbb{E}\|Z_u^{M,N}\|^p ds+C.\end{aligned}\tag{3.12}$$

For the term of J_1, by the Young's inequality, we have

$$\begin{aligned}J_1=&\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\langle Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N},F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\rangle ds\\ \leq&\mathbb{E}\int_0^t p\|Z_s^{M,N}\|^{p-2}\cdot\|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|\cdot\|F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\|ds\\ \leq&\mathbb{E}\int_0^t (p-2)\|Z_s^{M,N}\|^p\,ds+\underbrace{2\mathbb{E}\int_0^t\|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|^{\frac{p}{2}}\cdot\|F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\|^{\frac{p}{2}}ds}_{J_{11}}.\end{aligned}$$

Finally, according to the Hölder's inequality and (3.3), we obtain

$$\begin{aligned}J_{11}=&2\mathbb{E}\int_0^t \|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|^{\frac{p}{2}}\cdot\|F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\|^{\frac{p}{2}}ds\\ \leq&2h^{\frac{p}{2}}(\Delta)\mathbb{E}\int_0^t \|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|^{\frac{p}{2}}ds.\end{aligned}$$

However, to prove J_{11}, we first claim the following result

$$\mathbb{E}\|Z_t^{M,N}-Z_{\lfloor t\rfloor}^{M,N}\|^p\leq C\Delta^{\frac{\beta p}{2}}. \tag{3.13}$$

Since definition of the $Z_t^{M,N}$, that is (3.8), we have

$$\begin{aligned}Z_t^{M,N}-Z_{\lfloor t\rfloor}^{M,N}=&\underbrace{(E_N(t)-E_N(\lfloor t\rfloor))Z_0^{M,N}}_{I_1}\\ &+\underbrace{\int_0^t E_N(t-s)F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)ds-\int_0^{\lfloor t\rfloor}E_N(\lfloor t\rfloor-s)F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)ds}_{I_2}.\end{aligned}$$

Therefore,

$$\mathbb{E}\|Z_t^{M,N}-Z_{\lfloor t\rfloor}^{M,N}\|^p\leq C\mathbb{E}\|I_1\|^p+C\mathbb{E}\|I_2\|^p. \tag{3.14}$$

For the term I_2, we get

$$\begin{aligned}\mathbb{E}\|I_2\|^p=&\mathbb{E}\Big\|\int_0^t e^{-(t-s)A_N}F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\,ds-\int_0^{\lfloor t\rfloor}e^{-(\lfloor t\rfloor-s)A_N}F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\,ds\Big\|^p\\ \leq&\underbrace{C\mathbb{E}\Big\|\int_{\lfloor t\rfloor}^t e^{-(t-s)A_N}F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)ds\Big\|^p}_{I_{21}}\\ &+\underbrace{C\mathbb{E}\Big\|\int_0^{\lfloor t\rfloor}(e^{-(t-s)A_N}-e^{-(\lfloor t\rfloor-s)A_N})F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\,ds\Big\|^p}_{I_{22}}.\end{aligned}$$

The boundedness of the semigroup $E(t)$ in V promises

$$\begin{aligned}I_{21}=&C\mathbb{E}\Big\|\int_{\lfloor t\rfloor}^t e^{-(t-s)A_N}F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)ds\Big\|^p\\ \leq&C\Delta^{p-1}\mathbb{E}\int_{\lfloor t\rfloor}^t\|F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\|^p ds\\ \leq&\Delta^p h^p(\Delta).\end{aligned} \tag{3.15}$$

Since the (2.2) and (2.3), for any $\beta\in(0,\frac{1}{2})$, we obtain

$$
\begin{aligned}
I_{22}=&C\mathbb{E}\Big\|\int_0^{\lfloor t\rfloor}(e^{-(t-s)A_N}-e^{-(\lfloor t\rfloor-s)A_N})F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\,ds\Big\|^p\\
\le&C\mathbb{E}\int_0^{\lfloor t\rfloor}\Big\|(e^{-(t-s)A_N}-e^{-(\lfloor t\rfloor-s)A_N})F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\Big\|^p ds\\
=&C\mathbb{E}\int_0^{\lfloor t\rfloor}\Big\|e^{-(\lfloor t\rfloor-s)A_N}(e^{-(t-\lfloor t\rfloor)A_N}-I)F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\Big\|^p ds\\
=&C\mathbb{E}\int_0^{\lfloor t\rfloor}\|A^\beta e^{-(\lfloor t\rfloor-s)A_N}\|_{\mathcal{L}(H)}^p\cdot\|A^{-\beta}(I-e^{-(t-\lfloor t\rfloor)A_N})\|_{\mathcal{L}(H)}^p\cdot\|F_\Delta^N(Z_{\lfloor s\rfloor}^{M,N}+\mathcal{O}_{\lfloor s\rfloor}^N)\|^p ds\\
\le&C\mathbb{E}\int_0^{\lfloor t\rfloor}(\lfloor t\rfloor-s)^{-p\beta}\cdot(t-\lfloor t\rfloor)^{p\beta}\cdot h^p(\Delta)ds\\
=&C(t-\lfloor t\rfloor)^{p\beta}h^p(\Delta)\cdot\frac{\lfloor t\rfloor^{1-p\beta}}{1-p\beta}\\
\le&C\Delta^{p\beta}h^p(\Delta).
\end{aligned}
\tag{3.16}
$$

Combining (3.15) with (3.16) and noting that $\beta\in(0,\frac{1}{2})$, we have

$$
\mathbb{E}\|I_2\|^p\le C\Delta^p h^p(\Delta)+C\Delta^{p\beta}h^p(\Delta)\le C\Delta^{\frac{\beta p}{2}}. \tag{3.17}
$$

For I_1, as the same technique of I_{22}, we have for any $t>0$,

$$
\begin{aligned}
\mathbb{E}\|I_1\|^p=&\mathbb{E}\|(e^{-tA_N}-e^{-\lfloor t\rfloor A_N})Z_0^{M,N}\|^p\\
\le&\|e^{-\lfloor t\rfloor A_N}\|_{\mathcal{L}(H)}^p\cdot\|A^{-\beta/2}(I-e^{-(t-\lfloor t\rfloor)A_N})\|_{\mathcal{L}(H)}^p\cdot\|A^{\beta/2}Z_0^{M,N}\|^p\\
\le&C(t-\lfloor t\rfloor)^{\frac{p\beta}{2}}\cdot\|A^{\beta/2}Z_0^{M,N}\|^p\le C\Delta^{\frac{p\beta}{2}}.
\end{aligned}
\tag{3.18}
$$

Inserting (3.17), (3.18) into (3.14), we have

$$
\mathbb{E}\|Z_t^{M,N}-Z_{\lfloor t\rfloor}^{M,N}\|^p\le C\Delta^{\frac{\beta p}{2}}.
$$

That is, (3.13) holds.

Recalling that $\Delta^{\frac{\beta}{2}}h(\Delta)\le K$, $(\beta\in(0,1/2))$ in (3.2), we obtain

$$
\begin{aligned}
J_{11}\le&2h^{\frac{p}{2}}(\Delta)\mathbb{E}\int_0^t\|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|^{\frac{p}{2}}ds\le 2h^{\frac{p}{2}}(\Delta)\int_0^T\mathbb{E}\|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|^{\frac{p}{2}}ds\\
\le&2h^{\frac{p}{2}}(\Delta)\int_0^T\Big(\mathbb{E}\|Z_s^{M,N}-Z_{\lfloor s\rfloor}^{M,N}\|^p\Big)^{\frac{1}{2}}ds\le CT\Delta^{\frac{\beta p}{4}}h^{\frac{p}{2}}(\Delta)\le C.
\end{aligned}
$$

Therefore,

$$
J_1\le\mathbb{E}\int_0^t(p-2)\|Z_s^{M,N}\|^p ds+C. \tag{3.19}
$$

Inserting (3.12) and (3.19) into (3.10), we have

$$\mathbb{E}\|Z_t^{M,N}\|^p \leq C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}\|Z_u^{M,N}\|^p ds.$$

As this holds for any $t\in[0,T]$, while the right-hand side is non-decreasing in t, we then see

$$\sup_{0\leq u\leq t} \mathbb{E}\|Z_u^{M,N}\|^p \leq C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}\|Z_u^{M,N}\|^p ds.$$

Finally, by the discrete Gronwall inequality, we have

$$\sup_{0\leq u\leq T} \mathbb{E}\|Z_u^{M,N}\|^p \leq C.$$

This proof is therefore completed. □

With Lemma 3.1, one can obtain the following corollary.

Corollary 3.1. *Under conditions in Lemma* 3.1, *for any* $p\in[2,\infty)$ *and* $\beta<\frac{1}{2}$ *we obtain,*

$$\sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\|Y_t^{M,N}\|_V^p + \sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\|A^{\frac{\beta}{2}} Y_t^{M,N}\|^p < \infty.$$

3.2 Main results: error bounds for the full discretization

In this section, we will prove our main result about the convergence rate of the full discretization.

Theorem 3.1 (Error bounds for the full discretization)**.** *Let Assumptions* 2.1-2.4 *hold. Then for any* $q\in[2,p)$ *and* $\beta\in(0,\frac{1}{2})$,

$$\sup_{0\leq m\leq M} \mathbb{E}\|X(t_m)-Y_{t_m}^{M,N}\|^q \leq C\big(N^{-q\beta}+\Delta^{q\beta}+(\mu^{-1}(h(\Delta)))^{-(p-q)}\big), \tag{3.20}$$

where C is an independent of N and M generic constant.

Before proving this main result, we give the coming useful lemma.

Lemma 3.2. *Under conditions in Theorem* 3.1, *for any* $\beta\in(0,\frac{1}{2})$, $\eta>\frac{1}{2}$ *and* $s,t\in(0,\Delta)$, *we have*

$$\mathbb{E}\Big\|A^{-\frac{\eta}{2}}\big(F(Y_t^{M,N})-F(Y_s^{M,N})\big)\Big\|^q \leq C(t-s)^{q\beta}, \quad 0\leq s<t\leq T. \tag{3.21}$$

Proof. According to Lemma 4.9 in [36], for any $\sigma\in[0,1]$, we have

$$\begin{aligned}
&\mathbb{E}\|A^{-\frac{\eta}{2}}(F(Y_t^{M,N})-F(Y_s^{M,N}))\|^q\\
=&\mathbb{E}\|A^{-\frac{\eta}{2}}F'(Y_s^{M,N}+\sigma(Y_t^{M,N}-Y_s^{M,N}))(Y_t^{M,N}-Y_s^{M,N})\|^q\\
\le& C\mathbb{E}\Big(1+\max\{\|Y_s^{M,N}+\sigma(Y_t^{M,N}-Y_s^{M,N})\|,\|A^{\frac{\beta}{2}}(Y_s^{M,N}+\sigma(Y_t^{M,N}-Y_s^{M,N}))\|\}^2\Big)\\
&\quad\times\|A^{-\frac{\beta}{2}}(Y_t^{M,N}-Y_s^{M,N})\|^q\\
\le& C\|A^{-\frac{\beta}{2}}(Y_t^{M,N}-Y_s^{M,N})\|^q,
\end{aligned}$$

where the last inequality, Lemma 3.1 and Corollary 3.1 are used.

Recalling the (3.6), we have

$$\begin{aligned}
Y_t^{M,N}-Y_s^{M,N}=&(E_N(t-s)-I)Y_s^{M,N}+\int_s^t E_N(t-r)F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})dr\\
&+\int_s^t E_N(t-r)P^N dW(r).
\end{aligned}$$

Taking the operate of $A^{-\frac{\beta}{2}}$ and expectation $\mathbb{E}$ on both sides, we thus have

$$\begin{aligned}
&\mathbb{E}\|A^{-\frac{\beta}{2}}(Y_t^{M,N}-Y_s^{M,N})\|^q\\
\le& C\mathbb{E}\|A^{-\frac{\beta}{2}}(E_N(t-r)-I)Y_s^{M,N}\|^q+C\mathbb{E}\Big\|\int_s^t A^{-\frac{\beta}{2}}E_N(t-s)F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})dr\Big\|^q\\
&+C\mathbb{E}\Big\|\int_s^t A^{-\frac{\beta}{2}}E_N(t-r)P^N dW(r)\Big\|^q\\
:=&K_1+K_2+K_3.
\end{aligned}$$

For the term K_1, according to (2.2) and Corollary 3.1, we have

$$\begin{aligned}
K_1=&C\mathbb{E}\|A^{-\frac{\beta}{2}}(E_N(t-s)-I)Y_s^{M,N}\|^q\\
=&C\mathbb{E}\|A^{-\beta}(E_N(t-s)-I)A^{\frac{\beta}{2}}Y_s^{M,N}\|^q\\
\le& C\|A^{-\beta}(E_N(t-s)-I)\|^q\cdot\mathbb{E}\|A^{\frac{\beta}{2}}Y_s^{M,N}\|^q\\
\le& C(t-s)^{q\beta}\le C\Delta^{q\beta}.
\end{aligned}$$

For the term K_2, with the help of (3.2) and (3.3), note that $\beta\in(0,\frac{1}{2})$, we get

$$\begin{aligned}
K_2=&C\mathbb{E}\Big\|\int_s^t A^{-\frac{\beta}{2}}E_N(t-r)F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})dr\Big\|^q\\
\le& C(t-s)^{q-1}\int_s^t\mathbb{E}\|A^{-\frac{\beta}{2}}E_N(t-r)F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})\|^q dr\\
\le& C(t-s)^{q-1}\int_s^t\|F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})\|^q dr\\
\le& C\Delta^q h^q(\Delta)\le C\Delta^{q\beta}.
\end{aligned}$$

Finally, by the Burkholder-Davis-Gundy inequality and (2.3), one have

$$\begin{aligned}
K_3 =&\mathbb{E}\Big\|\int_s^t A^{-\frac{\beta}{2}}E_N(t-r)P^N dW(r)\Big\|^q\\
=&C\Big(\int_s^t \|A^{-\frac{\beta}{2}}E_N(t-r)P^N\|_{\mathcal{L}_2}^2 dr\Big)^{\frac{q}{2}}\\
=&C\Big(\int_s^t \|A^{\frac{\beta-1}{2}}A^{\frac{1}{2}-\beta}E_N(t-r)P^N\|_{\mathcal{L}_2}^2 dr\Big)^{\frac{q}{2}}\\
\le&C\Big(\int_s^t \|A^{\frac{\beta-1}{2}}\|^2\cdot\|A^{\frac{1}{2}-\beta}E_N(t-r)\|^2 dr\Big)^{\frac{q}{2}}\\
\le&C\Big(\int_s^t (t-r)^{2\beta-1}dr\Big)^{\frac{q}{2}}\\
\le&C(t-s)^{q\beta}\le C\Delta^{q\beta}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\Big\|A^{-\frac{\beta}{2}}(Y_t^{M,N}-Y_s^{M,N})\Big\|^q\le C\Delta^{q\beta},$$

that is the conclusion holds. □

Proof of Theorem 3.1. Denoting

$$e_t^{M,N}:=X^N(t)-Y_t^{M,N},$$

we note that

$$\|X(t_m)-Y_{t_m}^{M,N}\|^q\le\|X(t_m)-X^N(t_m)\|^q+\|e_{t_m}^{M,N}\|^q,$$

where $e_t^{M,N}$ satisfied

$$\frac{\mathrm{d}}{\mathrm{d}t}e_t^{M,N}=-A_N e_t^{M,N}+F^N\big(X(t)\big)-F_\Delta^N(Y_{\lfloor t\rfloor}^{M,N}).$$

This in conjunction with (2.4) tells us that

$$\begin{aligned}
\|e_t^{M,N}\|^q=&q\int_0^t\|e_s^{M,N}\|^{q-2}\Big\langle e_s^{M,N},-A_N e_s^{M,N}+F^N(X(s))-F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\
=&q\int_0^t\|e_s^{M,N}\|^{q-2}\Big\langle e_s^{M,N},-A_N e_s^{M,N}-F^N(X^N(s))+F^N(Y_s^{M,N})+F^N(X(s))\\
&-F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds+q\int_0^t\|e_s^{M,N}\|^{q-2}\Big\langle X^N(s)-Y_s^{M,N},F^N(X^N(s))-F^N(Y_s^{M,N})\Big\rangle ds
\end{aligned}$$

$$
\begin{aligned}
&\le q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle e_s^{M,N}, -A_N e_s^{M,N} - F^N(X^N(s)) + F^N(Y_s^{M,N}) + F^N(X(s)) \\
&\qquad - F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds + pL\int_0^t \|e_s^{M,N}\|^q ds \\
&\le q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle e_s^{M,N}, F^N(X(s)) - F^N(X^N(s)) + F^N(Y_s^{M,N}) - F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds \\
&\qquad + qL\int_0^t \|e_s^{M,N}\|^q ds \\
&= q\int_0^t \|e_s^{M,N}\|^{q-2}\langle e_s^{M,N}, F^N(X(s)) - F^N(X^N(s))\rangle ds \\
&\qquad + q\int_0^t \|e_s^{M,N}\|^{q-2}\langle e_s^{M,N}, F^N(Y_s^{M,N}) - F^N(Y_{\lfloor s\rfloor}^{M,N})\rangle ds \\
&\qquad + q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle e_s^{M,N}, F^N(Y_{\lfloor s\rfloor}^{M,N}) - F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds + qL\int_0^t \|e_s^{M,N}\|^q ds \\
&:= \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + qL\int_0^t \|e_s^{M,N}\|^q ds.
\end{aligned} \tag{3.22}
$$

Similar to J_0 in [36], we have

$$
\begin{aligned}
\mathbb{E}[\tilde{J}_1] &= q\int_0^t \|e_s^{M,N}\|^{q-2}\langle e_s^{M,N}, F^N(X(s)) - F^N(X^N(s))\rangle ds \\
&\le (q-1)\int_0^t \mathbb{E}[\|e_s^{M,N}\|^q] ds + CN^{-q\beta}.
\end{aligned} \tag{3.23}
$$

Recalling (A.6) and (3.6), we obtain

$$
e_s^{M,N} = X^N(s) - Y_s^{M,N} = \int_0^s E(s-r)\big(F^N(X(r)) - F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})\big) dr
$$

and

$$
\begin{aligned}
\tilde{J}_2 &= q\int_0^t \|e_s^{M,N}\|^{q-2}\langle e_s^{M,N}, F^N(Y_s^{M,N}) - F^N(Y_{\lfloor s\rfloor}^{M,N})\rangle ds \\
&= q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(X(r)) - F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})\big) dr, F^N(Y_s^{M,N}) - F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds \\
&= q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(X(r)) - F^N(Y_r^{M,N})\big) dr, F^N(Y_s^{M,N}) - F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds \\
&\qquad + q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(Y_r^{M,N}) - F^N(Y_{\lfloor r\rfloor}^{M,N})\big) dr, F^N(Y_s^{M,N}) - F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds \\
&\qquad + q\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(Y_{\lfloor r\rfloor}^{M,N}) - F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})\big) dr, F^N(Y_s^{M,N}) - F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds \\
&:= \tilde{J}_{21} + \tilde{J}_{22} + \tilde{J}_{23}.
\end{aligned} \tag{3.24}
$$

For the term of $\tilde{J}_{21}$ and $\tilde{J}_{22}$, as similar to the J_{11} and J_{12} in [36], by using Theorem A.1, then we also have

$$\begin{aligned}\mathbb{E}[\tilde{J}_{21}]=&q\mathbb{E}\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(X(r))-F^N(Y_r^{M,N})\big)dr,\\&F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\ \le& C\int_0^t \mathbb{E}\|e_s^{M,N}\|^q ds+CN^{-q\beta}+C\Delta^{q\beta},\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}\mathbb{E}[\tilde{J}_{22}]=&q\mathbb{E}\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(Y_r^{M,N})-F^N(Y_{\lfloor r\rfloor}^{M,N})\big)dr,\\&F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\ \le& C\int_0^t \mathbb{E}[\|e_s^{M,N}\|^q]ds+C\Delta^{q\beta}.\end{aligned}\tag{3.26}$$

Now we estimate $\mathbb{E}[\tilde{J}_{23}]$. Utilizing (2.4), (3.13), the Hölder's inequality, Lemma A.2, and Lemma 3.1 we obtain

$$\begin{aligned}\mathbb{E}[\tilde{J}_{23}]=&q\mathbb{E}\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(Y_{\lfloor r\rfloor}^{M,N})-F_\Delta^N(Y_{\lfloor r\rfloor}^{M,N})\big)dr,\\&F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\ =&q\mathbb{E}\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)\big(F^N(Y_{\lfloor r\rfloor}^{M,N})-F^N(\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N}))\big)dr,\\&F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\ =&q\mathbb{E}\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle \int_0^s E(s-r)P^N\int_0^1 F'\big(\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})+\sigma(Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N}))\big)d\sigma\\&\cdot\big(Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\big)dr, F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\ =&q\mathbb{E}\int_0^t\int_0^s\int_0^1 \|e_s^{M,N}\|^{q-2}\Big\langle Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N}),\big(F'\big(\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})+\sigma(Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N}))\big)\big)^*\\&P^N E(s-r)\big[F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\big]\Big\rangle d\sigma drds\\ \le& q\mathbb{E}\int_0^t\int_0^s\int_0^1 \|e_s^{M,N}\|^{q-2}\cdot\|Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\|\cdot\big\|F'\big(\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})+\sigma(Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N}))\big)\\&P^N A^{\frac{\eta}{2}}E(s-r)A^{-\frac{\eta}{2}}\big[F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\big]\big\|d\sigma drds\\ \le& C\mathbb{E}\int_0^t\int_0^s \|e_s^{M,N}\|^{q-2}\|Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\|\cdot\big(1+\|Y_{\lfloor r\rfloor}^{M,N}\|^2+\|\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\|^2\big)\\&\times(s-r)^{-\frac{\eta}{2}}\big\|A^{-\frac{\eta}{2}}\big[F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\big]\big\|drds\end{aligned}$$

$$\begin{aligned}
&\leq C\mathbb{E}\int_0^t\int_0^s (s-r)^{-\frac{\eta}{2}}\big\|e_s^{M,N}\big\|^q drds + C\mathbb{E}\int_0^t\int_0^s (s-r)^{-\frac{\eta}{2}}\big\|Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\big\|^{q/2}\\
&\quad\times\big(1+\|Y_{\lfloor r\rfloor}^{M,N}\|^2\big)\big\|A^{-\frac{\eta}{2}}\big[F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\big]\big\|^{q/2}drds\\
&\leq C\mathbb{E}\int_0^t\big\|e_s^{M,N}\big\|^q ds + C\mathbb{E}\int_0^t\int_0^s (s-r)^{-\frac{\eta}{2}}\big\|Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\big\|^q drds\\
&\quad+C\mathbb{E}\int_0^t\int_0^s (s-r)^{-\frac{\eta}{2}}\big\|A^{-\frac{\eta}{2}}\big[F^N(Y_s^{M,N})-F^N(Y_{\lfloor s\rfloor}^{M,N})\big]\big\|^q drds\\
&\leq C\mathbb{E}\int_0^t\big\|e_s^{M,N}\big\|^q ds + C\mathbb{E}\int_0^t\int_0^s (s-r)^{-\frac{\eta}{2}}\big\|Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\big\|^q drds + C\Delta^{q\beta}. \qquad (3.27)
\end{aligned}$$

However, for the second term of (3.27), by the Hölder's inequality and Chebyshev's inequality, we have

$$\begin{aligned}
&\sup_{0\leq s\leq T}\mathbb{E}\big\|Y_{\lfloor s\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor s\rfloor}^{M,N})\big\|^q\\
&\leq\sup_{0\leq s\leq T}\mathbb{E}\Big[I_{\|Y_{\lfloor s\rfloor}^{M,N}\|>\mu^{-1}(h(\Delta))}\|Y_{\lfloor s\rfloor}^{M,N}\|^q\Big]\\
&\leq\sup_{0\leq s\leq T}\Big[\mathbb{P}\{\|Y_{\lfloor s\rfloor}^{M,N}\|>\mu^{-1}(h(\Delta))\}\Big]^{\frac{p-q}{p}}\Big(\mathbb{E}\|Y_{\lfloor s\rfloor}^{M,N}\|^p\Big)^{\frac{q}{p}}\\
&\leq\sup_{0\leq s\leq T}\Big[\mathbb{P}\{\|Y_{\lfloor s\rfloor}^{M,N}\|>\mu^{-1}(h(\Delta))\}\Big]^{\frac{p-q}{p}}\Big(\sup_{0\leq s\leq T}\mathbb{E}\|Y_{\lfloor s\rfloor}^{M,N}\|^p\Big)^{\frac{q}{p}}\\
&\leq C\Big(\frac{\sup_{0\leq s\leq T}\mathbb{E}\|Y_{\lfloor s\rfloor}^{M,N}\|^p}{(\mu^{-1}(h(\Delta)))^p}\Big)^{\frac{p-q}{p}}\leq C(\mu^{-1}(h(\Delta)))^{-(p-q)}.
\end{aligned}$$

Therefore, for any $1/2<\eta<1$, we have

$$\begin{aligned}
&C\mathbb{E}\int_0^t\int_0^s(s-r)^{-\frac{\eta}{2}}\big\|Y_{\lfloor r\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor r\rfloor}^{M,N})\big\|^q drds\\
&\leq C\int_0^t\sup_{0\leq u\leq T}\mathbb{E}\big\|Y_{\lfloor u\rfloor}^{M,N}-\pi_\Delta(Y_{\lfloor u\rfloor}^{M,N})\big\|^q\int_0^s(s-r)^{-\frac{\eta}{2}}drds\\
&\leq C(\mu^{-1}(h(\Delta)))^{-(p-q)}\cdot\frac{t^{2-\frac{\eta}{2}}}{(1-\frac{\eta}{2})(2-\frac{\eta}{2})}\\
&\leq C(\mu^{-1}(h(\Delta)))^{-(p-q)}.
\end{aligned}$$

Thus,

$$\mathbb{E}[\tilde{J}_{23}]\leq C\int_0^t\mathbb{E}[\|e_s^{M,N}\|^q]ds+C(\mu^{-1}(h(\Delta)))^{-(p-q)}+C\Delta^{q\beta}. \qquad (3.28)$$

Combining (3.25) (3.26) (3.28) into (3.24), therefore,

$$\mathbb{E}[\tilde{J}_2]\leq C\int_0^t\mathbb{E}[\|e_s^{M,N}\|^q]ds+CN^{-q\beta}+C\Delta^{q\beta}+C(\mu^{-1}(h(\Delta)))^{-(p-q)}. \qquad (3.29)$$

The term $\tilde{J}_3$ is also easy to be treated, after taking the Hölder's inequality and (3.13) into account:

$$\begin{aligned}\mathbb{E}[\tilde{J}_3]:=&q\mathbb{E}\int_0^t \|e_s^{M,N}\|^{q-2}\Big\langle e_s^{M,N}, F^N(Y_{\lfloor s\rfloor}^{M,N})-F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\Big\rangle ds\\ \leq&(q-2)\int_0^t \mathbb{E}\|e_s^{M,N}\|^q ds+\mathbb{E}\int_0^t \|F^N(Y_{\lfloor s\rfloor}^{M,N})-F_\Delta^N(Y_{\lfloor s\rfloor}^{M,N})\|^q ds\\ =&(q-2)\int_0^t \mathbb{E}\|e_s^{M,N}\|^q ds+C(\mu^{-1}(h(\Delta)))^{-(p-q)}.\end{aligned}$$

Plugging this and (3.23), (3.29) into (3.22), the desired error bound is derived by using the discrete version of Gronwall's inequality.

4 Numerical experiments

In this section, we consider the following stochastic Allen-Cahn equation with additive space-time white noise, described by

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+u-u^3+\dot{W}(t), & t\in(0,1], \quad x\in(0,1),\\ u(0,x)=0, & x\in(0,1),\\ u(t,0)=u(t,1)=0, & t\in(0,1].\end{cases} \tag{4.1}$$

Here $\{W(t)\}_{t\in[0,T]}$ is a cylindrical I-Wiener process represented by (2.6).

Here we choose the $\mu(x)=x^3$ and $h(\Delta)=\Delta^{-\frac{\beta}{2}}$, then

$$\mu^{-1}(h(\Delta))=\Delta^{-\frac{\beta}{6}}.$$

Meanwhile, if we consider the mean-square convergence, that is $q=2$, we have

$$(\mu^{-1}(h(\Delta)))^{\frac{-(p-2)}{2}}=\Delta^{\frac{\beta(p-2)}{12}}.$$

Therefore, once we have the boundedness of numerical solution in L^p sense for $p\geq 14$, one can obtain that the mean-square convergence order in time of the proposed scheme is β, which does not exceed $\frac{1}{2}$.

Since, in this paper, we mainly consider the convergence rate in time, we set the space step that $N=2^{11}$. Meanwhile, we use the following formula to calculate the convergence rate in time

$$\begin{aligned}\text{convergence order}=&\frac{\ln\Big(\mathbb{E}\|Y_T^{\Delta,N}-Y_T^{\Delta_{\text{exact}},N}\|^2\Big)}{\ln\Delta}\\ \approx&\frac{1}{\ln\Delta}\ln\Big(\frac{1}{K}\sum_{k=1}^{K}\|Y_{T,k}^{\Delta,N}-Y_{T,k}^{\Delta_{\text{exact}},N}\|^2\Big),\end{aligned}$$

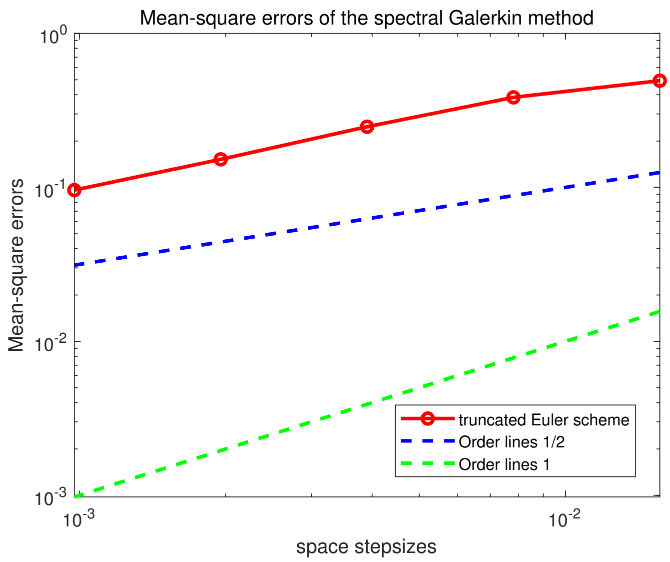

Figure 1: The convergence rate of the spectral Galerkin spatial discretization.

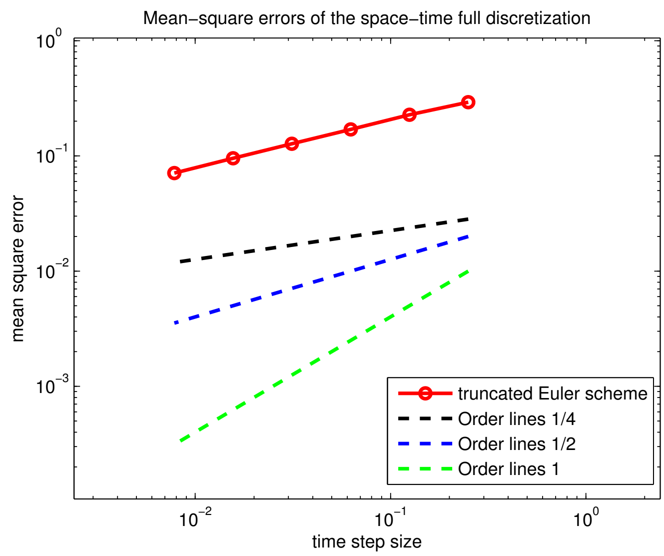

Figure 2: Mean-square errors of the spatio-temporal discretization.

where we set $\Delta_{\text{exact}} = 2^{-11}$ as the 'exact' time step-size and choose $\Delta \in \{2^{-2}, 2^{-3}, \cdots, 2^{-7}\}$. Here $T=1$ is the final time, we take $K=1000$ as the number of the simulation trajectories. We can find the convergence rate in time is closed to $\frac{1}{2}$ from Fig. 1, which agrees with our strong convergence result in Theorem 3.1. Meanwhile, we have chosen $\beta = \frac{1}{2}, \frac{1}{4}$, and

$\frac{1}{8}$, but we have observed that the convergence rate remains unchanged. The proposed scheme exhibits a mean-square convergence order in time that is close to $\frac{1}{2}$.

Appendix

Lemma A.1 ([36]). *For any $p\in[2,\infty)$ and $\beta\in(0,\frac{1}{2})$, the stochastic convolution $\{\mathcal{O}_t\}_{t\in[0,T]}$ satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|\mathcal{O}_t\|^p\Big]<\infty \quad \text{with } \mathcal{O}_t:=\int_0^t E(t-s)dW(s), \tag{A.1a}$$

$$\mathbb{E}\|\mathcal{O}_t-\mathcal{O}_s\|^p\le C(t-s)^{\frac{\beta p}{2}}, \quad 0\le s<t\le T. \tag{A.1b}$$

Lemma A.2 ([36]). *Under Assumptions* 2.1-2.4*, stochastic Allen-Cahn equation* (2.1) *possesses a unique mild solution $X:[0,T]\times\Omega\to V$ with continuous sample path, determined by,*

$$X(t)=E(t)X_0+\int_0^t E(t-s)F(X(s))ds+\mathcal{O}_t \quad \mathbb{P}\text{-a.s.}. \tag{A.2}$$

For $p\in[2,\infty)$ there exists a positive constant C such that, for any $\beta\in[0,\frac{1}{2})$,

$$\sup_{t\in[0,T]}\mathbb{E}\|X(t)\|^p\le C\big(1+\|X_0\|^p\big). \tag{A.3}$$

Moreover, we have

$$\mathbb{E}\|X_t-X_s\|^p\le C_2(t-s)^{\frac{\beta p}{2}}, \quad 0\le s<t\le T. \tag{A.4}$$

The spectral Galerkin approximation of (2.1) results in the following finite-dimensional SPDEs,

$$\begin{cases} \mathrm{d}X^N(t)+A_NX^N(t)\mathrm{d}t=F^N(X^N(t))\mathrm{d}t+P^N\mathrm{d}W(t), & t\in(0,T], \\ X^N(0)=P^NX_0, \end{cases} \tag{A.5}$$

where we write $F^N:=P^NF$ for short. By the variation of the constant, (A.5) admits a unique mild solution in $\mathbb{H}^N$ and can be written as

$$X^N(t)=E_N(t)P^NX_0+\int_0^t E_N(t-s)P^NF(X^N(s))ds+\int_0^t E_N(t-s)P^N\mathrm{d}W(s), \quad \mathbb{P}\text{-a.s.}. \tag{A.6}$$

The convergence analysis for the spectral Galerkin discretization (A.5) can be founded in [36]. We also put it here as an important theorem.

Theorem A.1 (Spatial error estimate). *Let Assumptions* 2.1-2.4 *hold. Let $X(t)$ and $X^N(t)$ be defined through* (2.1) *and* (A.6)*, respectively. Then it holds, for any $\beta\in(0,\frac{1}{2})$, $p\in[2,\infty)$ and $N\in\mathbb{N}$,*

$$\sup_{t\in[0,T]}\mathbb{E}\|X(t)-X^N(t)\|^p\le CN^{-p\beta}. \tag{A.7}$$

Lemma A.3. *Let the function* $f(x)=c_1x-c_2x^3$ *satisfied*

$$\langle x,f(x+y)\rangle \le K(\|x\|^2+1+\|y\|^K),$$

where $K=\max\{6,c_1+|c_2|^2+|c_2|^2\}$. *Then, for the truncated function* $f_\Delta(x)$, *we also have the following inequality*

$$\langle x,f_\Delta(x+y)\rangle \le K_1(\|x\|^2+1+\|y\|^{K_1}),$$

where $K_1=\max\{6,2+|c_1|^2+|c_2|^2\}$.

Proof: When $\|x+y\|\le \mu^{-1}(h(\Delta))$, the $f_\Delta(x+y)=f(x+y)$, then the assertion is clearly hold. Next, we only to prove the case of $\|x+y\|>\mu^{-1}(h(\Delta))$,

$$\begin{aligned}
&\langle x,f_\Delta(x+y)\rangle\\
=&\langle x,c_1\pi_\Delta(x+y)-c_2(\pi_\Delta(x+y))^3\rangle\\
\le& c_1\|x\|\|\pi_\Delta(x+y)\|-c_2\langle x,(\pi_\Delta(x+y))^3\rangle\\
\le&|c_1|^2\|x\|^2+\|\pi_\Delta(x+y)\|^2-c_2\langle x,(\pi_\Delta(x+y))^3\rangle\\
\le&|c_1|^2\|x\|^2+\|x+y\|^2-c_2\langle x,(\pi_\Delta(x+y))^3\rangle\\
\le&(2+|c_1|^2)\|x\|^2+2\|y\|^2-c_2\langle x,(\pi_\Delta(x+y))^3\rangle.
\end{aligned}$$

However, for the second term, we have

$$\begin{aligned}
&-c_2\langle x,(\pi_\Delta(x+y))^3\rangle=-c_2\Big\langle x,\Big(\frac{\mu^{-1}(h(\Delta))}{\|x+y\|}\cdot(x+y)\Big)^3\Big\rangle\\
=&-c_2\Big(\frac{\mu^{-1}(h(\Delta))}{\|x+y\|}\Big)^3\langle x,(x+y)^3\rangle\\
=&-c_2\Big(\frac{\mu^{-1}(h(\Delta))}{\|x+y\|}\Big)^3\|x\|^2\Big[\|x\|^2+3\langle x,y\rangle+3\|y\|^2\Big]-c_2\Big(\frac{\mu^{-1}(h(\Delta))}{\|x+y\|}\Big)^3\langle x,y^3\rangle\\
\le&-c_2\Big(\frac{\mu^{-1}(h(\Delta))}{\|x+y\|}\Big)^3\langle x,y^3\rangle\le c_2\Big|\frac{\mu^{-1}(h(\Delta))}{\|x+y\|}\Big|^3\|x\|\|y\|^3\\
\le& c_2\|x\|\|y\|^3\le|c_2|^2\|x\|^2+\|y\|^6.
\end{aligned}$$

Therefore, we obtain

$$\langle x,f_\Delta(x+y)\rangle\le(2+|c_1|^2+|c_2|^2)\|x\|^2+2(1+|y|^6)\le K_1(\|x\|^2+1+\|y\|^{K_1}).$$

This proof is therefore completed. □

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