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A Weak Galerkin Finite Element Method for a $H(\mathbf{curl})$-Elliptic Problem

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Abstract. In this paper, we develop and analyze a weak Galerkin (WG) finite element method for solving a $H(\mathbf{curl})$-elliptic problem. With the aid of the weak curl operator and a stabilizer term, we first design a WG discretization. Then, by using an auxiliary problem and establishing an error equation, we achieve the optimal order error estimates in both the energy norm and L^2 norm for the WG method. At last, we report some numerical experiments to confirm the theoretical results.

AMS subject classifications: 65N30, 65F10, 65N55

Key words: $H(\mathbf{curl})$-elliptic problem, weak Galerkin finite element method, weak curl operator, error estimate.

1 Introduction

Let Ω be a bounded Lipschitz polyhedron in $\mathbb{R}^3$ with boundary $\partial\Omega$ and unit outward normal vector $\boldsymbol{n}_{\partial\Omega}$. We consider to establish a weak Galerkin (WG) finite element discretization for the following $H(\mathbf{curl})$-elliptic problem

$$\nabla\times\alpha(\nabla\times\boldsymbol{u})+\beta\boldsymbol{u}=\boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\boldsymbol{n}_{\partial\Omega}\times\boldsymbol{u}=\mathbf{0} \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where the coefficients $\alpha>0$ and $\beta>0$ are material parameters and $\boldsymbol{f}\in(L^2(\Omega))^3$ can be regarded as the source function. For simplicity, in this paper, we suppose that $\alpha=1$ and $\beta>0$ is a positive constant. The boundary value problem (1.1a)-(1.1b) can be used to

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numerically simulate several classical electromagnetic field problems, such as the eddy current model [1], the time-domain electromagnetic field problem [8], etc.

In recent decades, a variety of numerical methods have been proposed for solving the electromagnetic field problems, such as $H(\mathbf{curl})$-conforming finite element method [1,11, 17], discontinuous Galerkin (DG) method [20, 21], and mixed finite element method [9, 12, 15] and so on.

The weak Galerkin (WG) finite element method, developed in recent years, is an effective numerical method for solving partial differential equations. It was first introduced by Wang and Ye [26] for second order elliptic problems. In that work, a weak function space was introduced, and the corresponding weak gradient operators and discrete weak gradient operators were defined to approximate the classical gradient operator. Since the boundary component of the weak function may not be necessarily related to the trace of the interior component, to guarantee the existence and uniqueness of WG numerical solution, the mesh partition and the approximation polynomial should have some restrictions. Later, a new class of WG method with a stabilized term was developed by Mu, Wang and Ye [18], which made it flexible for using polyhedra of arbitrary shape. In recent years, the WG method has received significant attention and has been successfully applied to solve many problems, for example, elliptic problems [13, 28, 31], biharmonic equation [4], Helmholtz equation [7, 10], Laplacian eigenvalue problem [30], Biot consolidation problem [3], parabolic equations [6,16,22], Darcy equations [14], electric interface model [5], poroelasticity problem [24], Cahn-Hilliard-Cook equation [2], etc.

There is little literature studying the WG method of Maxwell equations. For example, Mu, Wang, Ye and Zhang [19] gave a WG method for the time-harmonic Maxwell equations and established error estimates of optimal order in appropriate norms. Shields, Li and Machorro [23] adapted a WG discretization to time-dependent Maxwell equations and provided error analyses for semi-discrete and fully-discrete schemes. Wang [25] solved the time-harmonic Maxwell equations with complex coefficients by applying a WG method and presented the corresponding optimal order of convergence in various norms. Xie, Tang and Tang [29] approximate the solution of indefinite time-harmonic Maxwell equations by the WG method and give the error estimate. However, to the best of our knowledge, there is no published literature to discuss the WG method for $H(\mathbf{curl})$-elliptic problem.

In this paper, we develop a WG method for solving the $H(\mathbf{curl})$-elliptic problem (1.1a)-(1.1b). Inspired by [18] and [23], we prove the corresponding well-posedness and we also obtain the optimal error estimates of the weak Galerkin finite element approximation solution in both energy norm and L^2 norm by introducing an auxiliary problem and establish the corresponding error equation. In more detail, the contributions of this paper are,

(1). there is a lower order term in our WG discrete variational problem which is missed in [18] and [23] since the model problem we considered is different from the ones in [18] and [23];