Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 4 (2014), pp. 417-424

DOI: 10.4208/ata.2014.v30.n4.9

Nonconstant Harmonic Functions on the Level 3 Sierpinski Gasket

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Received 16 November 2014; Accepted (in revised version) 5 December 2014

Abstract. We give a detailed description of nonconstant harmonic functions on the level 3 Sierpinski gasket. Then we extend the method on β-set with $1/3<\beta<1/2$.

Key Words: Nonconstant harmonic function, level 3 Sierpinski gasket, β-set.

AMS Subject Classifications: 28A80

1 Introduction

Kigami (see [2–4]) used a constructive limit-of-difference-quotients method to define the Laplacian on p.c.f. self-similar sets. His theory can be used to compute values of junction points for harmonic functions.

A. Öberg, R. S. Strichartz and A. Q. Yingst [5] gave a detailed description of nonconstant harmonic functions on the Sierpinski gasket. It would be nice to give a detailed description of nonconstant harmonic functions on general p.c.f. self-similar sets, but it is not clear at present how to do this. Ultimately, the goal is to give details of nonconstant harmonic functions on wider classes of fractals. To advance these goals it is worthwhile to have an example different from the Sierpinski gasket worked out in detail.

The level 3 Sierpinski gasket K is generated by the i.f.s (see [1] for the definition) consisting of 6 mappings in the plane,

$$F_j x=\frac{1}{3}x+\frac{2}{3}p_j, \quad j=1,2,3,4,5,6,$$

where p_1, p_2, p_3 are vertices of an equilateral triangle, $p_4=(p_2+p_3)/2$, $p_5=(p_1+p_3)/2$, $p_6=(p_1+p_2)/2$ (see Fig. 1). Note that $V_0=\{p_1,p_2,p_3\}$, and the scaling factor on K is $7/15$.

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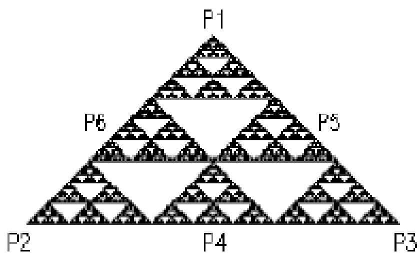

Figure 1: The construction of the level 3 Sierpinski gasket.

We approximate the fractal K by a sequence of graphs Γ_0, Γ_1, $\cdots$, with vertices $V_0 \subseteq V_1 \subseteq V_2 \cdots$, where $V_{k+1} = \cup_{j\in S} F_j V_k$. The edge relation for Γ_m, denoted $x \sim_m y$, for $x,y \in V_m$ and $x \neq y$, is defined by the existence of a word $w = (w_1,\cdots,w_m)$ with length $|w| = m$ such that $x,y \in F_w K$, where $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$. The simple energy form on Γ_m is

$$E_m(u,v) = \sum_{x\sim_m y} (u(x)-u(y))(v(x)-v(y)), \tag{1.1}$$

and the renormalization energy ε_m is given by

$$\varepsilon_m(u,v) = \Big(\frac{15}{7}\Big)^m E_m(u,v), \tag{1.2}$$

where u and v denote continuous functions on K and, by abuse of notation, their restriction to V_m.

We regard V_0 as the boundary of each graph V_m, and also of K. A function h on V_m (for $m \geq 1$) is called graph harmonic if it satisfies

$$h(x) = \begin{cases} \dfrac{1}{6}\sum\limits_{y\sim_m x} h(y) & \text{for } \sharp\{y : y \sim_m x\} = 6, \\ \dfrac{1}{4}\sum\limits_{y\sim_m x} h(y) & \text{for } \sharp\{y : y \sim_m x\} = 4, \end{cases} \tag{1.3}$$

for all non-boundary point x. It is easy to see this is equivalent to the property that h minimizes the energy $E_m(u,u)$ among all functions u with the same boundary values.

The following contents can be found in Kigami [2–4] and Strichartz [6].

For any continuous function u on K, the sequence $\varepsilon_m(u,u)$ is monotone increasing, so

$$\varepsilon(u,u) = \lim_{m\to\infty} \varepsilon_m(u,u) \tag{1.4}$$

is well-defined in $[0,\infty]$, and $\varepsilon(u,u) = 0$ if and only if u is constant. Let $dom(\varepsilon)$ denote the set of continuous functions for which $\varepsilon(u,u) < \infty$. Then $dom(\varepsilon)$ modulo constants is a Hilbert space with inner product

$$\varepsilon(u,v) = \lim_{m\to\infty} \varepsilon_m(u,v). \tag{1.5}$$