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A Two-Stage Fourth-Order Gas-Kinetic CPR Method for Subsonic Flows on Hexahedral Meshes

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Abstract. A compact high-order gas-kinetic scheme (GKS) is developed for three-dimensional subsonic inviscid and viscous flows on hexahedral meshes, which achieves fourth-order accuracy in both space and time. The scheme combines a compact and efficient correction procedure via reconstruction (CPR) framework with a time-evolving gas-kinetic flux, in which the inviscid and viscous fluxes are coupled and computed uniformly. With the CPR framework, the current scheme avoids the difficulty of compact fourth-order reconstruction encountered by the traditional finite volume GKS. Moreover, both the flux and its time-derivative are available in the gas-kinetic flux so that an efficient two-stage temporal discretization can be adopted to achieve fourth-order time accuracy, which is more efficient than the traditional Runge-Kutta CPR method. In addition, with the help of isoparametric transformation, the current scheme can treat curved boundaries with high-order curved meshes. Typical numerical tests demonstrate the good performance of the current scheme.

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Key words: Gas-kinetic scheme, correction procedure via reconstruction, two-stage fourth-order temporal discretization, curved mesh.

1 Introduction

In recent years, high-order methods on unstructured meshes have attracted increasing attention in the community of computational fluid dynamics to accurately capture the

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aerodynamic forces and heating on the flying objects with complex geometries [1]. With the same computational cost, high-order methods can achieve higher accuracy and efficiency than second-order methods. A variety of high-order methods have been developed during the past decades, such as the Discontinuous Galerkin (DG) method [2–4], the spectral difference (SD) method [5, 6] and the spectral volume (SV) method [7, 8]. These mentioned methods are characterized by compactness with inner degrees of freedom (IDOFs), when compared with the finite volume (FV) method [9, 10].

In 2007, Huynh proposed a simple and efficient method, named as the flux reconstruction (FR) method [11, 12]. Wang, Gao and Haga et al. extended it to triangular and mixed grids with the name of lifting collocation penalty (LCP) method [13–16]. FR and LCP were also renamed as correction procedure via reconstruction (CPR) [17]. It can recover many typical high-order methods, including DG, SD and SV, by choosing corresponding correction functions, with the advantage of simplicity and efficiency. Achieving high-order accuracy not only depends on high-order discretization of space, but also high-order time-evolution. Riemann solvers for Euler equations are widely used to compute the inviscid flux, while the viscous flux needs to be treated separately. As these traditional flux solvers are usually only first-order time accurate, the multi-stage Runge-Kutta (R-K) methods are usually adopted for high-order time accuracy, except for the ADER solver [18, 19] and the generalized Riemann problem (GRP) solver.

The Bhatnagar-Gross-Krook (BGK) model provides a mesoscopic description of flows through the gas distribution function, which can recover the Naiver-Stokes (N-S) equations by the first-order Chapman-Enskog expansion. Based on the BGK equation, the gas-kinetic scheme (GKS) has been developed [20, 21] and applied in a variety of flows. Its success is attributed to the adoption of the local integral solution of the BGK equation to compute the macroscopic flux, which describes a multiscale evolution from a kinetic scale to a hydrodynamic scale [22]. Thus the flux uniformly couples the inviscid and viscous effects and also contains an adaptive numerical dissipation. More importantly, the flux function is explicitly evolving in both space and time, which guarantees the genuine multidimensionality and makes it possible to use less quadrature points or stages for high-order accuracy [23].

Through a second-order Taylor expansion of the gas distribution function, Li et al. proposed a third-order GKS successfully, which achieves third-order time accuracy within a single stage [24–26]. This method has been extended to a series of framework, such as the compact FV-GKS [27], DG-GKS [28], SV-GKS [29] and CPR-GKS [30]. To achieve high-order time accuracy more efficiently, professor Jiequan Li developed a two-stage fourth-order (S2O4) temporal discretization for the GRP solver with the use of both the flux and its time-derivative [31–33]. It has been extended to the DG framework, which shows higher efficiency than RKDG [34, 35]. It has also been adopted to develop a two-stage fourth-order FV GKS based on the second-order GKS flux [36–39], as the time-derivative is also available. For three-dimensional flows, a compact third-order gas-kinetic scheme has also been developed recently by using the S2O4 time-stepping method [40]. However, for FV GKS, high-order compact reconstruction is still a chal-