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A Locking-Free Weak Galerkin Finite Element Method for Linear Elasticity Problems Based on a Reconstruction Operator

Fuchang Huo[1], Ruishu Wang[1,*], Yanqiu Wang[2,3] and Ran Zhang[1]

[1] *School of Mathematics, Jilin University, Changchun 130012, Jilin, China.*
[2] *School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, Jiangsu, China.*
[3] *Jiangsu Key Laboratory for NSLSCS, Nanjing Normal University, Nanjing 210023, Jiangsu, China.*

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Abstract. The weak Galerkin (WG) finite element method has shown great potential in solving various type of partial differential equations. In this paper, we propose an arbitrary order locking-free WG method for solving linear elasticity problems, with the help of an appropriate $H(div)$-conforming displacement reconstruction operator. Optimal order locking-free error estimates in both the H^1-norm and the L^2-norm are proved, i.e., the error is independent of the *Lamé* constant λ. Moreover, the term $\lambda\|\nabla\cdot\mathbf{u}\|_k$ does not need to be bounded in order to achieve these estimates. We validate the accuracy and the robustness of the proposed locking-free WG algorithm by numerical experiments.

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Key words: Weak Galerkin finite element methods, linear elasticity problems, $H(div)$-conforming displacement reconstruction, locking-free.

1 Introduction

In this paper, we consider the linear elasticity problems as follows: Find displacement vector $\mathbf{u}$ satisfying

$$-\nabla\cdot\sigma(\mathbf{u})=\mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{u}=\widehat{\mathbf{g}}, \quad \text{on } \Gamma_D, \tag{1.2}$$

$$\sigma(\mathbf{u})\mathbf{n}=\widehat{\mathbf{t}}, \quad \text{on } \Gamma_N, \tag{1.3}$$

*Corresponding author. *Email addresses:* `huofc22@mails.jlu.edu.cn` (F. Huo), `wangrs_math@jlu.edu.cn` (R. Wang), `yqwang@njnu.edu.cn` (Y. Wang), `zhangran@jlu.edu.cn` (R. Zhang)

where Ω is an open bounded, connected domain in $\mathbb{R}^d$ $(d=2,3)$, and the boundary $\Gamma=\partial\Omega$ is Lipschitz continuous. $\mathbf{f}$ is the body force, $\widehat{\mathbf{g}}$ is the boundary displacement function, $\widehat{\mathbf{t}}$ is the traction force, $\mathbf{n}$ is the unit outward normal direction on boundary Γ. Here Γ_D and Γ_N are two subsets of Γ, satisfying $|\Gamma_D|>0$, $\Gamma_D\bigcap\Gamma_N=\emptyset$, and $\Gamma_D\bigcup\Gamma_N=\Gamma$. In addition, $\sigma(\mathbf{u})$ is the Cauchy stress tensor given by

$$\sigma(\mathbf{u})=2\mu\varepsilon(\mathbf{u})+\lambda(\nabla\cdot\mathbf{u})\mathbf{I},$$

where $\varepsilon(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$ is the strain tensor, μ and λ are *Lamé* constants, satisfying $0<\mu_1\le\mu\le\mu_2\ll\infty$ and $0<\lambda<\infty$.

The weak formulation of (1.1)-(1.3) can be written as: Find $\mathbf{u}\in[H^1(\Omega)]^d$ satisfying $\mathbf{u}=\widehat{\mathbf{g}}$ on Γ_D and

$$2\mu(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))+\lambda(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v})+\langle\widehat{\mathbf{t}},\mathbf{v}\rangle_{\Gamma_N},\quad\forall\mathbf{v}\in[H_D^1(\Omega)]^d,\tag{1.4}$$

where $H^1(\Omega)$ and $H_D^1(\Omega)$ are the standard Sobolev spaces defined as follows:

$$\begin{aligned}&H^1(\Omega)=\{v\in L^2(\Omega):\ \nabla v\in[L^2(\Omega)]^d\},\\&H_D^1(\Omega)=\{v\in H^1(\Omega):\ v|_{\Gamma_D}=0\}.\end{aligned}$$

In elasticity theory, it is known that the "locking" phenomenon [1,3,8] arises when the *Lamé* constant λ approaches infinity. Conventional finite element scheme often fails to converge to the exact solution or does not reach optimal convergence in such cases. This phenomenon is primarily attributed to the dependence of the finite element error estimates on the *Lamé* constant λ. Consequently, the coefficients of the error estimates tend towards infinity when $\lambda\to\infty$, significantly impacting the computational accuracy and efficiency of the finite element scheme. In order to overcome the locking phenomenon, some effective techniques have been proposed in various discretizations, such as, the mixed finite element method [19], the nonconforming finite element method [11], the discontinuous Galerkin method [13,32], the virtual element method [2,12], and the weak Galerkin (WG) finite element method [25,33], etc. Among them, methods based on the primal formulation generally require regularity assumptions.

The main purpose of this paper is to propose an arbitrary order locking-free WG method for linear elasticity problems (1.1)-(1.3). The WG method is an extension of the classical Galerkin finite element method. It employs weak functions and introduces weak differential operators to replace the traditional differential operators. A stabilizer is added to ensure the weak continuity of the numerical solution. Comparing to the standard finite element method, it is usually much more convenient to design and implement high-order WG schemes. The WG method was first proposed by J. Wang and X. Ye for solving second order elliptic problems [26], and later applied to Navier-Stokes equations [34], Brinkman equations [20], Maxwell's equations [22], biharmonic equations [21,35], eigenvalue problems [6], Stokes-Darcy problems [7,24], lower regularity problems [31], convective Brinkman-Forchheimer equations [30], wave equation [9], etc.

The WG method has been used to solve linear elasticity problems based on primal formulation. [18, 25] presented locking-free WG methods equivalent to mixed formulations, [14,33] proposed locking-free and lowest-order WG approaches using local Raviart-Thomas spaces to approximate the gradient of the displacement. In recent research, [29] constructed a penalty-free WG method on quadrilateral meshes. However, it should be noted that this method relies on utilizing high-order regularity assumptions. [15] developed a robust lowest-order WG approach for the grad-div formulation.

In this paper, we focus on developing locking-free WG finite element schemes for the stain-div formulation of linear elasticity problems with mixed boundary conditions. Motivated by the pressure-robust methods [16, 17] for solving Stokes problems, we employ the WG method combined with the $H(div)$-conforming displacement reconstruction technique to solve linear elasticity problems (1.1)-(1.3). In order to address the issue of locking, we modify the WG test functions by utilizing an $H(div)$-conforming displacement reconstruction operator. The error estimates are independent of the parameter λ, i.e., the scheme is locking-free. Compared with the existing WG schemes for linear elasticity problems, our method neither requires transformation into a displacement-pressure mixed formulation nor high-order regularity assumptions. Furthermore, the scheme is simple and capable of achieving high-order approximation accuracy.

The paper is structured as follows. In Section 2, the WG finite element scheme of linear elasticity problems (1.1)-(1.3) is presented. In Section 3, we study the property of the $H(div)$-conforming displacement reconstruction operator. In Section 4, error equation and error estimates are established. In Section 5, we present numerical results to demonstrate the validity of theoretical analysis.

2 The WG finite element scheme

In this section, we present the WG scheme for (1.1)-(1.3) and study the well-posedness of the WG scheme.

Let $\mathcal{T}_h$ be a shape regular simplicial partition [27] of the domain $\Omega\subset\mathbb{R}^d\,(d=2,3)$. For each $T\in\mathcal{T}_h$, the diameter of T is denoted by h_T. The mesh size of the partition $\mathcal{T}_h$ is defined by $h=\max_{T\in\mathcal{T}_h}h_T$. Additionally, $\mathcal{E}_h$ represents the set of all edges or faces in the partition $\mathcal{T}_h$, while $\mathcal{E}_h^0$ denotes the set of all interior edges or faces in $\mathcal{T}_h$. Let $k\geq 1$ be an arbitrary positive integer, and $P_k(T)$ refers to the set of polynomials defined on T with a maximum degree of k.

For any integer $k\geq 1$, define the weak finite element space V_h as

$$V_h=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\ \mathbf{v}_0|_T\in[P_k(T)]^d,\ \mathbf{v}_b|_e\in[P_k(e)]^d,\ T\in\mathcal{T}_h,\ e\in\mathcal{E}_h\}.$$

A subspace of V_h is defined as

$$V_h^0=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h:\ \mathbf{v}_b=\mathbf{0}\text{ on }\Gamma_D\}.$$

Define the weak differential operators and the $H(div)$-conforming displacement reconstruction operator as follows.

Definition 2.1. [26, 28] For $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h$, define $\nabla_w\mathbf{v}$ and $\nabla_w\cdot\mathbf{v}$ on each $T\in\mathcal{T}_h$, respectively, to be the unique matrix-valued polynomial in $[P_{k-1}(T)]^{d\times d}$ and the unique polynomial in $P_k(T)$ satisfying

$$(\nabla_w\mathbf{v},\varphi)_T=-(\mathbf{v}_0,\nabla\cdot\varphi)_T+\langle\mathbf{v}_b,\varphi\mathbf{n}\rangle_{\partial T},\quad \forall\varphi\in[P_{k-1}(T)]^{d\times d}, \tag{2.1}$$

$$(\nabla_w\cdot\mathbf{v},\phi)_T=-(\mathbf{v}_0,\nabla\phi)_T+\langle\mathbf{v}_b\cdot\mathbf{n},\phi\rangle_{\partial T},\quad \forall\phi\in P_k(T), \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal direction on ∂T.

Based on the definitions of the weak gradient operator and weak divergence operator, we define the weak strain tensor $\varepsilon_w(\mathbf{u})$ and the weak stress tensor $\sigma_w(\mathbf{u})$ as follows

$$\varepsilon_w(\mathbf{u})=\frac{1}{2}\left(\nabla_w\mathbf{u}+(\nabla_w\mathbf{u})^T\right), \tag{2.3}$$

$$\sigma_w(\mathbf{u})=2\mu\varepsilon_w(\mathbf{u})+\lambda(\nabla_w\cdot\mathbf{u})\mathbf{I}. \tag{2.4}$$

Denote by $RT_k(T)$ the Raviart–Thomas space on $T\in\mathcal{T}_h$ by

$$RT_k(T)=[P_k(T)]^d+\mathbf{x}P_k(T),\quad d=2,3.$$

Definition 2.2. [20, 23] Define the displacement reconstruction operator $\boldsymbol{\pi}_h^{RT}:V_h\to H(div;\Omega):=\{\mathbf{w}\in[L^2(\Omega)]^d:\nabla\cdot\mathbf{w}\in L^2(\Omega)\}$, such that $\boldsymbol{\pi}_h^{RT}(\mathbf{v})|_T\in RT_k(T)$ for all $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h$ and $T\in\mathcal{T}_h$ by

$$\int_T\boldsymbol{\pi}_h^{RT}(\mathbf{v})\cdot\mathbf{w}dT=\int_T\mathbf{v}_0\cdot\mathbf{w}dT,\quad \forall\mathbf{w}\in[P_{k-1}(T)]^d, \tag{2.5}$$

$$\int_e\boldsymbol{\pi}_h^{RT}(\mathbf{v})\cdot\mathbf{n}\phi ds=\int_e\mathbf{v}_b\cdot\mathbf{n}\phi ds,\quad \forall\phi\in P_k(e),e\subset\partial T, \tag{2.6}$$

where $\mathbf{n}$ is the unit outward normal direction on ∂T.

It is clear that one has $\nabla\cdot\boldsymbol{\pi}_h^{RT}(\mathbf{v})\in P_k(T)$ for all $T\in\mathcal{T}_h$. From the property of Raviart–Thomas elements, we know that $\boldsymbol{\pi}_h^{RT}(\mathbf{v})\cdot\mathbf{n}|_e\in P_k(e)$ for all $e\in\mathcal{E}_h$. Hence (2.6) immediately implies that

$$\boldsymbol{\pi}_h^{RT}(\mathbf{v})\cdot\mathbf{n}=\mathbf{v}_b\cdot\mathbf{n},\quad \forall e\in\mathcal{E}_h. \tag{2.7}$$

For $\mathbf{w},\mathbf{v}\in V_h$, we introduce some bilinear forms as follows:

$$\mathcal{S}_h(\mathbf{w},\mathbf{v})=\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle\mathbf{w}_0-\mathbf{w}_b,\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}, \tag{2.8}$$

$$\mathcal{A}_h(\mathbf{w},\mathbf{v})=2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(\mathbf{w}),\varepsilon_w(\mathbf{v}))_T+\lambda\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\mathbf{w},\nabla_w\cdot\mathbf{v})_T, \tag{2.9}$$

$$\mathcal{A}_h^s(\mathbf{w},\mathbf{v})=\mathcal{A}_h(\mathbf{w},\mathbf{v})+\mathcal{S}_h(\mathbf{w},\mathbf{v}). \tag{2.10}$$

For each edge or face $e\in\mathcal{E}_h$, denote by Q_b the L^2 orthogonal projection onto $[P_k(e)]^d$.

Now, we propose the following robust WG scheme.

Algorithm 1 Robust WG Algorithm

A new WG scheme for (1.1)-(1.3) is given by seeking $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h$ with $\mathbf{u}_b=Q_b\widehat{\mathbf{g}}$ on Γ_D such that

$$\mathcal{A}_h^s(\mathbf{u}_h,\mathbf{v})=(\mathbf{f},\boldsymbol{\pi}_h^{RT}(\mathbf{v}))+\langle\widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N},\quad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0. \tag{2.11}$$

Theorem 2.1. *The WG scheme* (2.11) *has a unique solution.*

Proof. For a finite dimensional linear equation, we just need to prove the uniqueness of the solution, and the existence follows.

Let $\mathbf{u}_h^{(j)}=\{\mathbf{u}_0^{(j)},\mathbf{u}_b^{(j)}\}\in V_h$, $j=1,2$ be two solutions of (2.11), we have $\mathbf{u}_b^{(j)}=Q_b\widehat{\mathbf{g}}$ on Γ_D and

$$\mathcal{A}_h^s(\mathbf{u}_h^{(j)},\mathbf{v})=(\mathbf{f},\boldsymbol{\pi}_h^{RT}(\mathbf{v}))+\langle\widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N},\quad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0,\quad j=1,2.$$

Let $\mathbf{w}=\mathbf{u}_h^{(1)}-\mathbf{u}_h^{(2)}$ be the difference between two solutions, then we obtain that $\mathbf{w}\in V_h^0$ and

$$\mathcal{A}_h^s(\mathbf{w},\mathbf{v})=0,\quad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0. \tag{2.12}$$

By choosing $\mathbf{v}=\mathbf{w}$ in (2.12), we arrive at

$$\mathcal{A}_h^s(\mathbf{w},\mathbf{w})=0.$$

From the definition of $\mathcal{A}_h^s(\cdot,\cdot)$, we get

$$2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(\mathbf{w}),\varepsilon_w(\mathbf{w}))_T+\lambda\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\mathbf{w},\nabla_w\cdot\mathbf{w})_T+\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle\mathbf{w}_0-\mathbf{w}_b,\mathbf{w}_0-\mathbf{w}_b\rangle_{\partial T}=0,$$

which leads to

$$\varepsilon_w(\mathbf{w})=\mathbf{0},\quad \text{in } T, \tag{2.13}$$
$$\nabla_w\cdot\mathbf{w}=0,\quad \text{in } T, \tag{2.14}$$
$$\mathbf{w}_0-\mathbf{w}_b=\mathbf{0},\quad \text{on } \partial T. \tag{2.15}$$

Using the definition of the discrete weak gradient (2.1), we have

$$\begin{aligned}0=(\varepsilon_w(\mathbf{w}),\varphi)_T&=(\nabla_w\mathbf{w},\frac{1}{2}(\varphi+\varphi^T))_T\\&=(\nabla\mathbf{w}_0,\frac{1}{2}(\varphi+\varphi^T))_T-\langle\mathbf{w}_0-\mathbf{w}_b,\frac{1}{2}(\varphi+\varphi^T)\mathbf{n}\rangle_{\partial T}\\&=(\varepsilon(\mathbf{w}_0),\varphi)_T-\langle\mathbf{w}_0-\mathbf{w}_b,\frac{1}{2}(\varphi+\varphi^T)\mathbf{n}\rangle_{\partial T},\quad \forall\varphi\in[P_{k-1}(T)]^{d\times d}.\end{aligned}$$

Letting $\varphi=\varepsilon(\mathbf{w}_0)$ in the above equation, together with (2.15), we obtain $\varepsilon(\mathbf{w}_0)=\mathbf{0}$ on each element T. This implies that $\mathbf{w}_0|_T\in RM$, where RM is the space of rigid body motions. Since $\mathbf{w}_0=\mathbf{w}_b$ on ∂T, we have that $\mathbf{w}_0$ is continuous in Ω. It follows that $\mathbf{w}_0|_\Omega\in RM$. By using $\mathbf{w}_0=\mathbf{w}_b$ on ∂T and $\mathbf{w}_b=\mathbf{0}$ on Γ_D, we know that $\mathbf{w}_0|_{\Gamma_D}=\mathbf{0}$. Thus, according to the second Korn's inequality [7,25], we get $\mathbf{w}_0\equiv\mathbf{0}$ in Ω. Hence, $\mathbf{w}_b\equiv\mathbf{0}$ in Ω. This completes the proof. $\square$

3 Properties of bilinear forms and the reconstruction operator

In this section, we investigate some properties of bilinear forms and the $H(div)$-conforming displacement reconstruction operator.

Define

$$|||\mathbf{v}|||=\left(\sum_{T\in\mathcal{T}_h}\|\varepsilon_w(\mathbf{v})\|_T^2+h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\right)^{\frac{1}{2}},\quad\forall\mathbf{v}\in V_h. \tag{3.1}$$

Similar to the proof of Theorem 2.1, it is not hard to see that $|||\cdot|||$ satisfies the following properties.

Lemma 3.1. $|||\cdot|||$ *is a norm in the space* V_h^0.

Lemma 3.2. *Let* $\delta=\min\{2\mu,1\}>0$, *there holds*

$$\delta|||\mathbf{v}|||^2\le\mathcal{A}_h^s(\mathbf{v},\mathbf{v}),\quad\forall\mathbf{v}\in V_h^0. \tag{3.2}$$

For $T\in\mathcal{T}_h$, let Q_0 be the L^2 projection onto $[P_k(T)]^d$, $\mathcal{P}_h$ be the L^2 projection onto $P_k(T)$, and $\mathbf{\Pi}_h$ be the L^2 projection onto $[P_{k-1}(T)]^{d\times d}$. Recall that for $e\in\mathcal{E}_h$, Q_b is the L^2 projection onto $[P_k(e)]^d$. Combining Q_0 and Q_b, we define $Q_h\mathbf{u}=\{Q_0\mathbf{u},Q_b\mathbf{u}\}\in V_h$.

Lemma 3.3. [25] *For any* $\mathbf{v}\in[H^1(\Omega)]^d$, *we have*

$$\nabla_w\cdot(Q_h\mathbf{v})=\mathcal{P}_h(\nabla\cdot\mathbf{v}), \tag{3.3}$$
$$\nabla_w(Q_h\mathbf{v})=\mathbf{\Pi}_h(\nabla\mathbf{v}). \tag{3.4}$$

What's more, we obtain

$$\varepsilon_w(Q_h\mathbf{v})=\mathbf{\Pi}_h(\varepsilon(\mathbf{v})). \tag{3.5}$$

Proof. For all $\varphi\in P_k(T)$, from (2.2) and the integration by parts, we have

$$\begin{aligned}(\nabla_w\cdot(Q_h\mathbf{v}),\varphi)_T&=-(Q_0\mathbf{v},\nabla\varphi)_T+\langle Q_b\mathbf{v}\cdot\mathbf{n},\varphi\rangle_{\partial T}\\&=-(\mathbf{v},\nabla\varphi)_T+\langle\mathbf{v}\cdot\mathbf{n},\varphi\rangle_{\partial T}\\&=(\nabla\cdot\mathbf{v},\varphi)_T\\&=(\mathcal{P}_h(\nabla\cdot\mathbf{v}),\varphi)_T,\end{aligned}$$

which leads to (3.3).

For any $\tau \in [P_{k-1}(T)]^{d\times d}$, by using (2.1) and the integration by parts, we obtain

$$\begin{aligned}(\nabla_w(Q_h\mathbf{v}),\tau)_T &= -(Q_0\mathbf{v},\nabla\cdot\tau)_T + \langle Q_b\mathbf{v},\tau\mathbf{n}\rangle_{\partial T}\\ &= -(\mathbf{v},\nabla\cdot\tau)_T + \langle \mathbf{v},\tau\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\mathbf{v},\tau)_T\\ &= (\mathbf{\Pi}_h(\nabla\mathbf{v}),\tau)_T,\end{aligned}$$

which proves (3.4).

Similarly, for all $\tau \in [P_{k-1}(T)]^{d\times d}$, according to (2.1) and (3.4), we get

$$(\varepsilon_w(Q_h\mathbf{v}),\tau)_T = (\nabla_w(Q_h\mathbf{v}),\frac{1}{2}(\tau+\tau^T))_T = (\mathbf{\Pi}_h(\varepsilon(\mathbf{v})),\tau)_T,$$

which gives (3.5). □

Now, we present some results of the $H(div)$-conforming displacement reconstruction operator.

Lemma 3.4. [5,10] *For any $\boldsymbol{\omega} \in RT_k(T)$, there hold*

$$\nabla\cdot\boldsymbol{\omega} \in P_k(T), \tag{3.6}$$

$$(\boldsymbol{\omega}\cdot\mathbf{n})|_{\partial T} \in P_k(\partial T), \tag{3.7}$$

$$\|\boldsymbol{\omega}\|_T \le \left(\|Q_0^{k-1}\boldsymbol{\omega}\|_T^2 + \sum_{e\subset\partial T} h_T\|\boldsymbol{\omega}\cdot\mathbf{n}\|_e^2\right)^{\frac{1}{2}}, \tag{3.8}$$

where

$$P_k(\partial T) = \left\{\phi\in L^2(\partial T):\ \phi|_e \in P_k(e),\ \forall e\subset\partial T\right\},$$

and Q_0^{k-1} denotes the L^2 projection onto $[P_{k-1}(T)]^d$, $\mathbf{n}$ is the unit outward normal direction on ∂T.

Proof. The proof of (3.6)-(3.7) can be found in Proposition 3.2 from [5, Chapter 3]. The proof of (3.8) can be found in Proposition 12.5 and Example 12.6 from [10]. □

Lemma 3.5. *For all $\mathbf{v} = \{\mathbf{v}_0,\mathbf{v}_b\} \in V_h$, there holds*

$$\nabla\cdot(\boldsymbol{\pi}_h^{RT}(\mathbf{v})) = \nabla_w\cdot\mathbf{v}. \tag{3.9}$$

Moreover, we have

$$\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\pi}_h^{RT}(\mathbf{v})-\mathbf{v}_0\|_T^2 \le C\sum_{T\in\mathcal{T}_h} h_T\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2 \le Ch^2|||\mathbf{v}|||^2. \tag{3.10}$$

Proof. From the definition of the weak divergence (2.2) and the definition of the reconstruction operator π_h^{RT} in (2.5)-(2.6), we obtain

$$\begin{aligned}(\nabla\cdot(\pi_h^{RT}(\mathbf{v})),q)_T &= -(\pi_h^{RT}(\mathbf{v}),\nabla q)_T + \langle \pi_h^{RT}(\mathbf{v})\cdot\mathbf{n},q\rangle_{\partial T}\\ &= -(\mathbf{v}_0,\nabla q)_T + \langle \mathbf{v}_b\cdot\mathbf{n},q\rangle_{\partial T}\\ &= (\nabla_w\cdot\mathbf{v},q)_T,\end{aligned}$$

for all $q\in P_k(T)$. Using Lemma 3.4, ones gets (3.9).

Using (3.8) and noticing that $\mathbf{v}_0|_T\in[P_k(T)]^d\subset RT_k(T)$, we have

$$\|\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})\|_T^2 \le \|Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\|_T^2 + \sum_{e\subset\partial T} h_T\|(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\cdot\mathbf{n}\|_e^2. \tag{3.11}$$

Next, we estimate the two terms in the right-hand side of (3.11) one by one. According to definition of π_h^{RT} in (2.5)-(2.6), we have

$$\begin{aligned}&\|Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\|_T^2\\ =&(Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})),Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})))_T\\ =&(Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})),\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))_T\\ =&(Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})),\mathbf{v}_0)_T-(Q_0^{k-1}(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})),\mathbf{v}_0)_T\\ =&0.\end{aligned} \tag{3.12}$$

From the property of $RT_k(T)$ in (3.7) and the definition of π_h^{RT} in (2.5)-(2.6), we get

$$\begin{aligned}&\|(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\cdot\mathbf{n}\|_e^2\\ =&\langle(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\cdot\mathbf{n},(\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\cdot\mathbf{n}\rangle_e\\ =&\langle(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n},(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\rangle_e\\ =&\|(\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\|_e^2.\end{aligned} \tag{3.13}$$

Substituting (3.12)-(3.13) into (3.11), we obtain

$$\sum_{T\in\mathcal{T}_h}\|\pi_h^{RT}(\mathbf{v})-\mathbf{v}_0\|_T^2 \le C\sum_{T\in\mathcal{T}_h} h_T\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2 \le Ch^2|||\mathbf{v}|||^2.$$

This completes the proof of the lemma. □

4 Error estimate

In this section, we establish the error equation and study the convergence rate of the WG scheme (2.11).

Let $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h$ be the discrete solution to the WG scheme (2.11) and $\mathbf{u}\in[H^2(\Omega)]^d$ be the exact solution to (1.1). Define the error function $\mathbf{e}_h$ as follows:

$$\mathbf{e}_h=\{\mathbf{e}_0,\mathbf{e}_b\}=\{Q_0\mathbf{u}-\mathbf{u}_0,Q_b\mathbf{u}-\mathbf{u}_b\}. \tag{4.1}$$

It is clear that $\mathbf{e}_h\in V_h^0$.

4.1 Error equation

The goal of this section is to construct the error equation between the discrete solution $\mathbf{u}_h$ and the exact solution $\mathbf{u}$.

Lemma 4.1. *Let* $\mathbf{u}\in[H^2(\Omega)]^d$ *be the exact solution to* (1.1)*, there holds*

$$\mathcal{A}_h^s(\mathbf{e}_h,\mathbf{v})=\Theta_{\mathbf{u}}(\mathbf{v}),\quad \forall \mathbf{v}\in V_h^0, \tag{4.2}$$

where

$$\Theta_{\mathbf{u}}(\mathbf{v})=\mathcal{G}_{\mathbf{u}}(\mathbf{v})-\mathcal{K}_{\mathbf{u}}(\mathbf{v})+\mathcal{S}_h(Q_h\mathbf{u},\mathbf{v}), \tag{4.3}$$

$$\mathcal{G}_{\mathbf{u}}(\mathbf{v})=2\mu\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b,(\varepsilon(\mathbf{u})-\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}, \tag{4.4}$$

$$\mathcal{K}_{\mathbf{u}}(\mathbf{v})=2\mu\sum_{T\in\mathcal{T}_h}(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0-\boldsymbol{\pi}_h^{RT}(\mathbf{v}))_T. \tag{4.5}$$

Proof. According to the property of $\mathbf{\Pi}_h$ in (3.5), the definition of weak gradient (2.1), and the integration by parts, we obtain

$$\begin{aligned}
&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(Q_h\mathbf{u}),\varepsilon_w(\mathbf{v}))_T\\
=&2\mu\sum_{T\in\mathcal{T}_h}(\mathbf{\Pi}_h\varepsilon(\mathbf{u}),\varepsilon_w(\mathbf{v}))_T\\
=&-2\mu\sum_{T\in\mathcal{T}_h}(\mathbf{v}_0,\nabla\cdot(\mathbf{\Pi}_h\varepsilon(\mathbf{u})))_T+2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_b,(\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\\
=&2\mu\sum_{T\in\mathcal{T}_h}(\nabla\mathbf{v}_0,\mathbf{\Pi}_h\varepsilon(\mathbf{u}))_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\\
=&2\mu\sum_{T\in\mathcal{T}_h}(\nabla\mathbf{v}_0,\varepsilon(\mathbf{u}))_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}.
\end{aligned} \tag{4.6}$$

Next, from (3.3), the property of reconstruction operator $\boldsymbol{\pi}_h^{RT}$ in (3.9), the definition of $\mathcal{P}_h$, we can deduce the following result:

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot(Q_h\mathbf{u}),\nabla_w\cdot\mathbf{v})_T\\
=&\sum_{T\in\mathcal{T}_h}(\mathcal{P}_h(\nabla\cdot\mathbf{u}),\nabla_w\cdot\mathbf{v})_T\\
=&\sum_{T\in\mathcal{T}_h}(\mathcal{P}_h(\nabla\cdot\mathbf{u}),\nabla\cdot\boldsymbol{\pi}_h^{RT}(\mathbf{v}))_T\\
=&\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{u},\nabla\cdot\boldsymbol{\pi}_h^{RT}(\mathbf{v}))_T.
\end{aligned} \tag{4.7}$$

Testing (1.1) by $\pi_h^{RT}(\mathbf{v})$ to get

$$\begin{aligned}(\mathbf{f},\pi_h^{RT}(\mathbf{v}))&=-(\nabla\cdot\sigma(\mathbf{u}),\pi_h^{RT}(\mathbf{v}))\\&=-2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\pi_h^{RT}(\mathbf{v}))-\lambda(\nabla(\nabla\cdot\mathbf{u}),\pi_h^{RT}(\mathbf{v}))\\&=-2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0)+2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))-\lambda(\nabla(\nabla\cdot\mathbf{u}),\pi_h^{RT}(\mathbf{v})),\end{aligned} \tag{4.8}$$

where we have used the fact that $\nabla\cdot((\nabla\cdot\mathbf{u})\mathbf{I})=\nabla(\nabla\cdot\mathbf{u})$. Applying the integration by parts and (2.7), we know

$$-2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0)=2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon(\mathbf{u}),\nabla\mathbf{v}_0)_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0,(\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}, \tag{4.9}$$

and

$$\begin{aligned}-\lambda(\nabla(\nabla\cdot\mathbf{u}),\pi_h^{RT}(\mathbf{v}))&=\lambda\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{u},\nabla\cdot\pi_h^{RT}(\mathbf{v}))_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\nabla\cdot\mathbf{u},\pi_h^{RT}(\mathbf{v})\cdot\mathbf{n}\rangle_{\partial T}\\&=\lambda\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{u},\nabla\cdot\pi_h^{RT}(\mathbf{v}))_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\nabla\cdot\mathbf{u},\mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{4.10}$$

By substituting (4.6), (4.7), (4.9), and (4.10) into (4.8), we have

$$\begin{aligned}&(\mathbf{f},\pi_h^{RT}(\mathbf{v}))\\=&2\mu\sum_{T\in\mathcal{T}_h}\Big((\varepsilon_w(Q_h\mathbf{u}),\varepsilon_w(\mathbf{v}))_T+\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}-\langle\mathbf{v}_0,(\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\Big)\\&+\lambda\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot(Q_h\mathbf{u}),\nabla_w\cdot\mathbf{v})_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\nabla\cdot\mathbf{u},\mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}+2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0-\pi_h^{RT}(\mathbf{v})).\end{aligned}$$

Note that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\langle\sigma(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial T}&=2\mu\sum_{T\in\mathcal{T}_h}\langle\varepsilon(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial T}+\lambda\sum_{T\in\mathcal{T}_h}\langle((\nabla\cdot\mathbf{u})\mathbf{I})\mathbf{n},\mathbf{v}_b\rangle_{\partial T}\\&=2\mu\sum_{T\in\mathcal{T}_h}\langle\varepsilon(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial T}+\lambda\sum_{T\in\mathcal{T}_h}\langle\nabla\cdot\mathbf{u},\mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{4.11}$$

With the fact that $\sum_{T\in\mathcal{T}_h}\langle\sigma(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial T}=\langle\widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N}$, we get

$$\begin{aligned}&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(Q_h\mathbf{u}),\varepsilon_w(\mathbf{v}))_T+\lambda\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot(Q_h\mathbf{u}),\nabla_w\cdot\mathbf{v})_T\\=&(\mathbf{f},\pi_h^{RT}(\mathbf{v}))-2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))+2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0,(\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\\&-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}+\lambda\sum_{T\in\mathcal{T}_h}\langle\nabla\cdot\mathbf{u},\mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}\\&-\left(2\mu\sum_{T\in\mathcal{T}_h}\langle\varepsilon(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial T}+\lambda\sum_{T\in\mathcal{T}_h}\langle\nabla\cdot\mathbf{u},\mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}-\langle\widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N}\right)\\=&(\mathbf{f},\pi_h^{RT}(\mathbf{v}))+\langle\widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N}-2\mu(\nabla\cdot\varepsilon(\mathbf{u}),\mathbf{v}_0-\pi_h^{RT}(\mathbf{v}))\\&+2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0-\mathbf{v}_b,(\varepsilon(\mathbf{u})-\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T},\end{aligned}$$

which implies that

$$\mathcal{A}_h^s(Q_h\mathbf{u},\mathbf{v}) = (\mathbf{f},\boldsymbol{\pi}_h^{RT}(\mathbf{v})) + \langle \widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N} + \mathcal{G}_{\mathbf{u}}(\mathbf{v}) - \mathcal{K}_{\mathbf{u}}(\mathbf{v}) + \mathcal{S}_h(Q_h\mathbf{u},\mathbf{v}). \tag{4.12}$$

Subtracting (2.11) from (4.12), we obtain (4.2). □

4.2 Error estimate in the H^1-norm

In this section, we present the error estimate in the H^1-norm.

Lemma 4.2. [25,27] *Let $\mathcal{T}_h$ be a shape regular partition. Let $\mathbf{w}\in[H^{l+1}(\Omega)]^d$, $1\le l\le k$, and $s=0,1$. There hold*

$$\sum_{T\in\mathcal{T}_h} h_T^{2s}\|\mathbf{w}-Q_0\mathbf{w}\|_{T,s}^2 \le Ch^{2(l+1)}\|\mathbf{w}\|_{l+1}^2, \tag{4.13}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2s}\|\varepsilon(\mathbf{w})-\mathbf{\Pi}_h\varepsilon(\mathbf{w})\|_{T,s}^2 \le Ch^{2l}\|\mathbf{w}\|_{l+1}^2, \tag{4.14}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2s}\|\nabla\cdot\mathbf{w}-\mathcal{P}_h\nabla\cdot\mathbf{w}\|_{T,s}^2 \le Ch^{2l}\|\nabla\cdot\mathbf{w}\|_l^2 \le Ch^{2l}\|\mathbf{w}\|_{l+1}^2. \tag{4.15}$$

Lemma 4.3. *For any $\mathbf{w}\in[H^{l+1}(\Omega)]^d$, $1\le l\le k$, and $\mathbf{v}\in V_h$, the following estimates hold true*

$$|\mathcal{S}_h(Q_h\mathbf{w},\mathbf{v})| \le Ch^l\|\mathbf{w}\|_{l+1}|||\mathbf{v}|||, \tag{4.16}$$

$$|\mathcal{G}_{\mathbf{w}}(\mathbf{v})| \le Ch^l\|\mathbf{w}\|_{l+1}|||\mathbf{v}|||, \tag{4.17}$$

$$|\mathcal{K}_{\mathbf{w}}(\mathbf{v})| \le Ch^l\|\mathbf{w}\|_{l+1}|||\mathbf{v}|||. \tag{4.18}$$

Proof. According to (2.8), the Cauchy-Schwarz inequality, the triangle inequality, the trace inequality, and the estimate (4.13), one has

$$\begin{aligned}
|\mathcal{S}_h(Q_h\mathbf{w},\mathbf{v})| =& \left|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0\mathbf{w}-Q_b\mathbf{w},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}\right| \\
\le& C\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{w}-Q_b\mathbf{w}\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\right)^{\frac{1}{2}} \\
\le& C\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}(\|Q_0\mathbf{w}-\mathbf{w}\|_{\partial T}^2+\|Q_b\mathbf{w}-\mathbf{w}\|_{\partial T}^2)\right)^{\frac{1}{2}}|||\mathbf{v}||| \\
\le& C\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{w}-\mathbf{w}\|_{\partial T}^2\right)^{\frac{1}{2}}|||\mathbf{v}||| \\
\le& C\left(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0\mathbf{w}-\mathbf{w}\|_T^2+\|\nabla(Q_0\mathbf{w}-\mathbf{w})\|_T^2\right)^{\frac{1}{2}}|||\mathbf{v}||| \\
\le& Ch^l\|\mathbf{w}\|_{l+1}|||\mathbf{v}|||.
\end{aligned}$$

Now, we prove (4.17). From the Cauchy-Schwarz inequality, the trace inequality, and the estimate (4.14), we obtain

$$\begin{aligned}|\mathcal{G}_{\mathbf{w}}(\mathbf{v})| &= \left|2\mu\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b,(\varepsilon(\mathbf{w})-\mathbf{\Pi}_h\varepsilon(\mathbf{w}))\mathbf{n}\rangle_{\partial T}\right| \\ &\leq C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T\|\varepsilon(\mathbf{w})-\mathbf{\Pi}_h\varepsilon(\mathbf{w})\|_{\partial T}^2\right)^{\frac{1}{2}} \\ &\leq Ch^l\|\mathbf{w}\|_{l+1}|||\mathbf{v}|||.\end{aligned}$$

Next, we estimate the term $\mathcal{K}_{\mathbf{w}}(\mathbf{v})$. Let Q_0^{k-1} be the L^2 orthogonal projection onto $[P_{k-1}(T)]^d$, according to the definition of $\boldsymbol{\pi}_h^{RT}(\mathbf{v})$ in (2.5), we have

$$(Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{w})),\mathbf{v}_0-\boldsymbol{\pi}_h^{RT}(\mathbf{v}))_T=(Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{w})),\mathbf{v}_0)_T-(Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{w})),\mathbf{v}_0)_T=0. \tag{4.19}$$

Thus, it follows from the Cauchy-Schwarz inequality, and the estimates (3.10), (4.15) that

$$\begin{aligned}|\mathcal{K}_{\mathbf{w}}(\mathbf{v})| &= \left|2\mu\sum_{T\in\mathcal{T}_h}(\nabla\cdot\varepsilon(\mathbf{w}),\mathbf{v}_0-\boldsymbol{\pi}_h^{RT}(\mathbf{v}))_T\right| \\ &= \left|2\mu\sum_{T\in\mathcal{T}_h}(\nabla\cdot\varepsilon(\mathbf{w})-Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{w})),\mathbf{v}_0-\boldsymbol{\pi}_h^{RT}(\mathbf{v}))_T\right| \\ &\leq C\left(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot\varepsilon(\mathbf{w})-Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{w}))\|_T^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}\|\mathbf{v}_0-\boldsymbol{\pi}_h^{RT}(\mathbf{v})\|_T^2\right)^{\frac{1}{2}} \\ &\leq Ch^{l-1}\|\mathbf{w}\|_{l+1}h|||\mathbf{v}||| \\ &\leq Ch^l\|\mathbf{w}\|_{l+1}|||\mathbf{v}|||.\end{aligned}$$

This completes the proof of the lemma. □

Theorem 4.1. *Let $\mathbf{u}_h\in V_h$ be the discrete solution arising from the WG scheme* (2.11) *and $\mathbf{u}\in[H^{k+1}(\Omega)]^d$ be the exact solution of* (1.1). *Then, there holds*

$$|||Q_h\mathbf{u}-\mathbf{u}_h|||\leq Ch^k\|\mathbf{u}\|_{k+1}, \tag{4.20}$$

where the constant $C>0$ is independent of λ and the mesh size h.

Proof. By taking $\mathbf{v}=\mathbf{e}_h$ in (4.2), we get

$$\mathcal{A}_h^s(\mathbf{e}_h,\mathbf{e}_h)=\Theta_{\mathbf{u}}(\mathbf{e}_h).$$

Using Lemma 4.3 and Lemma 3.2, we obtain

$$\delta|||\mathbf{e}_h|||^2\leq\mathcal{A}_h^s(\mathbf{e}_h,\mathbf{e}_h)\leq Ch^k\|\mathbf{u}\|_{k+1}|||\mathbf{e}_h|||, \tag{4.21}$$

which gives (4.20). □

According to the results presented in Theorem 4.1, it is evident that the displacement error, when measured in the H^1-norm, demonstrates independence of the *Lamé* constant λ. This implies that the WG Algorithm 1 is robust about the *Lamé* constant λ.

4.3 Error estimate in the L^2-norm

In this section, we establish the error estimate in the L^2-norm. Consider the dual problem of seeking $\mathbf{\Phi}$ satisfying

$$-\nabla\cdot\sigma(\mathbf{\Phi})=\mathbf{e}_0, \quad \text{in } \Omega, \tag{4.22}$$

$$\mathbf{\Phi}=\mathbf{0}, \quad \text{on } \Gamma_D, \tag{4.23}$$

$$\sigma(\mathbf{\Phi})\mathbf{n}=\mathbf{0}, \quad \text{on } \Gamma_N. \tag{4.24}$$

Assume that the dual problem (4.22)-(4.24) satisfies the following regularity estimate:

$$\|\mathbf{\Phi}\|_2+\lambda\|\nabla\cdot\mathbf{\Phi}\|_1\le C\|\mathbf{e}_0\|. \tag{4.25}$$

According to [3,4], it is apparent that the regularity assumption is reasonable on convex polygons.

Theorem 4.2. *Let $\mathbf{u}_h\in V_h$ be the numerical solution of the WG scheme* (2.11) *and $\mathbf{u}\in[H^{k+1}(\Omega)]^d$ be the exact solution of* (1.1). *There exists a constant $C>0$ independent of λ and the mesh size h, such that*

$$\|Q_0\mathbf{u}-\mathbf{u}_0\|\le Ch^{k+1}\|\mathbf{u}\|_{k+1}. \tag{4.26}$$

Proof. Testing (4.22) by $\mathbf{e}_0$ and using the integration by parts, the fact that $\sum_{T\in\mathcal{T}_h}\langle\sigma(\mathbf{\Phi})\mathbf{n},\mathbf{e}_b\rangle_{\partial T}=0$, and (4.11), we get

$$\begin{aligned}
\|\mathbf{e}_0\|^2=&(-\nabla\cdot\sigma(\mathbf{\Phi}),\mathbf{e}_0)\\
=&-2\mu\sum_{T\in\mathcal{T}_h}(\nabla\cdot\varepsilon(\mathbf{\Phi}),\mathbf{e}_0)_T-\lambda\sum_{T\in\mathcal{T}_h}(\nabla(\nabla\cdot\mathbf{\Phi}),\mathbf{e}_0)_T\\
=&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon(\mathbf{\Phi}),\nabla\mathbf{e}_0)_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0,\varepsilon(\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\
&+\lambda\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{\Phi},\nabla\cdot\mathbf{e}_0)_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0,(\nabla\cdot\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\
=&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon(\mathbf{\Phi}),\nabla\mathbf{e}_0)_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,\varepsilon(\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\
&+\lambda\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{\Phi},\nabla\cdot\mathbf{e}_0)_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,(\nabla\cdot\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\
=&I_1+I_2,
\end{aligned} \tag{4.27}$$

where

$$\begin{aligned}
I_1=&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon(\mathbf{\Phi}),\nabla\mathbf{e}_0)_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,\varepsilon(\mathbf{\Phi})\mathbf{n}\rangle_{\partial T},\\
I_2=&\lambda\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{\Phi},\nabla\cdot\mathbf{e}_0)_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,(\nabla\cdot\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}.
\end{aligned}$$

By (4.6), we have

$$\begin{aligned}I_1=&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(\mathbf{e}_h),\varepsilon_w(Q_h\mathbf{\Phi}))_T-2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,(\varepsilon(\mathbf{\Phi})-\mathbf{\Pi}_h\varepsilon(\mathbf{\Phi}))\mathbf{n}\rangle_{\partial T}\\=&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(\mathbf{e}_h),\varepsilon_w(Q_h\mathbf{\Phi}))_T-\mathcal{G}_{\mathbf{\Phi}}(\mathbf{e}_h).\end{aligned}\tag{4.28}$$

From the definition of $\mathcal{P}_h$, the integration by parts, the definition of weak divergence (2.2), and (3.3), we obtain

$$\begin{aligned}I_2=&\lambda\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathbf{\Phi},\nabla\cdot\mathbf{e}_0)_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,(\nabla\cdot\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\=&\lambda\sum_{T\in\mathcal{T}_h}(\mathcal{P}_h\nabla\cdot\mathbf{\Phi},\nabla\cdot\mathbf{e}_0)_T-\lambda\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,(\nabla\cdot\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\=&-\lambda\sum_{T\in\mathcal{T}_h}(\nabla(\mathcal{P}_h\nabla\cdot\mathbf{\Phi}),\mathbf{e}_0)_T+\lambda\sum_{T\in\mathcal{T}_h}\langle(\mathcal{P}_h\nabla\cdot\mathbf{\Phi})\mathbf{n},\mathbf{e}_0\rangle_{\partial T}\\&-\lambda\sum_{T\in\mathcal{T}_h}\langle\mathbf{e}_0-\mathbf{e}_b,(\nabla\cdot\mathbf{\Phi})\mathbf{n}\rangle_{\partial T}\\=&\lambda\sum_{T\in\mathcal{T}_h}(\mathcal{P}_h\nabla\cdot\mathbf{\Phi},\nabla_w\cdot\mathbf{e}_h)_T-\lambda\sum_{T\in\mathcal{T}_h}\langle(\nabla\cdot\mathbf{\Phi}-\mathcal{P}_h\nabla\cdot\mathbf{\Phi})\mathbf{n},\mathbf{e}_0-\mathbf{e}_b\rangle_{\partial T}\\=&\lambda\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\mathbf{e}_h,\nabla_w\cdot(Q_h\mathbf{\Phi}))_T-\lambda\sum_{T\in\mathcal{T}_h}\langle(\nabla\cdot\mathbf{\Phi}-\mathcal{P}_h\nabla\cdot\mathbf{\Phi})\mathbf{n},\mathbf{e}_0-\mathbf{e}_b\rangle_{\partial T}.\end{aligned}\tag{4.29}$$

According to error equation (4.2), we have

$$\begin{aligned}&2\mu\sum_{T\in\mathcal{T}_h}(\varepsilon_w(\mathbf{e}_h),\varepsilon_w(Q_h\mathbf{\Phi}))_T+\lambda\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\mathbf{e}_h,\nabla_w\cdot(Q_h\mathbf{\Phi}))_T\\=&\mathcal{G}_{\mathbf{u}}(Q_h\mathbf{\Phi})-\mathcal{K}_{\mathbf{u}}(Q_h\mathbf{\Phi})+\mathcal{S}_h(Q_h\mathbf{u},Q_h\mathbf{\Phi})-\mathcal{S}_h(\mathbf{e}_h,Q_h\mathbf{\Phi}).\end{aligned}\tag{4.30}$$

Furthermore, combining (4.27)-(4.30), we arrive at

$$\begin{aligned}\|\mathbf{e}_0\|^2=&\mathcal{G}_{\mathbf{u}}(Q_h\mathbf{\Phi})-\mathcal{K}_{\mathbf{u}}(Q_h\mathbf{\Phi})+\mathcal{S}_h(Q_h\mathbf{u},Q_h\mathbf{\Phi})-\mathcal{S}_h(\mathbf{e}_h,Q_h\mathbf{\Phi})\\&-\mathcal{G}_{\mathbf{\Phi}}(\mathbf{e}_h)-\lambda\sum_{T\in\mathcal{T}_h}\langle(\nabla\cdot\mathbf{\Phi}-\mathcal{P}_h\nabla\cdot\mathbf{\Phi})\mathbf{n},\mathbf{e}_0-\mathbf{e}_b\rangle_{\partial T}.\end{aligned}\tag{4.31}$$

Next, we estimate each item in the right-hand side of (4.31) one by one.

(i) It follows from Lemma 4.3 and Theorem 4.1 that

$$|\mathcal{G}_{\mathbf{\Phi}}(\mathbf{e}_h)|\le Ch\|\mathbf{\Phi}\|_2|||\mathbf{e}_h|||\le Ch^{k+1}\|\mathbf{\Phi}\|_2\|\mathbf{u}\|_{k+1},\tag{4.32}$$

$$|\mathcal{S}_h(\mathbf{e}_h,Q_h\mathbf{\Phi})|\le Ch\|\mathbf{\Phi}\|_2|||\mathbf{e}_h|||\le Ch^{k+1}\|\mathbf{\Phi}\|_2\|\mathbf{u}\|_{k+1}.\tag{4.33}$$

(ii) Using the Cauchy-Schwarz inequality, the trace inequality, the estimate (4.15), and

Theorem 4.1, we arrive at

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\lambda\langle\nabla\cdot\mathbf{\Phi}-\mathcal{P}_h(\nabla\cdot\mathbf{\Phi}),(\mathbf{e}_0-\mathbf{e}_b)\cdot\mathbf{n}\rangle_{\partial T}\right|\\
\le& C\lambda\left(\sum_{T\in\mathcal{T}_h}h_T\|\nabla\cdot\mathbf{\Phi}-\mathcal{P}_h(\nabla\cdot\mathbf{\Phi})\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|(\mathbf{e}_0-\mathbf{e}_b)\cdot\mathbf{n}\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le& C\lambda h\|\nabla\cdot\mathbf{\Phi}\|_1|\!|\!|\mathbf{e}_h|\!|\!|\\
\le& C\lambda h^{k+1}\|\nabla\cdot\mathbf{\Phi}\|_1\|\mathbf{u}\|_{k+1}.
\end{aligned}\tag{4.34}
$$

(iii) From the Cauchy-Schwarz inequality, the triangle inequality, the trace inequality, and the estimate (4.13), we obtain

$$
\begin{aligned}
|\mathcal{S}_h(Q_h\mathbf{u},Q_h\mathbf{\Phi})|=&\left|\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle Q_0\mathbf{u}-Q_b\mathbf{u},Q_0\mathbf{\Phi}-Q_b\mathbf{\Phi}\rangle_{\partial T}\right|\\
\le& C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}(\|Q_0\mathbf{u}-\mathbf{u}\|_{\partial T}^2+\|Q_b\mathbf{u}-\mathbf{u}\|_{\partial T}^2)\right)^{\frac{1}{2}}\\
&\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}(\|Q_0\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2+\|Q_b\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2)\right)^{\frac{1}{2}}\\
\le& C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0\mathbf{u}-\mathbf{u}\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le& Ch^{k+1}\|\mathbf{u}\|_{k+1}\|\mathbf{\Phi}\|_2.
\end{aligned}\tag{4.35}
$$

Similarly, by the Cauchy-Schwarz inequality, the triangle inequality, the trace inequality, and the estimate (4.14), it follows that

$$
\begin{aligned}
|\mathcal{G}_{\mathbf{u}}(Q_h\mathbf{\Phi})|=&\left|2\mu\sum_{T\in\mathcal{T}_h}\langle Q_0\mathbf{\Phi}-Q_b\mathbf{\Phi},(\varepsilon(\mathbf{u})-\mathbf{\Pi}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\right|\\
\le& C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T\|\varepsilon(\mathbf{u})-\mathbf{\Pi}_h\varepsilon(\mathbf{u})\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le& Ch^{k+1}\|\mathbf{u}\|_{k+1}\|\mathbf{\Phi}\|_2.
\end{aligned}\tag{4.36}
$$

(iv) We turn to estimate the term $\mathcal{K}_{\mathbf{u}}(Q_h\mathbf{\Phi})$. From the definition of Q_0^{k-1} and the definition of $\boldsymbol{\pi}_h^{RT}(\mathbf{v})$ in (2.5), we have

$$
\sum_{T\in\mathcal{T}_h}(Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{u})),Q_0\mathbf{\Phi}-\boldsymbol{\pi}_h^{RT}(Q_h\mathbf{\Phi}))_T=\sum_{T\in\mathcal{T}_h}(Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{u})),Q_0\mathbf{\Phi}-Q_0\mathbf{\Phi})_T=0.
$$

Furthermore, by the Cauchy-Schwarz inequality, the estimates (3.10) and (4.13), we obtain

$$
\begin{aligned}
|\mathcal{K}_{\mathbf{u}}(Q_h\mathbf{\Phi})| &= \left|2\mu(\nabla\cdot\varepsilon(\mathbf{u}),Q_0\mathbf{\Phi}-\boldsymbol{\pi}_h^{RT}(Q_h\mathbf{\Phi}))\right| \\
&= \left|2\mu\sum_{T\in\mathcal{T}_h}(\nabla\cdot\varepsilon(\mathbf{u})-Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{u})),Q_0\mathbf{\Phi}-\boldsymbol{\pi}_h^{RT}(Q_h\mathbf{\Phi}))\right| \\
&\le C\left(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot\varepsilon(\mathbf{u})-Q_0^{k-1}(\nabla\cdot\varepsilon(\mathbf{u}))\|_T^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T\|Q_0\mathbf{\Phi}-Q_b\mathbf{\Phi}\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\le Ch^{k+1}\|\mathbf{u}\|_{k+1}\|\mathbf{\Phi}\|_2. \qquad (4.37)
\end{aligned}
$$

Combining the estimates (4.32)-(4.37) in (i)-(iv) and the regularity assumption (4.25), we arrive at

$$
\begin{aligned}
\|\mathbf{e}_0\|^2 &\le Ch^{k+1}\|\mathbf{\Phi}\|_2\|\mathbf{u}\|_{k+1}+C\lambda h^{k+1}\|\nabla\cdot\mathbf{\Phi}\|_1\|\mathbf{u}\|_{k+1} \\
&\le Ch^{k+1}(\|\mathbf{\Phi}\|_2+\lambda\|\nabla\cdot\mathbf{\Phi}\|_1)\|\mathbf{u}\|_{k+1} \\
&\le Ch^{k+1}\|\mathbf{u}\|_{k+1}\|\mathbf{e}_0\|,
\end{aligned}
$$

which yields the error estimate (4.26). This completes the proof of the theorem. □

Remark 4.1. Based on the proofs of Theorem 4.1 and Theorem 4.2, it becomes apparent that the introduction of an $H(div)$-conforming displacement reconstruction operator in our scheme effectively eradicates the dependence of displacement error on λ. Moreover, it is essential to highlight that our method does not require the imposition of high-order regularity assumptions, and it exhibits robustness even in scenarios where $\lambda\|\nabla\cdot\mathbf{u}\|_k$ is unbounded.

5 Numerical results

This section provides numerical examples to validate the theoretical conclusions on the WG scheme (2.11) for the elasticity problems. We use triangular meshes as shown in Fig. 1.

Example 5.1 (Convergence test)**.** Consider the elasticity problems (1.1)-(1.3) in square domain $\Omega=(0,1)^2$. The exact solution $\mathbf{u}$ is chosen as follows

$$
\mathbf{u}=\begin{pmatrix}\sin(\pi x)\cos(\pi y)\\ \cos(\pi x)\sin(\pi y)\end{pmatrix},
$$

where $\Gamma_N=\{(x,y):y=1\}$, *Lamé* constants $\mu=0.5$, $\lambda=1$, and the right-hand side function $\mathbf{f}$, boundary functions $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{t}}$ are chosen to match the exact solution $\mathbf{u}$.

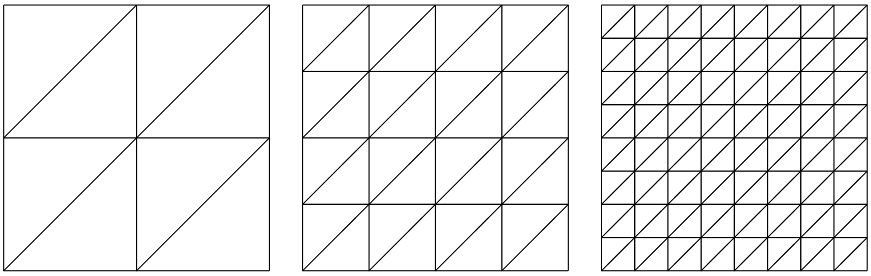

Figure 1: The first three levels of triangular grids in square domain.

Table 1: Error and convergence order of displacement $\mathbf{u}$ in Example 5.1.

Level	$\|\|\|Q_h\mathbf{u}-\mathbf{u}_h\|\|\|$	order	$\|\|Q_0\mathbf{u}-\mathbf{u}_0\|\|$	order
by the P_1-P_1 WG elements				
3	5.0109E-01	–	3.5196E-02	–
4	2.5987E-01	0.9473	9.3553E-03	1.9115
5	1.3099E-01	0.9883	2.3758E-03	1.9774
6	6.5582E-02	0.9981	5.9586E-04	1.9954
7	3.2790E-02	1.0001	1.4903E-04	1.9994
by the P_2-P_2 WG elements				
3	1.7095E-01	–	3.3804E-03	–
4	4.7386E-02	1.8511	4.6648E-04	2.8573
5	1.2343E-02	1.9408	6.0657E-05	2.9431
6	3.1379E-03	1.9758	7.7066E-06	2.9765
7	7.9022E-04	1.9894	9.7026E-07	2.9897

The results presented in Table 1 demonstrate the evaluation of error and numerical convergence order. Specifically, Table 1 shows that the approximate displacement $\mathbf{u}_h$ converges towards the exact solution $\mathbf{u}$, indicating the effectiveness of the WG method. Additionally, it is observed that the numerical convergence order in L^2-norm for the P_1-P_1 WG elements is $\mathcal{O}(h^2)$, while for the P_2-P_2 WG elements, it is $\mathcal{O}(h^3)$. This is consistent with theoretical analysis.

Example 5.2 (Locking-free test)**.** Consider the elasticity problems (1.1)-(1.3) on square domain $\Omega=(0,1)^2$ with triangular grids in Fig. 1. The exact solution $\mathbf{u}$ is chosen as follows [29]

$$\mathbf{u}=\begin{pmatrix} -(1-\cos(2\pi x))\sin(2\pi y) \\ (1-\cos(2\pi y))\sin(2\pi x) \end{pmatrix}+\frac{1}{\lambda+\mu}\begin{pmatrix} \sin(\pi x)\sin(\pi y) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix},$$

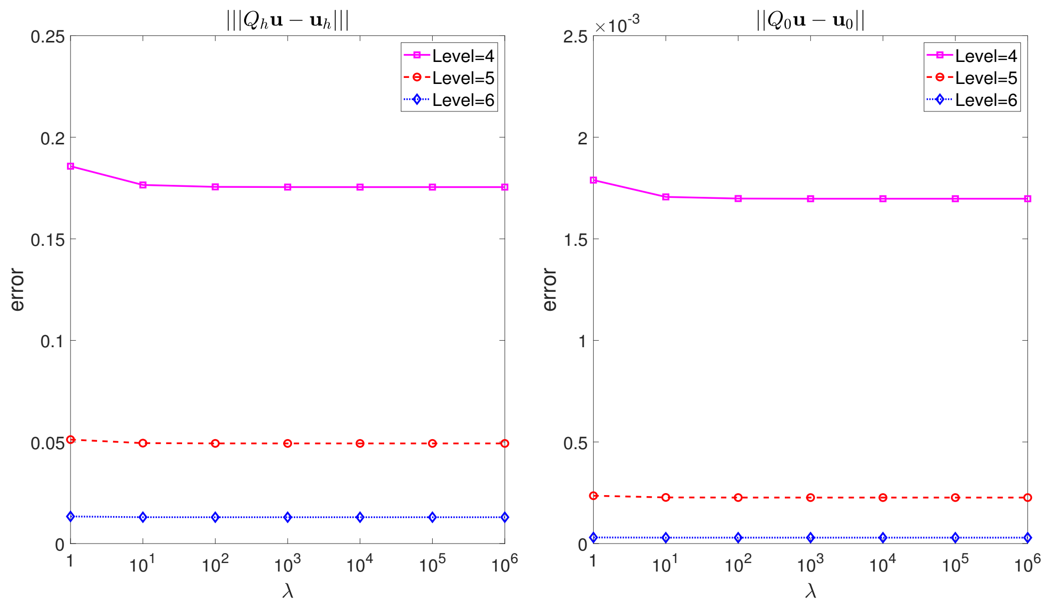

Figure 2: The error of displacement approximation by the P_2-P_2 WG elements under different parameter values λ from Example 5.2. Left: The displacement error in $|||\cdot|||$-norm. Right: The displacement error in L^2-norm.

where *Lamé* constant $\mu=0.5$ and

$$\begin{aligned}\mathbf{f}=&-4\pi^2\mu\begin{pmatrix}-\cos(2\pi x)\sin(2\pi y)-\sin(2\pi y)(\cos(2\pi x)-1)\\ \cos(2\pi y)\sin(2\pi x)+\sin(2\pi x)(\cos(2\pi y)-1)\end{pmatrix}\\ &+\frac{2\pi^2\mu}{\lambda+\mu}\begin{pmatrix}\sin(\pi x)\sin(\pi y)\\ \sin(\pi x)\sin(\pi y)\end{pmatrix}-\begin{pmatrix}\pi^2\cos(\pi x)\cos(\pi y)-\pi^2\sin(\pi x)\sin(\pi y)\\ \pi^2\cos(\pi x)\cos(\pi y)-\pi^2\sin(\pi x)\sin(\pi y)\end{pmatrix},\\ \widehat{\mathbf{g}}=&\begin{pmatrix}-(1-\cos(2\pi x))\sin(2\pi y)\\ (1-\cos(2\pi y))\sin(2\pi x)\end{pmatrix}+\frac{1}{\lambda+\mu}\begin{pmatrix}\sin(\pi x)\sin(\pi y)\\ \sin(\pi x)\sin(\pi y)\end{pmatrix}.\end{aligned}$$

In this example, we set $\Gamma_N=\{(x,y):y=1\}$.

In the computation, we take $\lambda=1$, $\lambda=10^2$, $\lambda=10^4$, and $\lambda=10^6$, respectively. The numerical results by the P_1-P_1 WG elements are provided in Table 2, whereas the numerical results by the P_2-P_2 WG elements are illustrated in Fig. 2.

Table 2 demonstrates that optimal convergence rate is achieved in all cases, and the error and the convergence rate are independent of the parameter λ. To further illustrate the locking-free property of the WG numerical scheme, we plot the errors in various norms by the P_2-P_2 WG elements under different parameter values λ in Fig. 2. Our

Table 2: Error and convergence order of displacement $\mathbf{u}$ by the P_1-P_1 WG elements in Example 5.2.

Level	$\|\|\|Q_h\mathbf{u}-\mathbf{u}_h\|\|\|$	order	$\|\|Q_0\mathbf{u}-\mathbf{u}_0\|\|$	order
		$\lambda=1$		
3	2.3641E+00	–	1.5365E-01	–
4	9.7568E-01	1.2768	3.7766E-02	2.0245
5	4.4411E-01	1.1355	9.3642E-03	2.0119
6	2.1522E-01	1.0451	2.3338E-03	2.0045
7	1.0665E-01	1.0129	5.8282E-04	2.0016
		$\lambda=10^2$		
3	2.2934E+00	–	1.3931E-01	–
4	9.4204E-01	1.2837	3.4328E-02	2.0208
5	4.2593E-01	1.1451	8.5193E-03	2.0106
6	2.0582E-01	1.0493	2.1242E-03	2.0038
7	1.0191E-01	1.0141	5.3058E-04	2.0012
		$\lambda=10^4$		
3	2.2934E+00	–	1.3925E-01	–
4	9.4207E-01	1.2836	3.4321E-02	2.0205
5	4.2594E-01	1.1452	8.5183E-03	2.0105
6	2.0582E-01	1.0493	2.1240E-03	2.0038
7	1.0191E-01	1.0141	5.3058E-04	2.0011
		$\lambda=10^6$		
3	2.2934E+00	–	1.3925E-01	–
4	9.4207E-01	1.2836	3.4321E-02	2.0205
5	4.2594E-01	1.1452	8.5183E-03	2.0105
6	2.0582E-01	1.0493	2.1240E-03	2.0038
7	1.0191E-01	1.0141	5.3076E-04	2.0007

numerical results reveal that there are no significant differences among the displacement errors in various norms under different parameter values λ, showing that the WG scheme remains unaffected by the parameter λ, thus validating its locking-free characteristic.

Example 5.3 (Locking-free test with unbounded $\lambda\|\nabla\cdot\mathbf{u}\|_k$)**.** To better demonstrate the robustness of the WG Algorithm 1, we consider the case of $\lambda\|\nabla\cdot\mathbf{u}\|_k$ unbounded in the following example. Besides, we introduce the standard WG scheme as Algorithm 2 for comparison.

In this example, we consider the elasticity problems (1.1)-(1.3) on square domain $\Omega=$

Algorithm 2 Standard WG Algorithm

A standard WG scheme for (1.1)-(1.3) is given by finding $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h$ with $\mathbf{u}_b=Q_b\widehat{\mathbf{g}}$ on Γ_D such that

$$\mathcal{A}_h^s(\mathbf{u}_h,\mathbf{v})=(\mathbf{f},\mathbf{v}_0)+\langle\widehat{\mathbf{t}},\mathbf{v}_b\rangle_{\Gamma_N},\quad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0. \tag{5.1}$$

$(0,1)^2$ with triangular grids in Fig. 1. The exact solution $\mathbf{u}$ is chosen as follows

$$\mathbf{u}=\begin{pmatrix}\sin(\pi x)\sin(\pi y)\\ \sin(\pi x)\sin(\pi y)\end{pmatrix},$$

where *Lamé* constant $\mu=0.5$ and

$$\mathbf{f}=-\mu\begin{pmatrix}-2\pi^2\sin(\pi x)\sin(\pi y)\\ -2\pi^2\sin(\pi x)\sin(\pi y)\end{pmatrix}-(\lambda+\mu)\begin{pmatrix}\pi^2\cos(\pi x)\cos(\pi y)-\pi^2\sin(\pi x)\sin(\pi y)\\ \pi^2\cos(\pi x)\cos(\pi y)-\pi^2\sin(\pi x)\sin(\pi y)\end{pmatrix},$$

$$\widehat{\mathbf{g}}=\begin{pmatrix}\sin(\pi x)\sin(\pi y)\\ \sin(\pi x)\sin(\pi y)\end{pmatrix}.$$

Here $\Gamma_N=\{(x,y):y=1\}$, then

$$\widehat{\mathbf{t}}=\begin{pmatrix}\mu(\pi\cos(\pi x)\sin(\pi y)+\pi\cos(\pi y)\sin(\pi x))\\ \lambda(\pi\cos(\pi x)\sin(\pi y)+\pi\cos(\pi y)\sin(\pi x))+2\mu\pi\cos(\pi y)\sin(\pi x)\end{pmatrix}.$$

In the computation, we repeat the computational procedure for various parameter values with $\lambda=1,10^2,10^4,10^6$. The obtained numerical results are presented in Tables 3-4. In the case of P_1-P_1 and P_2-P_2 WG elements, it is observed that the displacement errors resulting from the standard WG Algorithm 2 exhibit an increasing trend as the parameter λ is incremented. However, the WG Algorithm 1 shows a distinctive characteristic: the displacement errors remain unaffected by variations in the *Lamé* constant λ. This finding signifies the robustness of the WG Algorithm 1 with respect to the *Lamé* constant λ.

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Table 3: Error and convergence order of displacement $\mathbf{u}$ in Example 5.3 by the P_1-P_1 WG elements.

	Robust WG Algor0ithm 1				Standard WG Algorithm 2			
Level	$\|\|\|Q_h\mathbf{u}-\mathbf{u}_h\|\|\|$	order	$\|Q_0\mathbf{u}-\mathbf{u}_0\|$	order	$\|\|\|Q_h\mathbf{u}-\mathbf{u}_h\|\|\|$	order	$\|Q_0\mathbf{u}-\mathbf{u}_0\|$	order
				$\lambda=1$				
3	5.3102E-01	–	3.5163E-02	–	6.0233E-01	–	3.6772E-02	–
4	2.6944E-01	0.9788	9.5251E-03	1.8842	2.7940E-01	1.1082	9.6341E-03	1.9324
5	1.3510E-01	0.9960	2.4439E-03	1.9626	1.3641E-01	1.0344	2.4510E-03	1.9748
6	6.7529E-02	1.0004	6.1509E-04	1.9903	6.7697E-02	1.0107	6.1555E-04	1.9934
7	3.3745E-02	1.0009	1.5396E-04	1.9982	3.3766E-02	1.0035	1.5399E-04	1.9990
				$\lambda=10^2$				
3	5.2631E-01	–	2.1875E-02	–	1.6757E+01	–	5.8882E-01	–
4	2.6835E-01	0.9718	6.4525E-03	1.7613	4.3171E+00	1.9566	7.6122E-02	2.9514
5	1.3451E-01	0.9964	1.7134E-03	1.9130	1.0975E+00	1.9759	9.7537E-03	2.9643
6	6.7200E-02	1.0012	4.3629E-04	1.9735	2.8170E-01	1.9619	1.2835E-03	2.9259
7	3.3569E-02	1.0013	1.0964E-04	1.9924	7.6314E-02	1.8841	1.8684E-04	2.7802
				$\lambda=10^4$				
3	5.2659E-01	–	2.1904E-02	–	1.6651E+03	–	5.8561E+01	–
4	2.6843E-01	0.9721	6.4705E-03	1.7592	4.2840E+02	1.9586	7.5509E+00	2.9552
5	1.3453E-01	0.9966	1.7192E-03	1.9121	1.0828E+02	1.9841	9.5583E-01	2.9818
6	6.7203E-02	1.0014	4.3786E-04	1.9732	2.7195E+01	1.9934	1.2014E-01	2.9921
7	3.3570E-02	1.0014	1.1005E-04	1.9924	6.8126E+00	1.9970	1.5055E-02	2.9963
				$\lambda=10^6$				
3	5.2659E-01	–	2.1904E-02	–	1.6650E+05	–	5.8558E+03	–
4	2.6843E-01	0.9721	6.4707E-03	1.7592	4.2837E+04	1.9586	7.5505E+02	2.9552
5	1.3453E-01	0.9966	1.7193E-03	1.9121	1.0828E+04	1.9841	9.5579E+01	2.9818
6	6.7203E-02	1.0014	4.3788E-04	1.9732	2.7193E+03	1.9934	1.2013E+01	2.9921
7	3.3570E-02	1.0014	1.1001E-04	1.9929	6.8121E+02	1.9971	1.5054E+00	2.9964

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Table 4: Error and convergence order of displacement $\mathbf{u}$ in Example 5.3 by the P_2-P_2 WG elements.

	Robust WG Algorithm 1				Standard WG Algorithm 2			
Level	$\|\|\|\mathbf{u}_h-Q_h\mathbf{u}\|\|\|$	order	$\|\|\mathbf{u}_0-Q_0\mathbf{u}\|\|$	order	$\|\|\|\mathbf{u}_h-Q_h\mathbf{u}\|\|\|$	order	$\|\|\mathbf{u}_0-Q_0\mathbf{u}\|\|$	order
				$\lambda=1$				
3	1.6748E-01	–	3.3340E-03	–	1.6459E-01	–	3.2961E-03	–
4	4.7979E-02	1.8035	4.7096E-04	2.8236	4.7756E-02	1.7851	4.6915E-04	2.8126
5	1.2662E-02	1.9219	6.1785E-05	2.9303	1.2647E-02	1.9169	6.1720E-05	2.9262
6	3.2378E-03	1.9674	7.8795E-06	2.9711	3.2369E-03	1.9661	7.8774E-06	2.9700
7	8.1768E-04	1.9854	9.9374E-07	2.9872	8.1762E-04	1.9851	9.9367E-07	2.9869
				$\lambda=10^2$				
3	1.6689E-01	–	3.2633E-03	–	7.2538E-01	–	2.8693E-02	–
4	4.8585E-02	1.7803	4.7060E-04	2.7938	9.8058E-02	2.8870	1.8321E-03	3.9692
5	1.2897E-02	1.9135	6.2202E-05	2.9195	1.6597E-02	2.5627	1.2654E-04	3.8558
6	3.3052E-03	1.9642	7.9554E-06	2.9670	3.5499E-03	2.2251	1.0514E-05	3.5892
7	8.3541E-04	1.9842	1.0045E-06	2.9854	8.5085E-04	2.0608	1.0924E-06	3.2667
				$\lambda=10^4$				
3	1.6683E-01	–	3.2621E-03	–	7.3837E+01	–	2.8771E+00	–
4	4.8586E-02	1.7798	4.7057E-04	2.7933	9.1304E+00	3.0156	1.7991E-01	3.9992
5	1.2899E-02	1.9133	6.2204E-05	2.9193	1.1346E+00	3.0084	1.1235E-02	4.0011
6	3.3059E-03	1.9642	7.9561E-06	2.9669	1.4156E-01	3.0027	7.0206E-04	4.0003
7	8.3560E-04	1.9841	1.0046E-06	2.9854	1.7698E-02	2.9998	4.3885E-05	3.9998
				$\lambda=10^6$				
3	1.6683E-01	–	3.2621E-03	–	7.3869E+03	–	2.8773E+02	–
4	4.8586E-02	1.7798	4.7057E-04	2.7933	9.1361E+02	3.0153	1.7994E+01	3.9992
5	1.2899E-02	1.9133	6.2204E-05	2.9193	1.1354E+02	3.0084	1.1237E+00	4.0011
6	3.3059E-03	1.9641	7.9565E-06	2.9668	1.4164E+01	3.0030	7.0215E-02	4.0004
7	8.3584E-04	1.9838	1.0054E-06	2.9844	1.7693E+00	3.0010	4.3883E-03	4.0001

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