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A Weak Galerkin Finite Element Method Coupled with Mortar Spectral Element Method for Schrödinger Eigenvalue Problem with an Inverse Square Potential

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Abstract. In this paper, we introduce a weak Galerkin (WG) finite element method coupled with mortar spectral element method (MSEM) to solve the Schrödinger eigenvalue problem with an inverse square potential. For the domain around the inverse square potential, we use the mortar spectral element method to simulate the singularities in eigenfunctions caused by the inverse square potential, while we employ the WG method in the remaining domain. This coupled method can effectively handle the singularity arising from the inverse square potential. Notably, hanging nodes are allowed on the coupled interface. Compared to the conforming finite element method coupled with MSEM, our approach is not constrained by the mesh size of the mortar spectral element. This flexibility permits the use of fine meshes in the WG domain, thereby enhancing accuracy. We provide hp error analysis for both eigenfunctions and eigenvalues. Numerical experiments demonstrate the hp convergence of the theoretical results.

AMS subject classifications: 65N15, 65N25, 65N30, 65N35
Key words: Weak Galerkin finite element method, mortar spectral element method, Schrödinger eigenvalue problem, error estimates.

1 Introduction

The Schrödinger operator is extremely important in science and exists in several different forms. The Schrödinger operator with an inverse square singular potential has recently garnered significant attention due to its fundamental role in both mathematics and

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physics. Mathematically, the inverse square potential exhibits the same homogeneity or "differential order" as the Laplacian, however, it typically induces strong singularities in the Schrödinger eigenfunctions, and, as such, cannot be considered as a lower-order perturbation term [9,13,14,20]. On the other hand, in non-relativistic quantum mechanics, the inverse square potential represents an intermediate threshold between regular and singular potentials [11,15]. Additionally, the inverse square potential figures prominently in nuclear physics, molecular physics, and quantum cosmology [11,15].

Based on the variational form, Galerkin-type numerical schemes can be designed. However, even with adaptive schemes, low-order methods exhibit only limited convergence rates [21,22,31]. Similarly, due to the strong singularities of eigenfunctions induced by the singular potential, classical high-order methods, including spectral/spectral element methods, generally fail to achieve exponential convergence rates [8,17,23]. In the literature, the idea of incorporating singular terms into the basis functions has been used, which comes at the expenses of compromising the sparsity of the resulting algebraic matrix system. Although this approach improves the convergence rate to some extent, its effectiveness depends on the number of singular terms introduced in [16].

In [23], the authors proposed a new method: the mortar spectral element method (MSEM), which can effectively handle all singularities and construct orthogonal basis functions, thereby achieving very sparse (sometimes diagonal) matrices. Specifically, at each singular point including the origin, an additional disk or sector element is used. This element employs a class of non-polynomial spectral basis functions to model the singularity. The remaining domain are divided into quadrilaterals or triangles, with some boundaries being curved. Conforming basis functions are then applied to the elements in the rest of subdomains. These two types of elements are connected using mortar techniques. Exponential convergence rates $e^{-\sigma\sqrt{DoF}}$ are observed for different eigenvalues in various numerical experiments, with σ being nearly uniform. Numerical tests were also conducted for the Schrödinger eigenvalue problem on the whole plane and other types of inverse square potential problems [24,25]. Additionally, Jia et al. [19] conducted a rigorous theoretical analysis of the numerical results in [23] and obtained optimal error estimates.

The mortar element method is a flexible technique for connecting different variational discretizations on each subdomain. In addition to the h finite element method and the spectral element method studied in [6], it can be applied within the frameworks of the hp finite element method [7], and the finite volume method [1]. Consequently, the mortar method finds significant applications in various areas, including those involving the curl and divergence operators [2,5], as well as fourth-order problems [4]. The weak Galerkin (WG) finite element method is an effective numerical method for solving partial differential equations. In [32], Wang and Ye first proposed the WG method for solving second-order elliptic problems. The WG method has also been applied to a range of different equations, such as: second-order elliptic problems [33,35], biharmonic equations [37,39], Stokes equations [26], Brinkman equations [27], Maxwell equations [29,36], eigenvalue problems [10,38], elasticity equations [18,34], and others.

In this paper, we propose a finite element method that couples the WG finite element method with the mortar spectral element method. For the domain surrounding the inverse square potential, we use the mortar spectral element method to simulate the singularities of the eigenfunctions, while in the remaining domain, we apply the WG method on a triangulation. Notably, we permit the use of hanging nodes on the coupled interface. Unlike using the conforming finite element method in the remaining domain, our method is not affected by the mesh size of the mortar spectral element, enabling us to improve accuracy by utilizing fine meshes in the rest of the domain.

The structure of the paper is as follows. In Section 2, we introduce the Schrödinger eigenvalue problem and describe the finite element space, along with the corresponding numerical scheme. In Section 3, we analyze the errors of eigenfunctions. Error estimates in the H^1 and L^2 norms for the source problem are provided in Section 4. In Section 5, we combine the error estimates from Sections 3 and 4 to obtain the H^1 and L^2 error estimates for the eigenfunctions. Furthermore, we provide an expression for the eigenvalue error and present the corresponding error estimates for eigenvalues. Multiple sets of numerical experiment results are presented in Section 6.

2 The coupled finite element method for Schrödinger eigenvalue problem

In this section, we begin by introducing the Schrödinger eigenvalue problem (2.1). We then proceed to define the finite element space. Finally, we present the numerical scheme. Throughout the paper, we employ the standard notations for Sobolev spaces. For any open bounded domain $D\subset\mathbb{R}^d$ $(d=2,3)$, $(\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{s,D}$ represent the inner product and norm in the Sobolev space $H^s(D)(s\geq 0)$, respectively.

2.1 Schrödinger eigenvalue problem

Let $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ be a bounded Lipschitz domain and the origin O is assumed to be in Ω. We consider the following Schrödinger eigenvalue problem with an inverse square potential:

$$\begin{cases} -\Delta u+\dfrac{c^2}{\|x\|^2}u=\lambda u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $c\neq 0$ is a constant.

Denote $r=\|x\|$ and the weighted Sobolev space

$$\mathscr{H}^1(\Omega)=\mathscr{H}^{1,c}(\Omega)=\left\{u\in L^2(\Omega),\ \|u\|_{\mathscr{H}^1(\Omega)}<\infty\right\},$$

where

$$\|u\|_{\mathscr{H}^1(\Omega)}:=\left(\|\nabla u\|^2_{\Omega}+c^2\|u\|^2_{r^{-2},\Omega}\right)^{1/2}.$$

The corresponding inner product is $\langle u,v\rangle_\Omega := (\nabla u,\nabla v)_\Omega + c^2(u,v)_{r^{-2},\Omega}$. Here $(\cdot,\cdot)_{r^{-2},\Omega}$ is the L^2 inner product with weight function r^{-2}, i.e., $(u,v)_{r^{-2},\Omega} = \int_\Omega r^{-2}uv d\Omega$. From [19], $\mathscr{H}^1(\Omega)$ is a subspace of $H^1(\Omega)$ and $\mathscr{H}^1(\Omega)=H^1(\Omega)$ when $d\geq 3$ owing to the Hardy-type inequality. Let $\mathscr{H}_0^1(\Omega)=\mathscr{H}^1(\Omega)\cap H_0^1(\Omega)$.

The corresponding variational form of (2.1) reads: Find $\lambda\in\mathbb{R}$ and $u\in\mathscr{H}_0^1(\Omega)\backslash\{0\}$ such that $b(u,u)=1$ and

$$a(u,v)=\lambda b(u,v), \quad \forall v\in\mathscr{H}_0^1(\Omega), \tag{2.2}$$

where

$$\begin{aligned} &a(u,v)=(\nabla u,\nabla v)_\Omega + c^2(u,v)_{r^{-2},\Omega},\\ &b(u,v)=(u,v)_\Omega. \end{aligned}$$

To get error estimates for the eigenvalue problem, we introduce the corresponding source problem:

$$\begin{cases} -\Delta u+\dfrac{c^2}{\|x\|^2}u=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The corresponding variational form reads: Find $u\in\mathscr{H}_0^1(\Omega)$ such that

$$a(u,v)=b(f,v), \quad \forall v\in\mathscr{H}_0^1(\Omega).$$

It is easy to verify that $a(\cdot,\cdot)$ is coercive and continuous over $\mathscr{H}_0^1(\Omega)$.

2.2 Sobolev orthogonal ball functions

Let $\mathbb{S}_R^{d-1}(y)$ and $\mathbb{B}_R^d(y)$ be the sphere and the ball in $\mathbb{R}^d$, respectively, both with radius R and center $y\in\mathbb{R}^d$. When y is the origin O, we can denote $\mathbb{S}_R^{d-1}=\mathbb{S}_R^{d-1}(O)$ and $\mathbb{B}_R^d=\mathbb{B}_R^d(O)$. Similarly, the subscript R can be omitted from the notation when $R=1$. For $x\in\mathbb{R}^d$, we can define its spherical-polar coordinates (r,ξ) as follows

$$x=r\xi, \quad r=\|x\|, \quad \xi\in\mathbb{S}^{d-1}.$$

Let $\mathbb{N}_0$ and $\mathbb{N}$ be the set of nonnegative integers and the set of positive integers, respectively. Denote $\Xi_n^d=\{1,2,\cdots a_n^d\}$ and $\beta_n=\sqrt{c^2+(n+d/2-1)^2}$, where $a_n^2=2-\delta_{n,0}$ and $a_n^3=2n+1$. The following ball functions are defined in [19] and [23]:

$$P_{k,\ell}^{\alpha,n}(x) := J_k^{\alpha,\beta_n}(2r^2-1)\, r^{\beta_n+1-d/2}Y_\ell^n(\xi), \quad x=r\xi\in\mathbb{B}^d, \ \ell\in\Xi_n^d, \ k,n\in\mathbb{N}_0,$$

where J_k^{α,β_n} and Y_ℓ^n are generalized Jacobi polynomials and harmonic polynomials, respectively. The specific definitions of generalized Jacobi polynomials and harmonic polynomials can be found in Section 2 of [19]. In addition, the spherical harmonics are an orthonormal basis for $L^2(\mathbb{S}^{d-1})$, i.e.,

$$(Y_\ell^n,Y_\iota^m)_{\mathbb{S}^{d-1}}=\delta_{n,m}\delta_{\ell,\iota}, \quad \ell\in\Xi_n^d, \ \iota\in\Xi_m^d, \ n,m\in\mathbb{N}_0.$$

For $\alpha > -1$, ball functions are orthogonal with respect to the weight function $\omega^\alpha(x) := \left(1-\|x\|^2\right)^\alpha$,

$$\left(P_{k,\ell}^{\alpha,n}, P_{j,l}^{\alpha,m}\right)_{\omega^\alpha,\mathbb{B}^d} = \delta_{n,m}\delta_{\ell,l}\delta_{k,j}\gamma_k^{\alpha,n}, \quad \ell\in\Xi_n^d,\ \iota\in\Xi_m^d,\ k,j,m,n\in\mathbb{N}_0,$$

where

$$\gamma_k^{\alpha,n} = \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta_n+1)}{2(2k+\alpha+\beta_n+1)\Gamma(k+1)\Gamma(k+\alpha+\beta_n+1)}.$$

Then we define

$$P_{k,\ell}^n(x) := \frac{2k+\beta_n}{k+\beta_n} P_{k,\ell}^{-1,n}(x), \quad x\in\mathbb{B}^d,\ \ell\in\Xi_n^d,\ k,n\in\mathbb{N}_0.$$

2.3 Finite element space

First, we introduce the finite element partition which has hanging nodes. To provide a clearer description, we will consider the two dimensional case as an example. As shown in Fig. 1, we divide the domain $\Omega\subset\mathbb{R}^2$ into three parts: Ω_1, $\Omega_2 = \cup_{i=1}^4 \Omega_2^{(i)} = [-a,a]^2 \setminus \Omega_1$ and Ω_3. Ω_1 is a circle with a radius R, Ω_2 consists of four quadrilaterals, and the remaining domain Ω_3 is composed of multiple simplex elements. Similarly, if $\Omega\subset\mathbb{R}^3$, Ω_1 is a ball and $\Omega_2 = [-a,a]^3\setminus\Omega_1$ consists of six curved hexahedra, the remaining domain Ω_3 is composed of multiple simplex elements. Let Γ_R and Γ be the interfaces between Ω_1 and Ω_2, and between Ω_2 and Ω_3, respectively. We assume that the origin O is the center of Ω_1.

Let $\mathcal{T}_h^\delta = \{K_i\}_{i\in\mathcal{I}}$, $\mathcal{I} = \{1,2,\cdots,5\}$ be a finite element partition of subdomain $\Omega_s = \Omega_1\cup\Omega_2$ satisfying the regularity in [19]. To address the singularity at the origin, the circle ($d=2$) or ball ($d=3$) $K_1 = \mathbb{B}_R^d$ should be included in $\mathcal{T}_h^\delta$, away from the boundary $\partial\Omega$. To

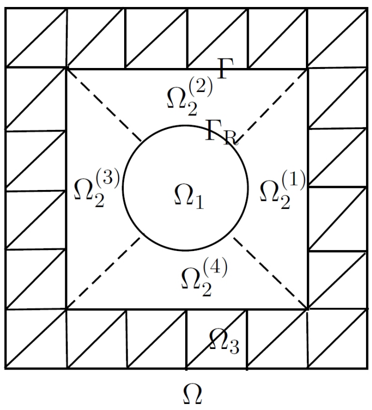

Figure 1: Domain decomposition of Ω.

better fit the mesh, we opt for four curved quadrilateral elements ($d=2$) or six curved hexahedral elements ($d=3$) around K_1, where $K_{i+1}=\Omega_2^{(i)}$, for $i=1,\cdots,4$. The physical elements $K_i\in\mathcal{T}_h^\delta$, for $i\in\mathcal{I}$, can be transformed to the reference element $\hat{K}$ by the following mapping. Let $\Phi_K:\hat{K}\to K$ be infinitely differentiable, i.e., $x=\Phi_K(\hat{x})$. Denote the Jacobi matrix of the transformation Φ_K and its determinant

$$F_K(\hat{x}):=\left(\frac{\partial\Phi_{K,i}}{\partial_{\hat{x}_j}}(\hat{x}_j)\right)_{1\le i,j\le d},\quad J_K(\hat{x}):=\det(F_K(\hat{x})).$$

The local mesh size $h_K(x)$ for $x=\Phi_K(\hat{x})$ is defined as follows

$$h_K(x):=\left\|\left[F_K\circ\Phi_K^{-1}\right](x)\right\|,\quad h_K:=\sup_{x\in K}h_K(x).$$

For simplicity, we write h_{K_i} as h_i. It is obvious that h_1 is actually the radius of the element K_1. For the other K_i, $i\in\mathcal{I}\backslash\{1\}$, the mesh size is equal, so we denote $h_2=h_{K_i}$, $i\in\mathcal{I}\backslash\{1\}$ and $h_\delta=\max\{h_1,h_2\}$.

In the remaining domain Ω_3, we use simplex elements. Let $\mathcal{T}_h^w$ be a finite element partition of Ω_3 satisfying the regularity in [33]. For each element $T\in\mathcal{T}_h^w$, we use h_T to represent the diameter of T. Denote by $h_3=\max_{T\in\mathcal{T}_h^w}h_T$ the mesh size of $\mathcal{T}_h^w$. To demonstrate the flexibility of our method, hanging nodes are allowed on the coupled interface, so we need h_3 to be fine enough, i.e., $h_3\ll h_\delta$. Let $\mathcal{E}_h$ be the set of all edges or faces in $\mathcal{T}_h^w$. In addition, $\mathcal{E}_h^0=\mathcal{E}_h\backslash\partial\Omega_3$ is the set of all interior edges or faces in $\mathcal{T}_h^w$. We denote $\mathcal{E}_h\backslash\Gamma$ as the set of all edges or faces in $\mathcal{T}_h^w$ that do not intersect with Γ, and we denote $\mathcal{E}_h\cap\Gamma$ as the set of all edges or faces in $\mathcal{T}_h^w$ that intersect with Γ.

Denote by P_N and Q_N the sets of polynomials with a total degree not exceeding N and with separate degrees not exceeding N, respectively. Let C be a positive constant independent of the mesh sizes and degrees of polynomials.

To construct finite element space, we first define the basis functions of the MSEM on the reference element. For $N\in\mathbb{N}_0$, define

$$\widehat{W}_N(\hat{K}):=\begin{cases}\operatorname{span}\left\{P_{k,\ell}^n:\ell\in\Xi_n^d,\,0\le k,n,\,2k+n\le N\right\}, & \hat{K}=\mathbb{B}^d,\\ Q_N(\hat{K}), \quad \hat{K}=[-1,1]^d.\end{cases}$$

Therefore, the basis function can be mapped from the reference element:

$$W_N(K):=\operatorname{span}\left\{\varphi:\varphi\circ\Phi_K\in\widehat{W}_N(\hat{K})\right\}.$$

The approximation space in Ω_2 is given by

$$X_{\delta,2}:=\operatorname{span}\left\{v_\delta\in C^0(\Omega_2):v_\delta|_{K_i}\in W_{N_2}(K_i),\,i\in\mathcal{I}\backslash\{1\}\right\},\quad N_2\in\mathbb{N}_0.$$

Next, we introduce the finite element space in Ω_s:

$$X_\delta^*:=\left\{v_\delta:v_\delta|_{K_1}\in W_{N_1}(K_1),\,v_\delta|_{\Omega_2}\in X_{\delta,2}\right\},\quad N_1\in\mathbb{N}_0.$$

Because $v_\delta \in X_\delta^*$ does not ensure any continuity across ∂K_1, additional conditions must be imposed. Let Γ_R^+ and Γ_R^- be the inner and outer surfaces of K_1, respectively. We select Γ_R^+ as the non-mortar boundary, then the non-mortar space is the restriction of X_δ^* on Γ_R^+:

$$V_{N_1} = \left\{ v_\delta|_{\Gamma_R^+},\ v_\delta \in X_\delta^* \right\}.$$

In fact, V_{N_1} is the set of spherical harmonics defined on Γ_R with degree no greater than N_1. Finally, the mortar approximation space in Ω_s is defined by

$$X_\delta = \left\{ v_\delta \in X_\delta^* : \int_{\Gamma_R} \left(v_\delta|_{\Gamma_R^+} - v_\delta|_{\Gamma_R^-} \right) \cdot \psi d\sigma(x) = 0,\ \forall \psi \in V_{N_1} \right\},$$

where $d\sigma(x)$ is the surface measure on Γ_R. The integral equation that v_δ satisfies is called the mortar condition.

Next, we introduce the WG space in Ω_3. We define the WG space V_w as

$$V_w = \{ v_w = \{v_0, v_b\} : v_0|_T \in P_{N_3}(T),\ T \in \mathcal{T}_h^w,\ v_b|_e \in P_{N_3}(e),\ e \in \mathcal{E}_h \setminus \Gamma,\ v_b|_e \in P_{N_2}(e),\ e \in \mathcal{E}_h \cap \Gamma \},$$

where $N_3 \in \mathbb{N}$.

Definition 2.1. For $v_w \in V_w$, its weak gradient $\nabla_w v_w|_T \in [P_{N_3-1}(T)]^d$ is defined as follows:

$$(\nabla_w v_w, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{q} \in [P_{N_3-1}(T)]^d, \tag{2.4}$$

where $\mathbf{n}$ is the outward unit normal vector.

With the above preparations, assume $N_1 = N_2 \gg N_3$ (such as $N_1 = N_2 = 15$, $N_3 \le 6$), we can introduce the coupled finite element space V_h in Ω:

$$V_h = \{ v_h = \{v_\delta, v_w\} \in X_\delta \times V_w,\ v_b|_\Gamma = v_\delta|_\Gamma,\ v_b = 0 \text{ on } \partial\Omega \}.$$

2.4 Numerical scheme

First, we define some bilinear forms over X_δ and V_w. For any $u_\delta, v_\delta \in X_\delta$,

$$\begin{aligned} a_\delta(u_\delta, v_\delta) &= \sum_{i \in \mathcal{I}} \left((\nabla u_\delta, \nabla v_\delta)_{K_i} + c^2 (u_\delta, v_\delta)_{r^{-2}, K_i} \right), \\ b_\delta(u_\delta, v_\delta) &= \sum_{i \in \mathcal{I}} (u_\delta, v_\delta)_{K_i}. \end{aligned}$$

And we denote $\|v_\delta\|_{X_\delta}^2 = a_\delta(v_\delta, v_\delta)$. For any $u_w, v_w \in V_w$,

$$\begin{aligned} a_w(u_w, v_w) &= \sum_{T \in \mathcal{T}_h^w} (\nabla_w u_w, \nabla_w v_w)_T + \sum_{T \in \mathcal{T}_h^w} c^2 (u_0, v_0)_{r^{-2}, T} + s(u_w, v_w), \\ s(u_w, v_w) &= \rho \sum_{T \in \mathcal{T}_h^w} h_T^{-1} N_3^2 \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T}, \\ b_w(u_w, v_w) &= \sum_{T \in \mathcal{T}_h^w} (u_0, v_0)_T, \end{aligned}$$

where $0<\rho\leq 1$ is a constant. In the following, we omit ρ in $s(\cdot,\cdot)$ because it is a constant and does not affect our theoretical analysis.

Now we define the bilinear forms over V_h. For any $u_h,v_h\in V_h$,

$$a_h(u_h,v_h)=a_\delta(u_\delta,v_\delta)+a_w(u_w,v_w),$$
$$b_h(u_h,v_h)=b_\delta(u_\delta,v_\delta)+b_w(u_w,v_w).$$

Next, we give the numerical scheme for the Schrödinger eigenvalue problem (2.1): Find $\lambda_h\in\mathbb{R}$ and $u_h\in V_h\setminus\{0\}$ such that $b_h(u_h,u_h)=1$ and

$$a_h(u_h,v_h)=\lambda_h b_h(u_h,v_h),\quad \forall v_h\in V_h. \tag{2.5}$$

3 Error analysis for eigenvalue problem

In this section, we conduct the numerical analysis of the Schrödinger eigenvalue problem.

3.1 V-norm and projection operators

In order to unify the norm over the $\mathscr{H}_0^1(\Omega)$ and V_h spaces, we define the following inner product over $V=\mathscr{H}_0^1(\Omega)+V_h$. For any $v,w\in V$,

$$(v,w)_V=a_\delta(v_\delta,w_\delta)+\sum_{T\in\mathcal{T}_h^w}\Big((\nabla v_0,\nabla w_0)_T+c^2(v_0,w_0)_{r^{-2},T}+h_T^{-1}N_3^2\langle v_0-v_b,w_0-w_b\rangle_{\partial T}\Big),$$

where v_δ represents the value on the interior of Ω_s and v_0 and v_b represent the values on the interior and boundary of cell $T\in\mathcal{T}_h^w$, respectively. The corresponding norm is:

$$\|v\|_V^2=\|v_\delta\|_{X_\delta}^2+\sum_{T\in\mathcal{T}_h^w}\|\nabla v_0\|_T^2+\sum_{T\in\mathcal{T}_h^w}c^2\|v_0\|_{r^{-2},T}^2+\sum_{T\in\mathcal{T}_h^w}h_T^{-1}N_3^2\|v_0-v_b\|_{\partial T}^2,\quad \forall v\in V.$$

Next, we define projection $\Pi_\delta:\mathscr{H}^1(\Omega_s)\to X_\delta$. In order to achieve higher-order convergence, we need to assume $v|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, where $s_1\geq 1$. The specific definitions of Π_δ and $\mathscr{H}^{s_1}(\Omega_1)$ can be found in Section 3 of [19]. According to [19, Theorem 3.4], we have the following estimate.

Lemma 3.1. *Assume $u\in\mathscr{H}_0^1(\Omega)$, $u|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\geq 1$, $u|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\geq 1$, there holds*

$$\|u-\Pi_\delta u\|_{X_\delta}\leq C\Big(h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{1-s_2}\|u\|_{H^{s_2}(\Omega_2)}\Big),$$

where $\mu_2=\min\{s_2,N_2+1\}$.

Let Q_0 be the L^2 orthogonal projection operator from $L^2(T)$ onto $P_{N_3}(T)$. For $e\in\mathcal{E}_h\setminus\Gamma$, let Q_b be the L^2 orthogonal projection operator from $L^2(e)$ onto $P_{N_3}(e)$. In particular, for $e\in\Gamma\cap\mathcal{E}_h$, we define $Q_bv|_e=\Pi_\delta v|_e\in P_{N_2}(e)$, and in order for $\Pi_\delta v$ to make sense, we need $v\in\mathscr{H}^1(\Omega)$. Let $\mathbb{Q}_h$ be the L^2 orthogonal projection operator from $[L^2(T)]^d$ onto $[P_{N_3-1}(T)]^d$. Combining Q_0 and Q_b, let $Q_h=\{Q_0,Q_b\}$ be the projection onto V_w.

The following estimates of Π_δ and Q_0 are crucial for the analysis and can be found in [12,30].

Lemma 3.2. *For $\Pi_\delta|_{\Omega_2}$, assume $u\in\mathscr{H}_0^1(\Omega)$, $u|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\geq 1$, we have*

$$\frac{N_2}{h_\delta}\|u-\Pi_\delta u\|_{\Omega_2}+\sqrt{\frac{N_2}{h_\delta}}\|u-\Pi_\delta u\|_{\partial\Omega_2}+\|\nabla(u-\Pi_\delta u)\|_{\Omega_2}\leq Ch_\delta^{\mu_2-1}N_2^{1-s_2}\|u\|_{s_2,\Omega_2},$$

where $\mu_2=\min\{N_2+1,s_2\}$. Similarly, for any $T\in\mathcal{T}_h^w$, assume that $u\in H^{s_3}(T)$, $s_3\geq 1$, there holds

$$\frac{N_3}{h_T}\|u-Q_0u\|_T+\sqrt{\frac{N_3}{h_T}}\|u-Q_0u\|_{\partial T}+\|\nabla(u-Q_0u)\|_T\leq Ch_T^{\mu_3-1}N_3^{1-s_3}\|u\|_{s_3,T},$$

where $\mu_3=\min\{N_3+1,s_3\}$.

Let T_Γ be the cell in $\mathcal{T}_h^w$ that has common edge with the interface Γ. We use $\mathcal{T}_\Gamma^w$ to denote the set of all cells T_Γ. For any cell $T\in\mathcal{T}_h^w\backslash\mathcal{T}_\Gamma^w$, we have the following commutativity of projection operators which can be found in [28, Lemma 5.1].

Lemma 3.3. *For each cell $T\in\mathcal{T}_h^w\backslash\mathcal{T}_\Gamma^w$, there holds*

$$\nabla_w Q_h v=\mathbb{Q}_h(\nabla v),\quad \forall v\in H^1(T).$$

But for $T_\Gamma\in\mathcal{T}_\Gamma^w$, the commutativity of projection operators above no longer holds.

Lemma 3.4. *For any cell $T_\Gamma\in\mathcal{T}_\Gamma^w$ and $v\in\mathscr{H}^1(\Omega)$, the following equation holds*

$$(\nabla_w Q_h v,\mathbf{q})_{T_\Gamma}=(\mathbb{Q}_h(\nabla v),\mathbf{q})_{T_\Gamma}+\langle\Pi_\delta v-v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\cap\Gamma},\quad \forall\mathbf{q}\in[P_{N_3-1}(T_\Gamma)]^d.$$

Proof. It follows from (2.4) that

$$\begin{aligned}(\nabla_w Q_h v,\mathbf{q})_{T_\Gamma}&=-(Q_0v,\nabla\cdot\mathbf{q})_{T_\Gamma}+\langle Q_bv,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma}\\&=-(v,\nabla\cdot\mathbf{q})_{T_\Gamma}+\langle v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\backslash(\partial T_\Gamma\cap\Gamma)}+\langle Q_bv,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\cap\Gamma}\\&=-(v,\nabla\cdot\mathbf{q})_{T_\Gamma}+\langle v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma}+\langle Q_bv-v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\cap\Gamma}\\&=(\nabla v,\mathbf{q})_{\partial T_\Gamma}+\langle Q_bv-v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\cap\Gamma}\\&=(\mathbb{Q}_h(\nabla v),\mathbf{q})_{T_\Gamma}+\langle Q_bv-v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\cap\Gamma}\\&=(\mathbb{Q}_h(\nabla v),\mathbf{q})_{T_\Gamma}+\langle\Pi_\delta v-v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_\Gamma\cap\Gamma}.\end{aligned}$$

Then we complete the proof. □

In addition, the following trace inequality holds.

Lemma 3.5. [12] *For any cell T, and $\phi\in P_N(T)$, there holds*

$$\|\phi\|_{\partial T}\leq Ch_T^{-1/2}N\|\phi\|_T.$$

3.2 Error analysis

For further error analysis, we need to verify that $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ are continuous and coercive with respect to the norm $\|\cdot\|_V$. Obviously, $a(\cdot,\cdot)$ is continuous and coercive over $\mathscr{H}_0^1(\Omega)$. From the definition of norm $\|\cdot\|_V$, we can see that $a_h(\cdot,\cdot)$ is continuous. It follows from [38, Lemma 3.2] that $a_h(\cdot,\cdot)$ is coercive over V_h.

We define the following operators $K: L^2(\Omega) \to \mathscr{H}_0^1(\Omega)$ and $K_h: V_h \to V_h$ such that

$$\begin{aligned} &a(Kf,v) = b(f,v), \quad \forall v \in \mathscr{H}_0^1(\Omega), \\ &a_h(K_h f_h, v_h) = b_h(f_h, v_h), \quad \forall v_h \in V_h. \end{aligned}$$

It follows from the Lax-Milgram Theorem that K and K_h are well-defined. We note that K is compact in $\mathscr{H}_0^1(\Omega)$. In fact, $\mathscr{H}_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, and $K: L^2(\Omega) \to \mathscr{H}_0^1(\Omega)$ is a continuous linear operator. Notice that K_h is a continuous linear and finite ranked operator, then K_h is compact in V_h.

In addition, we define the orthogonal projection operators $\Pi_c: V \to \mathscr{H}_0^1(\Omega)$ and $\Pi_h: V \to V_h$ with respect to the inner product $(\cdot,\cdot)_V$. According to Section 2 of [38] and [3, Theorem 7.1], to estimate the H^1 errors of eigenfunctions, we need only estimate $\|(K\Pi_c - K_h\Pi_h)|_{R(E_\mu(K))}\|_V = \|(K - K_h\Pi_h)|_{R(E_\mu(K))}\|_V$, where $R(E_\mu(K))$ is the generalized eigenfunction space corresponding to the eigenvalue μ of the operator K. Similarly, we define the orthogonal projection operator $\Pi_0: V \to L^2(\Omega)$ with respect to the inner product $b(\cdot,\cdot)$. To estimate the L^2 errors of eigenfunctions, it is sufficient to estimate $\|(\Pi_0 K - \Pi_0 K_h)|_{R(E_\mu(K))}\|$.

We consider the following variational form of source problem (2.3): Find $u \in \mathscr{H}_0^1(\Omega)$ and $Kf \in \mathscr{H}_0^1(\Omega)$ such that

$$\begin{cases} a(u,v) = b(f,v), & \forall v \in \mathscr{H}_0^1(\Omega), \\ a(Kf,v) = b(f,v), & \forall v \in \mathscr{H}_0^1(\Omega). \end{cases} \tag{3.1}$$

The corresponding numerical scheme is: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = b_h(f, v_h), \quad \forall v_h \in V_h. \tag{3.2}$$

Next, let's replace f with $\Pi_h f$. The corresponding numerical scheme is: Find $\widetilde{u}_h \in V_h$ and $K_h \Pi_h f \in V_h$ such that

$$\begin{cases} a_h(\widetilde{u}_h, v_h) = b_h(\Pi_h f, v_h), & \forall v_h \in V_h, \\ a_h(K_h \Pi_h f, v_h) = b_h(\Pi_h f, v_h), & \forall v_h \in V_h. \end{cases} \tag{3.3}$$

Then $\|(K - K_h\Pi_h)|_{R(E_\mu(K))}\|_V \le C\|u - \widetilde{u}_h\|_V \le C(\|u - u_h\|_V + \|u_h - \widetilde{u}_h\|_V)$. Notice that $\|u - u_h\|_V$ is the H^1 error estimate for source problem (2.3). Since $a_h(\cdot,\cdot)$ is coercive, we have

$$\begin{aligned} \|u_h - \widetilde{u}_h\|_V^2 &\le C a_h(u_h - \widetilde{u}_h, u_h - \widetilde{u}_h) \\ &= C b_h(f - \Pi_h f, u_h - \widetilde{u}_h) \\ &\le C\|f - \Pi_h f\|_V \|u_h - \widetilde{u}_h\|_V, \end{aligned}$$

which implies that $\|u_h-\widetilde{u}_h\|_V\le C\|f-\Pi_h f\|_V$.

Define the projection operator $I_h=\{\Pi_\delta,Q_h\}:\mathscr{H}_0^1(\Omega)\to V_h$. Since Π_h is orthogonal projection, then

$$\|u_h-\widetilde{u}_h\|_V\le C\|f-\Pi_h f\|_V\le C\|f-I_h f\|_V.$$

For $\|f-I_h f\|_V$, we have the following estimate.

Lemma 3.6. *Assume $f\in\mathscr{H}_0^1(\Omega)$, and $f|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge 1$, $f|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge 3/2$, $f|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge 3/2$, if $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^2$, there holds*

$$\|f-I_hf\|_V\le C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|f\|_{H^{s_2}(\Omega_2)}+h_3^{\mu_3-1}N_3^{3/2-s_3}\|f\|_{H^{s_3}(\Omega_3)}\right),$$

where $\mu_2=\min\{s_2,N_2+1\}$ and $\mu_3=\min\{s_3,N_3+1\}$. Furthermore, we have

$$\|f-I_hf\|_V\le Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|f\|_{H^{s_2}(\Omega_2)}+\|f\|_{H^{s_3}(\Omega_3)}\right).$$

Proof. It follows from Lemma 3.2 that

$$\begin{aligned}\|f-Q_hf\|_V^2=&\sum_{T\in\mathcal{T}_h^w}\left(\|\nabla(f-Q_0f)\|_T^2+c^2\|f-Q_0f\|_{r^{-2},T}^2\right)\\&+\sum_{T\in\mathcal{T}_h^w}\left(h_3^{-1}N_3^2\|Q_0f-Q_bf\|_{\partial T\backslash(\partial T\cap\Gamma)}^2+h_3^{-1}N_3^2\|Q_0f-\Pi_\delta f\|_{\partial T\cap\Gamma}^2\right)\\&\le C\left(h_3^{2(\mu_3-1)}N_3^{3-2s_3}\|f\|_{H^{s_3}(\Omega_3)}^2+h_3^{-1}N_3^2h_\delta^{2\mu_2-1}N_2^{1-2s_2}\|f\|_{H^{s_2}(\Omega_2)}^2\right).\end{aligned}$$

If $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^2$, then we have

$$\|f-Q_hf\|_V^2\le C\left(h_\delta^{2(\mu_2-1)}N_2^{3-2s_2}\|f\|_{H^{s_2}(\Omega_2)}^2+h_3^{2(\mu_3-1)}N_3^{3-2s_3}\|f\|_{H^{s_3}(\Omega_3)}^2\right).$$

It follows from Lemma 3.1 that

$$\begin{aligned}&\|f-I_hf\|_V\\=&\|f-\Pi_\delta f\|_V+\|f-Q_hf\|_V\\\le& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|f\|_{H^{s_2}(\Omega_2)}+h_3^{\mu_3-1}N_3^{3/2-s_3}\|f\|_{H^{s_3}(\Omega_3)}\right).\end{aligned}$$

Notice that we assume $N_1=N_2\gg N_3$, so the main error comes from the WG element, then we have

$$\|f-I_hf\|_V\le Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|f\|_{H^{s_2}(\Omega_2)}+\|f\|_{H^{s_3}(\Omega_3)}\right).$$

Then we complete the proof. □

Similarly, we estimate $\|(\Pi_0 K-\Pi_0 K_h)|_{R(E_\mu(K))}\|$. We replace $\Pi_h f$ with $I_h f$ in (3.3):

$$\begin{cases} a_h(\bar{u}_h,v_h)=b_h(I_hf,v_h), & \forall v_h\in V_h,\\ a_h(K_hI_hf,v_h)=b_h(I_hf,v_h), & \forall v_h\in V_h. \end{cases} \tag{3.4}$$

Combining (3.2) and (3.4), and $a_h(\cdot,\cdot)$ is coercive, we have

$$\begin{aligned} \|u_h-\bar{u}_h\|_V^2 &\le Ca_h(u_h-\bar{u}_h,u_h-\bar{u}_h)\\ &=Cb_h(f-I_hf,u_h-\bar{u}_h)\\ &\le C\|f-I_hf\|\|u_h-\bar{u}_h\|_V, \end{aligned}$$

which implies that $\|u_h-\bar{u}_h\|\le C\|u_h-\bar{u}_h\|_V\le C\|f-I_hf\|$. Then

$$\begin{aligned} \|(\Pi_0K-\Pi_0K_h)|_{R(E_\mu(K))}\| &\le \sup_{\substack{\|f\|=1,\\ f\in R(E_\mu(K))}}\|Kf-K_hI_hf\|+\sup_{\substack{\|f\|=1,\\ f\in R(E_\mu(K))}}\|K_hI_hf-K_hf\|\\ &\le C(\|u-\bar{u}_h\|+\|f-I_hf\|)\\ &\le C(\|u-u_h\|+\|u_h-\bar{u}_h\|+\|f-I_hf\|)\\ &\le C(\|u-u_h\|+\|f-I_hf\|). \end{aligned}$$

Notice that $\|u-u_h\|$ is the L^2 error estimate for source problem (2.3). For $\|f-I_hf\|$, the following estimate holds.

Lemma 3.7. *Assume $f\in\mathscr{H}_0^1(\Omega)$, and $f|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge 1$, $f|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge 3/2$, $f|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge 3/2$, there holds*

$$\|f-I_hf\|\le C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{1-s_2}\|f\|_{H^{s_2}(\Omega_2)}+h_3^{\mu_3}N_3^{-s_3}\|f\|_{H^{s_3}(\Omega_3)}\right),$$

where $\mu_2=\min\{s_2,N_2+1\}$ and $\mu_3=\min\{s_3,N_3+1\}$. Furthermore, we have

$$\|f-I_hf\|\le Ch_3^{\mu_3}N_3^{-s_3}\left(\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|f\|_{H^{s_2}(\Omega_2)}+\|f\|_{H^{s_3}(\Omega_3)}\right).$$

Proof. It follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned} &\|f-I_hf\|\\ =&\|f-\Pi_\delta f\|+\|f-Q_hf\|\\ \le& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{1-s_2}\|f\|_{H^{s_2}(\Omega_2)}+h_3^{\mu_3}N_3^{-s_3}\|f\|_{H^{s_3}(\Omega_3)}\right). \end{aligned}$$

Notice that $N_1=N_2\gg N_3$, so the main error comes from the WG element, then we have

$$\|f-I_hf\|\le Ch_3^{\mu_3}N_3^{-s_3}\left(\|f\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|f\|_{H^{s_2}(\Omega_2)}+\|f\|_{H^{s_3}(\Omega_3)}\right),$$

which completes the proof. □

Based on the above analysis, it is sufficient to estimate the H^1 and L^2 errors for the source problem (2.3). We will elaborate on this in the next section.

4 Error estimates for source problem

In this section, we derive the error equation of the source problem (2.3). Subsequently, we obtain the H^1 and L^2 error estimates for the source problem (2.3).

4.1 Error equation

Let $u\in\mathscr{H}_0^1(\Omega)$ be the exact solution of (2.3), and $u_h=\{u_\delta,u_w\}\in V_h$ be the numerical solution of (3.2). Denote $e_\delta=u|_{\Omega_s}-u_\delta$, $\widetilde{e}_\delta=\Pi_\delta u-u_\delta$, and $e_w=\{e_0,e_b\}=Q_hu-u_w=\{Q_0u-u_0,Q_bu-u_b\}$. In the following, we use $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ to represent the outward unit normal vectors of cells in Ω_1, Ω_2 and Ω_3, respectively. Denote $[v_\delta]=v_\delta|_{\Gamma_R^+}-v_\delta|_{\Gamma_R^-}$.

Lemma 4.1. *Let u be the exact solution of* (2.3)*. For any $v_h=\{v_\delta,v_w\}\in V_h$, the following error equation holds*

$$\begin{aligned}a_\delta(e_\delta,v_\delta)+a_w(e_w,v_w)=&\langle\nabla u\cdot\mathbf{n}_1,[v_\delta]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,v_0-v_b\rangle_{\partial T}\\&+\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w v_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+s(Q_hu,v_w).\end{aligned}\tag{4.1}$$

Proof. For any $\mathbf{q}\in[P_{N_3-1}(T)]^d$, it follows from Lemma 3.4 that

$$\sum_{T\in\mathcal{T}_h^w}(\nabla_wQ_hu,\mathbf{q})_T=\sum_{T\in\mathcal{T}_h^w}(\mathbb{Q}_h\nabla u,\mathbf{q})_T-\sum_{T\in\mathcal{T}_h^w}\langle u-\Pi_\delta u,\mathbf{q}\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}.$$

Let $\mathbf{q}=\nabla_w v_w$, we have

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h^w}(\nabla_wQ_hu,\nabla_wv_w)_T\\=&\sum_{T\in\mathcal{T}_h^w}(\mathbb{Q}_h\nabla u,\nabla_wv_w)_T-\sum_{T\in\mathcal{T}_h^w}\langle u-\Pi_\delta u,\nabla_wv_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\\=&-\sum_{T\in\mathcal{T}_h^w}(v_0,\nabla\cdot\mathbb{Q}_h\nabla u)_T+\sum_{T\in\mathcal{T}_h^w}\langle v_b,\mathbb{Q}_h\nabla u\cdot\mathbf{n}_3\rangle_{\partial T}-\sum_{T\in\mathcal{T}_h^w}\langle u-\Pi_\delta u,\nabla_wv_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\\=&\sum_{T\in\mathcal{T}_h^w}(\nabla v_0,\mathbb{Q}_h\nabla u)_T-\sum_{T\in\mathcal{T}_h^w}\langle v_0-v_b,\mathbb{Q}_h\nabla u\cdot\mathbf{n}_3\rangle_{\partial T}-\sum_{T\in\mathcal{T}_h^w}\langle u-\Pi_\delta u,\nabla_wv_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\\=&\sum_{T\in\mathcal{T}_h^w}(\nabla v_0,\nabla u)_T-\sum_{T\in\mathcal{T}_h^w}\langle v_0-v_b,\mathbb{Q}_h\nabla u\cdot\mathbf{n}_3\rangle_{\partial T}-\sum_{T\in\mathcal{T}_h^w}\langle u-\Pi_\delta u,\nabla_wv_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}.\end{aligned}\tag{4.2}$$

Notice that $\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,v_b\rangle_{\partial T}=\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,v_b\rangle_{\partial T\cap\Gamma}=-\langle\nabla u\cdot\mathbf{n}_2,v_\delta\rangle_\Gamma$. Testing (2.3) with $v_h=\{v_\delta,v_w\}$, then we have

$$\begin{aligned}&(\nabla u,\nabla v_\delta)_{\Omega_s}+c^2(u,v_\delta)_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla v_0)_T+\sum_{T\in\mathcal{T}_h^w}c^2(u,v_0)_{r^{-2},T}\\&-\langle\nabla u\cdot\mathbf{n}_1,[v_\delta]\rangle_{\Gamma_R}-\langle\nabla u\cdot\mathbf{n}_2,v_\delta\rangle_\Gamma-\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,v_0\rangle_{\partial T}\\=&(\nabla u,\nabla v_\delta)_{\Omega_s}+c^2(u,v_\delta)_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla v_0)_T+\sum_{T\in\mathcal{T}_h^w}c^2(u,v_0)_{r^{-2},T}\\&-\langle\nabla u\cdot\mathbf{n}_1,[v_\delta]\rangle_{\Gamma_R}-\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,v_0-v_b\rangle_{\partial T}=(f,v_\delta)_{\Omega_s}+(f,v_0)_{\Omega_3}.\end{aligned}\tag{4.3}$$

Substituting (4.2) into (4.3), the following equation holds

$$\begin{aligned}&(\nabla u,\nabla v_\delta)_{\Omega_s}+c^2(u,v_\delta)_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}(\nabla_w Q_h u,\nabla_w v_w)_T+\sum_{T\in\mathcal{T}_h^w}c^2(u,v_0)_{r^{-2},T}\\&-\langle\nabla u\cdot\mathbf{n}_1,[v_\delta]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}\langle(\mathbb{Q}_h\nabla u-\nabla u)\cdot\mathbf{n}_3,v_0-v_b\rangle_{\partial T}+\sum_{T\in\mathcal{T}_h^w}\langle u-\Pi_\delta u,\nabla_w v_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\\=&(f,v_\delta)_{\Omega_s}+(f,v_0)_{\Omega_3}.\end{aligned}$$

Then there holds

$$\begin{aligned}&a_\delta(u,v_\delta)+a_w(Q_hu,v_w)\\=&(\nabla u,\nabla v_\delta)_{\Omega_s}+c^2(u,v_\delta)_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}(\nabla_w Q_hu,\nabla_w v_w)_T+\langle\nabla u\cdot\mathbf{n}_1,[v_\delta]\rangle_{\Gamma_R}\\&+\sum_{T\in\mathcal{T}_h^w}c^2(Q_0u,v_0)_{r^{-2},T}+s(Q_hu,v_w)+\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w v_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\\=&(f,v_\delta)_{\Omega_s}+(f,v_0)_{\Omega_3}+\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,v_0-v_b\rangle_{\partial T}+s(Q_hu,v_w),\end{aligned}\tag{4.4}$$

where we have used the fact that $\sum_{T\in\mathcal{T}_h^w}c^2(Q_0u-u,v_0)_{r^{-2},T}=0$.

Subtracting (3.2) from (4.4), we have

$$\begin{aligned}a_\delta(e_\delta,v_\delta)+a_w(e_w,v_w)=&\langle\nabla u\cdot\mathbf{n}_1,[v_\delta]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,v_0-v_b\rangle_{\partial T}\\&+\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w v_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+s(Q_hu,v_w).\end{aligned}$$

Then we complete the proof. □

4.2 H^1 error estimate

In this subsection, we derive the H^1 error estimate for the source problem (2.3).

Lemma 4.2. *Let u be the exact solution of* (2.3). *Assume $u\in\mathscr{H}_0^1(\Omega)$, and $u|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\geq 1$, $u|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\geq 3/2$, $u|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\geq 3/2$, if $h_3^{-1}N_3^2\leq Ch_\delta^{-1}N_2^2$, there holds*

$$\begin{aligned}&\|\widetilde{e}_\delta\|_V+\|e_w\|_V\\ \leq& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}+h_3^{\mu_3-1}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\right),\end{aligned}$$

where $\mu_2=\min\{s_2,N_2+1\}$ and $\mu_3=\min\{s_3,N_3+1\}$. Furthermore, we have

$$\|\widetilde{e}_\delta\|_V+\|e_w\|_V\leq Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u\|_{H^{s_2}(\Omega_2)}+\|u\|_{H^{s_3}(\Omega_3)}\right).$$

Proof. Let $v_\delta=\widetilde{e}_\delta$ and $v_w=e_w$ in (4.1), then there holds

$$\begin{aligned}a_\delta(\widetilde{e}_\delta,\widetilde{e}_\delta)+a_w(e_w,e_w)=&\sum_{T\in\mathcal{T}_h^w}\left(\langle\Pi_\delta u-u,\nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}\right)\\&+a_\delta(\Pi_\delta u-u,\widetilde{e}_\delta)+\langle\nabla u\cdot\mathbf{n}_1,[\widetilde{e}_\delta]\rangle_{\Gamma_R}+s(Q_hu,e_w).\end{aligned}$$

Next, we estimate the six terms on the right side of the above equation. It follows from Lemma 3.1 that

$$|a_\delta(\Pi_\delta u-u,\widetilde{e}_\delta)|\leq C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+h_\delta^{\mu_2-1}N_2^{1-s_2}\|u\|_{H^{s_2}(\Omega_2)}\right)\|\widetilde{e}_\delta\|_V. \tag{4.5}$$

According to [19, Theorem 3.3], there holds

$$|\langle\nabla\cdot\mathbf{n}_1,[\widetilde{e}_\delta]\rangle_{\Gamma_R}|\leq Ch_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}\|\widetilde{e}_\delta\|_V. \tag{4.6}$$

From Lemma 3.2, we have

$$\begin{aligned}&|s(Q_hu,e_w)|\\=&\left|\sum_{T\in\mathcal{T}_h^w}h_T^{-1}N_3^2\langle Q_0u-Q_bu,e_0-e_b\rangle_{\partial T\setminus(\partial T\cap\Gamma)}+\sum_{T\in\mathcal{T}_h^w}h_T^{-1}N_3^2\langle Q_0u-\Pi_\delta u,e_0-e_b\rangle_{\partial T\cap\Gamma}\right|\\ \leq& C\left(h_T^{-1/2}N_3h_\delta^{\mu_2-1/2}N_2^{1/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}\|e_w\|_V+h_T^{\mu_3-1}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\|e_w\|_V\right).\end{aligned}$$

If $h_3^{-1}N_3^2\leq Ch_\delta^{-1}N_2^2$, then we have

$$|s(Q_hu,e_w)|\leq C\left(h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}\|e_w\|_V+h_3^{\mu_3-1}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\|e_w\|_V\right). \tag{4.7}$$

It follows from Lemmas 3.2 and 3.5 that

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right|\leq Ch_T^{-1/2}N_3h_\delta^{\mu_2-1/2}N_2^{1/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}\|e_w\|_V.$$

If $h_3^{-1}N_3^2 \le Ch_\delta^{-1}N_2^2$, then we have

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle \Pi_\delta u - u, \nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right| \le Ch_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}\|e_w\|_V. \tag{4.8}$$

From Lemma 3.2, the following estimate holds

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3, e_0 - e_b\rangle_{\partial T}\right| \le Ch_3^{\mu_3-1}N_3^{1/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\|e_w\|_V. \tag{4.9}$$

Notice that $\|\widetilde{e}_\delta\|_V^2 + \|e_w\|_V^2 \le C(a_\delta(\widetilde{e}_\delta,\widetilde{e}_\delta) + a_w(e_w,e_w))$. It follows from (4.5)-(4.9) and Young's inequality that

$$\begin{aligned}&\|\widetilde{e}_\delta\|_V + \|e_w\|_V\\ \le& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u\|_{H^{s_2}(\Omega_2)} + h_3^{\mu_3-1}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\right).\end{aligned}$$

Notice that $N_1 = N_2 \gg N_3$, so the main error comes from the WG element, then we have

$$\|\widetilde{e}_\delta\|_V + \|e_w\|_V \le Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u\|_{H^{s_2}(\Omega_2)} + \|u\|_{H^{s_3}(\Omega_3)}\right).$$

Then we complete the proof. □

Theorem 4.1. *Let u be the exact solution of (2.3) and $u_h = \{u_\delta, u_w\}$ be the numerical solution of (3.2). Assume $u \in \mathscr{H}_0^1(\Omega)$, and $u|_{\Omega_1} \in \mathscr{H}^{s_1}(\Omega_1)$, $s_1 \ge 1$, $u|_{\Omega_2} \in H^{s_2}(\Omega_2)$, $s_2 \ge 3/2$, $u|_{\Omega_3} \in H^{s_3}(\Omega_3)$, $s_3 \ge 3/2$, if $h_3^{-1}N_3^2 \le Ch_\delta^{-1}N_2^2$, there holds*

$$\begin{aligned}&\|u|_{\Omega_s} - u_\delta\|_V + \|u|_{\Omega_3} - u_w\|_V\\ \le& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u\|_{H^{s_2}(\Omega_2)} + h_3^{\mu_3-1}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\right),\end{aligned}$$

where $\mu_2 = \min\{s_2, N_2+1\}$ and $\mu_3 = \min\{s_3, N_3+1\}$. Furthermore, we have

$$\|u|_{\Omega_s} - u_\delta\|_V + \|u|_{\Omega_3} - u_w\|_V \le Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u\|_{H^{s_2}(\Omega_2)} + \|u\|_{H^{s_3}(\Omega_3)}\right).$$

Proof. It follows from Lemmas 3.1, 3.6 and 4.2 that

$$\begin{aligned}&\|u|_{\Omega_s} - u_\delta\|_V + \|u|_{\Omega_3} - u_w\|_V\\ \le& \|u|_{\Omega_s} - \Pi_\delta u\|_V + \|\widetilde{e}_\delta\|_V + \|u|_{\Omega_3} - Q_h u\|_V + \|e_w\|_V\\ \le& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u\|_{H^{s_2}(\Omega_2)} + h_3^{\mu_3-1}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\right).\end{aligned}$$

Notice that $N_1 = N_2 \gg N_3$, so the main error comes from the WG element, then we have

$$\|u|_{\Omega_s} - u_\delta\|_V + \|u|_{\Omega_3} - u_w\|_V \le Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u\|_{H^{s_2}(\Omega_2)} + \|u\|_{H^{s_3}(\Omega_3)}\right),$$

which completes the proof. □

4.3 L^2 error estimate

In this subsection, we obtain the L^2 error estimate for the source problem (2.3). To obtain the L^2 error estimate, we consider the following dual problem

$$\begin{cases} -\Delta\phi+\dfrac{c^2}{\|x\|^2}\phi=e, & \text{in } \Omega, \\ \phi=0, & \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

where $e=\{e_\delta,e_0\}=\{u|_{\Omega_s}-u_\delta,Q_0u-u_0\}$. And from [21, Theorem 2.3], the following regularity estimate holds

$$\|\phi\|_{\mathscr{H}^2(\Omega_1)}+\|\phi\|_{H^2(\Omega_2)}+\|\phi\|_{H^2(\Omega_3)}\le C\|e\|. \tag{4.11}$$

Lemma 4.3. *Let u be the exact solution of* (2.3) *and ϕ be the exact solution of* (4.10), *we have the following error equation*

$$\begin{aligned}\|e\|^2=&a_\delta(e_\delta,\phi-\Pi_\delta\phi)-\langle\nabla\phi\cdot\mathbf{n}_1,[e_\delta]\rangle_{\Gamma_R}+\langle\nabla\phi\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_\Gamma+s(Q_hu,Q_h\phi)-s(e_w,Q_h\phi)\\&+\langle\nabla u\cdot\mathbf{n}_1,[\Pi_\delta\phi]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}(\langle\phi-\Pi_\delta\phi,\nabla_we_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+\langle\Pi_\delta u-u,\nabla_wQ_h\phi\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma})\\&+\sum_{T\in\mathcal{T}_h^w}(\langle(\mathbb{Q}_h\nabla\phi-\nabla\phi)\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}+\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,Q_0\phi-Q_b\phi\rangle_{\partial T}).\end{aligned} \tag{4.12}$$

Proof. Testing (4.10) with e, then we have

$$\begin{aligned}\|e\|^2=&-(\Delta\phi,e_\delta)_{\Omega_s}+c^2(\phi,e_\delta)_{r^{-2},\Omega_s}-(\Delta\phi,e_0)_{\Omega_3}+c^2(\phi,e_0)_{r^{-2},\Omega_3}\\=&(\nabla\phi,\nabla e_\delta)_{\Omega_s}+c^2(\phi,e_\delta)_{r^{-2},\Omega_s}-\langle\nabla\phi\cdot\mathbf{n}_1,[e_\delta]\rangle_{\Gamma_R}-\langle\nabla\phi\cdot\mathbf{n}_2,e_\delta\rangle_\Gamma\\&+(\nabla\phi,\nabla e_0)_{\Omega_3}-\sum_{T\in\mathcal{T}_h^w}\langle\nabla\phi\cdot\mathbf{n}_3,e_0\rangle_{\partial T}+c^2(\phi,e_0)_{r^{-2},\Omega_3}.\end{aligned} \tag{4.13}$$

It follows from (4.2) that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h^w}(\nabla\phi,\nabla e_0)_T\\=&\sum_{T\in\mathcal{T}_h^w}(\nabla_wQ_h\phi,\nabla_we_w)_T+\sum_{T\in\mathcal{T}_h^w}\langle\mathbb{Q}_h\nabla\phi\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}+\sum_{T\in\mathcal{T}_h^w}\langle\phi-\Pi_\delta\phi,\nabla_we_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}.\end{aligned} \tag{4.14}$$

Notice that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^w}\langle\nabla\phi\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}=&\sum_{T\in\mathcal{T}_h^w}\langle\nabla\phi\cdot\mathbf{n}_3,e_0\rangle_{\partial T}-\sum_{T\in\mathcal{T}_h^w}\langle\nabla\phi\cdot\mathbf{n}_3,e_b\rangle_{\partial T\cap\Gamma}\\=&\sum_{T\in\mathcal{T}_h^w}\langle\nabla\phi\cdot\mathbf{n}_3,e_0\rangle_{\partial T}+\langle\nabla\phi\cdot\mathbf{n}_2,e_\delta\rangle_\Gamma+\langle\nabla\phi\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_\Gamma.\end{aligned} \tag{4.15}$$

Combining (4.13)-(4.15), thus we have

$$\begin{aligned}\|e\|^2 =&(\nabla\phi,\nabla e_\delta)_{\Omega_s}-\langle\nabla\phi\cdot\mathbf{n}_1,[e_\delta]\rangle_{\Gamma_R}+c^2(\phi,e_\delta)_{r^{-2},\Omega_s}+c^2(\phi,e_0)_{r^{-2},\Omega_3}\\&+\sum_{T\in\mathcal{T}_h^w}(\nabla_w Q_h\phi,\nabla_w e_w)_T+\sum_{T\in\mathcal{T}_h^w}\langle(\mathbb{Q}_h\nabla\phi-\nabla\phi)\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}\\&+\langle\nabla\phi\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_\Gamma+\sum_{T\in\mathcal{T}_h^w}\langle\phi-\Pi_\delta\phi,\nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}.\end{aligned}\tag{4.16}$$

Let $v_\delta=\Pi_\delta\phi$ and $v_w=Q_h\phi$ in error equation (4.1), then there holds

$$\begin{aligned}&a_\delta(e_\delta,\Pi_\delta\phi)+\sum_{T\in\mathcal{T}_h^w}(\nabla_w e_w,\nabla_w Q_h\phi)_T+\sum_{T\in\mathcal{T}_h^w}c^2(e_0,Q_0\phi)_T+s(e_w,Q_h\phi)\\=&\langle\nabla u\cdot\mathbf{n}_1,[\Pi_\delta\phi]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,Q_0\phi-Q_b\phi\rangle_{\partial T}\\&+\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w Q_h\phi\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+s(Q_h u,Q_h\phi).\end{aligned}\tag{4.17}$$

Using the fact that $\sum_{T\in\mathcal{T}_h^w}c^2(\phi-Q_0\phi,e_0)_{r^{-2},T}=0$, from (4.16) and (4.17), the following equation holds

$$\begin{aligned}\|e\|^2=&a_\delta(e_\delta,\phi-\Pi_\delta\phi)-\langle\nabla\phi\cdot\mathbf{n}_1,[e_\delta]\rangle_{\Gamma_R}+\langle\nabla\phi\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_\Gamma+s(Q_h u,Q_h\phi)-s(e_w,Q_h\phi)\\&+\langle\nabla u\cdot\mathbf{n}_1,[\Pi_\delta\phi]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}\left(\langle\phi-\Pi_\delta\phi,\nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+\langle\Pi_\delta u-u,\nabla_w Q_h\phi\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right)\\&+\sum_{T\in\mathcal{T}_h^w}\left(\langle(\mathbb{Q}_h\nabla\phi-\nabla\phi)\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}+\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,Q_0\phi-Q_b\phi\rangle_{\partial T}\right).\end{aligned}$$

Then we complete the proof. □

Lemma 4.4. *Let u be the exact solution of* (2.3) *and $u_h=\{u_\delta,u_w\}$ be the numerical solution of* (3.2). *Assume $u\in\mathscr{H}_0^1(\Omega)$, and $u|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge 1$, $u|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge 3/2$, $u|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge 3/2$, if $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^{3/2}$ and $h_3^{-1}N_3^{4/3}\le Ch_\delta^{-1}N_2$, there holds*

$$\|e\|\le Ch_3^{\mu_3}N_3^{1-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u\|_{H^{s_2}(\Omega_2)}+\|u\|_{H^{s_3}(\Omega_3)}\right),$$

where $\mu_3=\min\{s_3,N_3+1\}$.

Proof. Next, we estimate the terms on the right side of the expression (4.12). First, using Lemma 3.1, we have

$$\begin{aligned}|a_\delta(e_\delta,\phi-\Pi_\delta\phi)|\le&\|e_\delta\|_V\|\phi-\Pi_\delta\phi\|_{X_\delta}\\\le&C\left(h_\delta N_1^{-1}\|\phi\|_{\mathscr{H}^2(\Omega_1)}+h_\delta N_2^{-1}\|\phi\|_{H^2(\Omega_2)}\right)\|e_\delta\|_V.\end{aligned}\tag{4.18}$$

It follows from [19, Theorem 3.3] that

$$\left|\langle \nabla\phi\cdot\mathbf{n}_1,[e_\delta]\rangle_{\Gamma_R}\right| \le C h_\delta N_1^{-1}\|\phi\|_{\mathscr{H}^2(\Omega_1)}\|e_\delta\|_V. \tag{4.19}$$

From Lemma 3.2, the following estimate holds

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle(\nabla\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_3,e_0-e_b\rangle_{\partial T}\right| \le C h_3 N_3^{-3/2}\|\phi\|_{H^2(\Omega_3)}\|e_w\|_V. \tag{4.20}$$

It follows from Lemmas 3.2 and 3.5 that

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle\phi-\Pi_\delta\phi,\nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right| \le C h_3^{-1/2}N_3 h_\delta^{3/2}N_2^{-3/2}\|\phi\|_{H^2(\Omega_2)}\|e_w\|_V.$$

If $h_\delta^{3/2}N_2^{-3/2}\le C h_3^{3/2}N_3^{-2}$, there holds

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle\phi-\Pi_\delta\phi,\nabla_w e_w\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right| \le C h_3 N_3^{-1}\|\phi\|_{H^2(\Omega_2)}\|e_w\|_V. \tag{4.21}$$

Using the fact that $\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,\phi-Q_b\phi\rangle_{\partial T}=\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,\phi-\Pi_\delta\phi\rangle_{\partial T\cap\Gamma}$ and Lemma 3.2, we have

$$\begin{aligned}&\left|\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,Q_0\phi-Q_b\phi\rangle_{\partial T}\right|\\ \le{}& C\left(h_\delta^{3/2}N_2^{-3/2}h_3^{\mu_3-3/2}N_3^{3/2-s_3}\|u\|_{H^{s_3}(\Omega_3)}\|\phi\|_{H^2(\Omega_2)}+h_3^{\mu_3}N_3^{-s_3}\|u\|_{H^{s_3}(\Omega_3)}\|\phi\|_{H^2(\Omega_3)}\right).\end{aligned}$$

If $h_\delta^{3/2}N_2^{-3/2}\le C h_3^{3/2}N_3^{-3/2}$, then we have

$$\left|\sum_{T\in\mathcal{T}_h^w}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}_3,Q_0\phi-Q_b\phi\rangle_{\partial T}\right| \le C h_3^{\mu_3}N_3^{-s_3}\|u\|_{H^{s_3}(\Omega_3)}\left(\|\phi\|_{H^2(\Omega_2)}+\|\phi\|_{H^2(\Omega_3)}\right). \tag{4.22}$$

Since ϕ is continuous on Γ_R, it follows from Lemma 3.1 and [19, Theorem 3.3] that

$$\begin{aligned}&\left|\langle\nabla u\cdot\mathbf{n}_1,[\Pi_\delta\phi]\rangle_{\Gamma_R}\right|\\ ={}&\left|\langle\nabla u\cdot\mathbf{n}_1,[\Pi_\delta\phi-\phi]\rangle_{\Gamma_R}\right|\\ \le{}& C h_\delta^{s_1-1}N_1^{1-s_1}\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}\left(h_\delta N_1^{-1}\|\phi\|_{\mathscr{H}^2(\Omega_1)}+h_\delta N_2^{-1}\|\phi\|_{H^2(\Omega_2)}\right).\end{aligned} \tag{4.23}$$

Next, we estimate $\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w Q_h\phi\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+\langle\nabla\phi\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_\Gamma$. Notice that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,\nabla_w Q_h\phi\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+\langle\nabla\phi\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_\Gamma\\ =&\sum_{T\in\mathcal{T}_h^w}\langle\Pi_\delta u-u,(\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}+\sum_{T\in\mathcal{T}_h^w}\langle(\nabla\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_{\partial T\cap\Gamma}.\end{aligned}$$

Then we estimate the two terms on the right side one by one. It follows from Lemma 3.4 that

$$\sum_{T\in\mathcal{T}_\Gamma^w}(\nabla_w Q_h\phi,\mathbf{q})_T=\sum_{T\in\mathcal{T}_\Gamma^w}(\mathbb{Q}_h(\nabla\phi),\mathbf{q})_T+\sum_{T\in\mathcal{T}_\Gamma^w}\langle\Pi_\delta\phi-\phi,\mathbf{q}\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma},$$

for any $\mathbf{q}\in[P_{N_3-1}(T)]^d$. Let $\mathbf{q}=\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi$, we have $\sum_{T\in\mathcal{T}_\Gamma^w}\|\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi\|_T^2=\sum_{T\in\mathcal{T}_\Gamma^w}\langle\Pi_\delta\phi-\phi,(\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}$. Then from Lemma 3.5, there holds

$$\begin{aligned}&\sum_{T\in\mathcal{T}_\Gamma^w}\|\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi\|_{\partial T\cap\Gamma}^2\\ \le& Ch_T^{-1}N_3^2\sum_{T\in\mathcal{T}_\Gamma^w}\|\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi\|_T^2\\ \le& Ch_T^{-1}N_3^2\left(\sum_{T\in\mathcal{T}_\Gamma^w}\|\phi-\Pi_\delta\phi\|_{\partial T\cap\Gamma}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_\Gamma^w}\|\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi\|_{\partial T\cap\Gamma}^2\right)^{1/2},\end{aligned}$$

which implies that

$$\sum_{T\in\mathcal{T}_\Gamma^w}\|\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi\|_{\partial T\cap\Gamma}^2\le Ch_T^{-2}N_3^4\sum_{T\in\mathcal{T}_\Gamma^w}\|\phi-\Pi_\delta\phi\|_{\partial T\cap\Gamma}^2.$$

Thus, from Lemma 3.2, we have

$$\begin{aligned}&\left|\sum_{T\in\mathcal{T}_\Gamma^w}\langle\Pi_\delta u-u,(\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right|\\ \le& C\left(\sum_{T\in\mathcal{T}_\Gamma^w}\|\Pi_\delta u-u\|_{\partial T\cap\Gamma}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_\Gamma^w}h_T^{-2}N_3^4\|\phi-\Pi_\delta\phi\|_{\partial T\cap\Gamma}^2\right)^{1/2}\\ \le& Ch_\delta^{\mu_2+1}N_2^{-1-s_2}h_3^{-1}N_3^2\|u\|_{H^{s_2}(\Omega_2)}\|\phi\|_{H^2(\Omega_2)}.\end{aligned}$$

If $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^{3/2}$, there holds

$$\left|\sum_{T\in\mathcal{T}_\Gamma^w}\langle\Pi_\delta u-u,(\nabla_w Q_h\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_3\rangle_{\partial T\cap\Gamma}\right|\le Ch_\delta^{\mu_2}N_2^{1/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}\|\phi\|_{H^2(\Omega_2)}. \tag{4.24}$$

It follows from Lemma 3.2 that

$$\left|\sum_{T\in\mathcal{T}_\Gamma^w}\langle(\nabla\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_{\partial T\cap\Gamma}\right|\le Ch_\delta^{\mu_2-1/2}N_2^{1/2-s_2}h_3^{1/2}N_3^{-1/2}\|u\|_{H^{s_2}(\Omega_2)}\|\phi\|_{H^2(\Omega_2)}.$$

Notice that $h_3^{1/2}\le h_\delta^{1/2}$ and $N_3^{-1/2}\le 1$, we have

$$\left|\sum_{T\in\mathcal{T}_\Gamma^w}\langle(\nabla\phi-\mathbb{Q}_h\nabla\phi)\cdot\mathbf{n}_2,\Pi_\delta u-u\rangle_{\partial T\cap\Gamma}\right|\le Ch_\delta^{\mu_2}N_2^{1/2-s_2}\|u\|_{H^{s_2}(\Omega_2)}\|\phi\|_{H^2(\Omega_2)}. \tag{4.25}$$

It follows from Lemma 3.2 that

$$\begin{aligned}
&|s(Q_hu,Q_h\phi)|\\
\le&C\left(h_3^{\mu_3}N_3^{1-s_3}+h_\delta^{3/2}N_2^{-3/2}h_3^{\mu_3-3/2}N_3^{5/2-s_3}\right)\|u\|_{H^{s_3}(\Omega_3)}\left(\|\phi\|_{H^2(\Omega_2)}+\|\phi\|_{H^2(\Omega_3)}\right)\\
&+C\left(h_\delta^{\mu_2-1/2}N_2^{1/2-s_2}h_3^{1/2}N_3^{1/2}+h_\delta^{\mu_2+1}N_2^{-1-s_2}h_3^{-1}N_3^2\right)\|u\|_{H^{s_2}(\Omega_2)}\left(\|\phi\|_{H^2(\Omega_2)}+\|\phi\|_{H^2(\Omega_3)}\right).
\end{aligned}$$

If $h_\delta^{3/2}N_2^{-3/2}\le Ch_3^{3/2}N_3^{-3/2}$ and $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^2$, then the following estimate holds

$$\begin{aligned}
&|s(Q_hu,Q_h\phi)|\\
\le&C\left(h_\delta^{\mu_2}N_2^{1-s_2}+h_3^{\mu_3}N_3^{1-s_3}\right)\left(\|u\|_{H^{s_2}(\Omega_2)}+\|u\|_{H^{s_3}(\Omega_3)}\right)\left(\|\phi\|_{H^2(\Omega_2)}+\|\phi\|_{H^2(\Omega_3)}\right).
\end{aligned} \tag{4.26}$$

From Lemma 3.2, there holds

$$|s(e_w,Q_h\phi)|\le Ch_3N_3^{-1/2}\|\phi\|_{H^2(\Omega_3)}\|e_w\|_V+Ch_3^{-1/2}N_3h_\delta^{3/2}N_2^{-3/2}\|\phi\|_{H^2(\Omega_2)}\|e_w\|_V.$$

If $h_\delta^{3/2}N_2^{-3/2}\le Ch_3^{3/2}N_3^{-3/2}$, then we have

$$|s(e_w,Q_h\phi)|\le Ch_3N_3^{-1/2}\left(\|\phi\|_{H^2(\Omega_2)}+\|\phi\|_{H^2(\Omega_3)}\right)\|e_w\|_V. \tag{4.27}$$

Notice that $N_1=N_2\gg N_3$, so the main error comes from the WG element. It follows from (4.11), (4.18)-(4.27) and Lemma 4.2 that

$$\|e\|\le Ch_3^{\mu_3}N_3^{1-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u\|_{H^{s_2}(\Omega_2)}+\|u\|_{H^{s_3}(\Omega_3)}\right),$$

which completes the proof. □

According to Lemmas 3.2 and 4.4, the following estimate holds.

Theorem 4.2. *Let u be the exact solution of* (2.3) *and $u_h=\{u_\delta,u_w\}$ be the numerical solution of* (3.2). *Assume $u\in\mathscr{H}_0^1(\Omega)$, and $u|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge 1$, $u|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge 3/2$, $u|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge 3/2$, if $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^{3/2}$ and $h_3^{-1}N_3^{4/3}\le Ch_\delta^{-1}N_2$, there holds*

$$\|u|_{\Omega_s}-u_\delta\|+\|u|_{\Omega_3}-u_0\|\le Ch_3^{\mu_3}N_3^{1-s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u\|_{H^{s_2}(\Omega_2)}+\|u\|_{H^{s_3}(\Omega_3)}\right),$$

where $\mu_3=\min\{s_3,N_3+1\}$.

5 Error estimates for eigenvalue problem

In this section, we provide the H^1 and L^2 error estimates for eigenfunctions. Furthermore, we present an expression for the error of the eigenvalue, followed by giving the error estimates for eigenvalues.

5.1 Error estimate for eigenfunction

Based on the error analysis in Section 3, the following estimate is valid, following from Lemma 3.6 and Theorem 4.1.

Theorem 5.1. *Suppose λ_j is an eigenvalue of* (2.2) *with multiplicity m, and $R(E_\mu(K))$ is the corresponding m-dimensional eigenspace. Suppose $\{\lambda_{i,h}\}_{i=1}^m$ are the eigenvalues of* (2.5) *approximating λ_j, and $u_{i,h}$ is an eigenfunction corresponding to $\lambda_{i,h}$, for $i=1,2,\cdots,m$. For any $u_j \in R(E_\mu(K))$, assume $u_j \in \mathscr{H}_0^1(\Omega)$, and $u_j|_{\Omega_1} \in \mathscr{H}^{s_1}(\Omega_1)$, $s_1 \geq 1$, $u_j|_{\Omega_2} \in H^{s_2}(\Omega_2)$, $s_2 \geq 3/2$, $u_j|_{\Omega_3} \in H^{s_3}(\Omega_3)$, $s_3 \geq 3/2$. For any $i=1,2,\cdots,m$, if $h_3^{-1}N_3^2 \leq Ch_\delta^{-1}N_2^2$, there exists an eigenfunction $u_j \in R(E_\mu(K))$ such that*

$$
\begin{aligned}
&\|u_j - u_{i,h}\|_V \\
=&\|u_j|_{\Omega_s} - u_{i,\delta}\|_V + \|u_j|_{\Omega_3} - u_{i,w}\|_V \\
\leq& C\left(h_\delta^{s_1-1}N_1^{1-s_1}\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)} + h_\delta^{\mu_2-1}N_2^{3/2-s_2}\|u_j\|_{H^{s_2}(\Omega_2)} + h_3^{\mu_3-1}N_3^{3/2-s_3}\|u_j\|_{H^{s_3}(\Omega_3)}\right),
\end{aligned}
$$

where $\mu_2 = \min\{s_2, N_2+1\}$ and $\mu_3 = \min\{s_3, N_3+1\}$. Furthermore, we have

$$
\|u_j - u_{i,h}\|_V \leq Ch_3^{\mu_3-1}N_3^{3/2-s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u_j\|_{H^{s_2}(\Omega_2)} + \|u_j\|_{H^{s_3}(\Omega_3)}\right).
$$

In addition, from Lemma 3.7 and Theorem 4.2, we have the following estimate.

Theorem 5.2. *Suppose λ_j is an eigenvalue of* (2.2) *with multiplicity m, and $R(E_\mu(K))$ is the corresponding m-dimensional eigenspace. Suppose $\{\lambda_{i,h}\}_{i=1}^m$ are the eigenvalues of* (2.5) *approximating λ_j, and $u_{i,h}$ is an eigenfunction corresponding to $\lambda_{i,h}$, for $i=1,2,\cdots,m$. For any $u_j \in R(E_\mu(K))$, assume $u_j \in \mathscr{H}_0^1(\Omega)$, and $u_j|_{\Omega_1} \in \mathscr{H}^{s_1}(\Omega_1)$, $s_1 \geq 1$, $u_j|_{\Omega_2} \in H^{s_2}(\Omega_2)$, $s_2 \geq 3/2$, $u_j|_{\Omega_3} \in H^{s_3}(\Omega_3)$, $s_3 \geq 3/2$. For any $i=1,2,\cdots,m$, if $h_3^{-1}N_3^2 \leq Ch_\delta^{-1}N_2^{3/2}$ and $h_3^{-1}N_3^{4/3} \leq Ch_\delta^{-1}N_2$, there exists an eigenfunction $u_j \in R(E_\mu(K))$ such that*

$$
\|u_j - u_{i,h}\| \leq Ch_3^{\mu_3}N_3^{1-s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u_j\|_{H^{s_2}(\Omega_2)} + \|u_j\|_{H^{s_3}(\Omega_3)}\right),
$$

where $\mu_3 = \min\{s_3, N_3+1\}$.

5.2 Error estimate for eigenvalue

In this subsection, we derive the expression for the eigenvalue error and provide error estimates for eigenvalues.

Lemma 5.1. *Suppose (λ,u) is an eigenpair of* (2.2)*, and (λ_h,u_h) is an eigenpair of* (2.5)*. For any $v_h\in V_h$, there holds*

$$\lambda-\lambda_h=a(u,u)-a_h(v_h,v_h)+a_h(u_h-v_h,u_h-v_h)-\lambda_h b(u-u_h,u-u_h)+2\lambda_h b_h(u_h,v_h-u).$$

Proof. Notice that $b(u,u)=b_h(u_h,u_h)=1$, for any $v_h\in V_h$, then we have

$$\begin{aligned}
&\lambda-\lambda_h\\
=&a(u,u)-a_h(u_h,u_h)-2a_h(u_h,v_h)+2\lambda_h b_h(u_h,v_h)\\
=&a(u,u)-a_h(v_h,v_h)+a_h(u_h,u_h)-2a_h(u_h,v_h)+a_h(v_h,v_h)-2a_h(u_h,u_h)+2\lambda_h b_h(u_h,v_h)\\
=&a(u,u)-a_h(v_h,v_h)+a_h(u_h-v_h,u_h-v_h)-2\lambda_h b_h(u_h,u_h)+2\lambda_h b_h(u_h,v_h)\\
=&a(u,u)-a_h(v_h,v_h)+a_h(u_h-v_h,u_h-v_h)-\lambda_h b(u,u)-\lambda_h b_h(u_h,u_h)+2\lambda_h b_h(u_h,v_h)\\
=&a(u,u)-a_h(v_h,v_h)+a_h(u_h-v_h,u_h-v_h)-\lambda_h b(u-u_h,u-u_h)+2\lambda_h b_h(u_h,v_h-u),
\end{aligned}$$

which completes the proof. □

For further theoretical analysis, we extend the definition of the weak gradient.

Definition 5.1. For $v\in H^1(T)$, its weak gradient $\nabla_w v|_T\in[P_{N_3-1}(T)]^d$ is defined as follows:

$$(\nabla_w v,\mathbf{q})_T=-(v,\nabla\cdot\mathbf{q})_T+\langle v,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}=(\nabla v,\mathbf{q})_T,\quad \forall\mathbf{q}\in[P_{N_3-1}(T)]^d, \tag{5.1}$$

where $\mathbf{n}$ is the outward unit normal vector.

Furthermore, for any $u\in\mathscr{H}^1(\Omega)$, $v_h\in V_h$, we have

$$\begin{aligned}
a_h(u,v_h)&=(\nabla u,\nabla v_\delta)_{\Omega_s}+(u,v_\delta)_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}(\nabla_w u,\nabla_w v_w)_T+\sum_{T\in\mathcal{T}_h^w}c^2(u,v_0)_{r^{-2},T}\\
&=(\nabla u,\nabla v_\delta)_{\Omega_s}+(u,v_\delta)_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla_w v_w)_T+\sum_{T\in\mathcal{T}_h^w}c^2(u,v_0)_{r^{-2},T},
\end{aligned}$$

here are no stabilizer $s(u,v_h)$, since $s(u,v_h)=\sum_{T\in\mathcal{T}_h^w}\langle u-u,v_0-v_b\rangle_{\partial T}=0$. Similarly, we can extend the definition of $a_h(\cdot,\cdot)$ to $\mathscr{H}^1(\Omega)+V_h$. Then we define $P_h:\mathscr{H}_0^1(\Omega)\to V_h$ such that

$$a_h(P_h u,v_h)=a_h(u,v_h),\quad \forall v_h\in V_h. \tag{5.2}$$

It is easy to verify that $a_h(u,\cdot)$ is continuous over V_h, then according to the Lax-Milgram Theorem, P_h is well-defined. For any $v_h=\{v_\delta,v_w\}\in V_h$, we define

$$\begin{aligned}
\theta_u(v_h)=&\|\nabla u-\nabla v_\delta\|_{\Omega_s}^2+c^2\|u-v_\delta\|_{r^{-2},\Omega_s}^2+s(v_w,v_w)\\
&+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\nabla_w v_w\|_T^2+\sum_{T\in\mathcal{T}_h^w}c^2\|u-v_0\|_{r^{-2},T}^2.
\end{aligned}$$

Lemma 5.2. *For any $u\in\mathscr{H}_0^1(\Omega)$, there holds*

$$a(u,u)-a_h(P_h u,P_h u)=\theta_u(P_h u)\le\theta_u(I_h u).$$

Proof. For any $v_h=\{v_\delta,v_w\}\in V_h$, we have

$$\begin{aligned}
&a(u,u)-a_h(v_h,v_h)\\
=&\|\nabla u\|^2_{\Omega_s}+c^2\|u\|^2_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}\|\nabla u\|^2_T+\sum_{T\in\mathcal{T}_h^w}c^2\|u\|^2_{r^{-2},T}\\
&-\|\nabla v_\delta\|^2_{\Omega_s}-c^2\|v_\delta\|^2_{r^{-2},\Omega_s}-\sum_{T\in\mathcal{T}_h^w}\|\nabla_w v_w\|^2_T-\sum_{T\in\mathcal{T}_h^w}c^2\|v_0\|^2_{r^{-2},T}-s(v_w,v_w)\\
=&\|\nabla u-\nabla v_\delta\|^2_{\Omega_s}+c^2\|u-v_\delta\|^2_{r^{-2},\Omega_s}+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\nabla_w v_w\|^2_T+\sum_{T\in\mathcal{T}_h^w}c^2\|u-v_0\|^2_{r^{-2},T}\\
&+2(\nabla u-\nabla v_\delta,\nabla v_\delta)_{\Omega_s}+2c^2(u-v_\delta,v_\delta)_{r^{-2},\Omega_s}+2\sum_{T\in\mathcal{T}_h^w}(\nabla u-\nabla_w v_w,\nabla_w v_w)_T\\
&+2\sum_{T\in\mathcal{T}_h^w}c^2(u-v_0,v_0)_{r^{-2},T}-s(v_w,v_w).
\end{aligned}\tag{5.3}$$

From (5.1) and (5.2), there holds

$$\begin{aligned}
&2(\nabla u-\nabla P_hu,\nabla v_\delta)_{\Omega_s}+2c^2(u-P_hu,v_\delta)_{r^{-2},\Omega_s}-2s(P_hu,v_w)\\
&+2\sum_{T\in\mathcal{T}_h^w}(\nabla_w u-\nabla_w P_hu,\nabla_w v_w)_T+2\sum_{T\in\mathcal{T}_h^w}c^2(u-P_hu,v_0)_{r^{-2},T}\\
=&2(\nabla u-\nabla P_hu,\nabla v_\delta)_{\Omega_s}+2c^2(u-P_hu,v_\delta)_{r^{-2},\Omega_s}-2s(P_hu,v_w)\\
&+2\sum_{T\in\mathcal{T}_h^w}(\nabla u-\nabla_w P_hu,\nabla_w v_w)_T+2\sum_{T\in\mathcal{T}_h^w}c^2(u-P_hu,v_0)_{r^{-2},T}=0.
\end{aligned}\tag{5.4}$$

Let $v_h=P_hu$ in (5.3) and (5.4), we have $a(u,u)-a_h(P_hu,P_hu)=\theta_u(P_hu)$. Then the following inequality holds $\theta_u(P_hu)\le\theta_u(I_hu)$. In fact,

$$\begin{aligned}
\theta_u(I_hu)=&\|\nabla u-\nabla I_hu\|^2_{\Omega_s}+c^2\|u-I_hu\|^2_{r^{-2},\Omega_s}+s(I_hu,I_hu)\\
&+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\nabla_w I_hu\|^2_T+\sum_{T\in\mathcal{T}_h^w}c^2\|u-I_hu\|^2_{r^{-2},T}\\
=&\|\nabla u-\nabla P_hu+\nabla P_hu-\nabla I_hu\|^2_{\Omega_s}+c^2\|u-P_hu+P_hu-I_hu\|^2_{r^{-2},\Omega_s}\\
&+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\nabla_w P_hu+\nabla_w P_hu-\nabla_w I_hu\|^2_T+\sum_{T\in\mathcal{T}_h^w}c^2\|u-P_hu+P_hu-I_hu\|^2_{r^{-2},T}\\
&+s(I_hu-P_hu+P_hu,I_hu-P_hu+P_hu)\\
=&\|\nabla u-\nabla P_hu\|^2_{\Omega_s}+\|\nabla P_hu-\nabla I_hu\|^2_{\Omega_s}+c^2\|u-P_hu\|^2_{r^{-2},\Omega_s}+c^2\|P_hu-I_hu\|^2_{r^{-2},\Omega_s}\\
&+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\nabla_w P_hu\|^2_T+\sum_{T\in\mathcal{T}_h^w}\|\nabla_w P_hu-\nabla_w I_hu\|^2_T+s(I_hu-P_hu,I_hu-P_hu)\\
&+\sum_{T\in\mathcal{T}_h^w}c^2\|u-P_hu\|^2_{r^{-2},T}+\sum_{T\in\mathcal{T}_h^w}c^2\|P_hu-I_hu\|^2_{r^{-2},T}+s(P_hu,P_hu),
\end{aligned}$$

where we have used the fact that $a_h(P_hu,P_hu-I_hu)=a_h(u,P_hu-I_hu)$. Thus,

$$a(u,u)-a_h(P_hu,P_hu)=\theta_u(P_hu)\le\theta_u(I_hu),$$

which completes the proof. □

Lemma 5.3. *For any $u\in\mathscr{H}_0^1(\Omega)$, the following estimate holds*

$$\sum_{T\in\mathcal{T}_h^w}\|\nabla_w(u-Q_hu)\|_T\le C\|u-I_hu\|_V.$$

Proof. From (2.4) and (5.1), for any $\mathbf{q}\in[P_{N_3-1}(T)]^d$, we have

$$(\nabla_w(u-Q_hu),\mathbf{q})_T=(\nabla(u-Q_0u),\mathbf{q})_T+\langle Q_0u-Q_bu,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}.$$

Let $\mathbf{q}=\nabla_w(u-Q_hu)$, from Lemma 3.5, then there holds

$$\begin{aligned}&\|\nabla_w(u-Q_hu)\|_T^2\\ \le&\|\nabla(u-Q_0u)\|_T\|\nabla_w(u-Q_hu)\|_T+Ch_T^{-1/2}N_3\|Q_0u-Q_bu\|_{\partial T}\|\nabla_w(u-Q_hu)\|_T,\end{aligned}$$

which implies that $\|\nabla_w(u-Q_hu)\|_T\le\|\nabla(u-Q_0u)\|_T+Ch_T^{-1/2}N_3\|Q_0u-Q_bu\|_{\partial T}$. Then we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^w}\|\nabla_w(u-Q_hu)\|_T\le&\sum_{T\in\mathcal{T}_h^w}\|\nabla(u-Q_0u)\|_T+C\sum_{T\in\mathcal{T}_h^w}h_T^{-1/2}N_3\|Q_0u-Q_bu\|_{\partial T}\\ \le&C\|u-I_hu\|_V.\end{aligned}$$

Then we complete the proof. □

Furthermore, the following estimate holds.

Lemma 5.4. *Let u be the eigenfunction of* (2.2). *Assume $u\in\mathscr{H}_0^1(\Omega)$, and $u|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge1$, $u|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge3/2$, $u|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge3/2$, if $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^2$, then we have*

$$\theta_u(I_hu)\le Ch_3^{2(\mu_3-1)}N_3^{3-2s_3}\left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u\|_{H^{s_2}(\Omega_2)}+\|u\|_{H^{s_3}(\Omega_3)}\right)^2,$$

where $\mu_3=\min\{s_3,N_3+1\}$.

Proof. Notice that $\nabla_wu=\mathbb{Q}_h(\nabla u)$. Using Lemma 5.3, we have

$$\begin{aligned}\theta_u(I_hu)=&\|\nabla(u-\Pi_\delta u)\|_{\Omega_s}^2+c^2\|u-\Pi_\delta u\|_{r^{-2},\Omega_s}^2+s(Q_hu,Q_hu)\\ &+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\nabla_wQ_hu\|_T^2+\sum_{T\in\mathcal{T}_h^w}c^2\|u-Q_0u\|_{r^{-2},T}^2\\ \le&\|\nabla(u-\Pi_\delta u)\|_{\Omega_s}^2+c^2\|u-\Pi_\delta u\|_{r^{-2},\Omega_s}^2+\sum_{T\in\mathcal{T}_h^w}c^2\|u-Q_0u\|_{r^{-2},T}^2\\ &+\sum_{T\in\mathcal{T}_h^w}\|\nabla_w(u-Q_hu)\|_T^2+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\mathbb{Q}_h\nabla u\|_T^2+s(Q_hu,Q_hu)\\ \le&C\|u-I_hu\|_V^2+\sum_{T\in\mathcal{T}_h^w}\|\nabla u-\mathbb{Q}_h\nabla u\|_T^2.\end{aligned}$$

It follows from Lemmas 3.2 and 3.6 that

$$\theta_u(I_h u) \le C h_3^{2(\mu_3-1)} N_3^{3-2s_3} \left(\|u\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u\|_{H^{s_2}(\Omega_2)} + \|u\|_{H^{s_3}(\Omega_3)} \right)^2,$$

which completes the proof. □

For any $v \in V$, we define the following norm: $\|v\|_h^2 = a_h(v,v)$.

Lemma 5.5. *Suppose (λ_j, u_j) is the j-th eigenpair of (2.2), and $(\lambda_{j,h}, u_{j,h})$ is the j-th eigenpair of (2.5). Assume $u_j \in \mathscr{H}_0^1(\Omega)$, and $u_j|_{\Omega_1} \in \mathscr{H}^{s_1}(\Omega_1)$, $s_1 \ge 1$, $u_j|_{\Omega_2} \in H^{s_2}(\Omega_2)$, $s_2 \ge 3/2$, $u_j|_{\Omega_3} \in H^{s_3}(\Omega_3)$, $s_3 \ge 3/2$, if $h_3^{-1} N_3^2 \le C h_\delta^{-1} N_2^2$, then we have*

$$a_h(u_{j,h} - P_h u_j, u_{j,h} - P_h u_j) \le C h_3^{2(\mu_3-1)} N_3^{3-2s_3} \left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u_j\|_{H^{s_2}(\Omega_2)} + \|u_j\|_{H^{s_3}(\Omega_3)} \right)^2,$$

where $\mu_2 = \min\{s_2, N_2+1\}$ and $\mu_3 = \min\{s_3, N_3+1\}$.

Proof. It follows from the triangle inequality that

$$a_h(u_{j,h} - P_h u_j, u_{j,h} - P_h u_j) = \|u_{j,h} - P_h u_j\|_h^2 \le C\left(\|u_j - u_{j,h}\|_h^2 + \|u_j - P_h u_j\|_h^2 \right).$$

There holds $\|u_j - P_h u_j\|_h^2 \le \|u_j - I_h u_j\|_h^2$. In fact,

$$\|u_j - I_h u_j\|_h^2 = \|u_j - P_h u_j + P_h u_j - I_h u_j\|_h^2 = \|u_j - P_h u_j\|_h^2 + \|P_h u_j - I_h u_j\|_h^2,$$

where we have used the fact that $a_h(u_j - P_h u_j, P_h u_j - I_h u_j) = 0$.

Compared to $\|u_j - u_{j,h}\|_V$ and $\|u_j - I_h u_j\|_V$, to estimate $\|u_j - u_{j,h}\|_h$ and $\|u_j - I_h u_j\|_h$, we just need to estimate $\sum_{T \in \mathcal{T}_h^w} \|\nabla_w (u_j - u_{j,h})\|_T$ and $\sum_{T \in \mathcal{T}_h^w} \|\nabla_w (u_j - Q_h u_j)\|_T$, respectively. It follows from Lemma 5.3 that

$$\begin{aligned}
\sum_{T \in \mathcal{T}_h^w} \|\nabla_w (u_j - u_{j,h})\|_T &\le \sum_{T \in \mathcal{T}_h^w} \|\nabla_w (u_j - Q_h u_j)\|_T + \sum_{T \in \mathcal{T}_h^w} \|\nabla_w (Q_h u_j - u_{j,h})\|_T \\
&\le \sum_{T \in \mathcal{T}_h^w} \|\nabla_w (u_j - Q_h u_j)\|_T + \|(Q_h u_j - u_{j,h})|_{\Omega_3}\|_V \\
&\le \|u_j - I_h u_j\|_V + \|u_j - u_{j,h}\|_V.
\end{aligned}$$

Based on the above analysis, we have

$$a_h(u_{j,h} - P_h u_j, u_{j,h} - P_h u_j) \le C\left(\|u_j - u_{j,h}\|_h^2 + \|u_j - I_h u_j\|_h^2 \right) \le C\left(\|u_j - u_{j,h}\|_V^2 + \|u_j - I_h u_j\|_V^2 \right).$$

It follows from Lemma 3.6 and Theorem 5.1 that

$$a_h(u_{j,h} - P_h u_j, u_{j,h} - P_h u_j) \le C h_3^{2(\mu_3-1)} N_3^{3-2s_3} \left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)} + \|u_j\|_{H^{s_2}(\Omega_2)} + \|u_j\|_{H^{s_3}(\Omega_3)} \right)^2,$$

which completes the proof. □

Lemma 5.6. *Let* (λ,u) *be an eigenpair of* (2.2), *then the following equation holds*

$$b(u,P_hu-u)=\frac{1}{\lambda}\Big(-a_h(u-P_hu,u-P_hu)-\langle\nabla u\cdot\mathbf{n}_1,[P_hu-u]\rangle_{\Gamma_R}+\sum_{T\in\mathcal{T}_h^w}(\nabla u-\mathbb{Q}_h\nabla u,\nabla(Q_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}\langle(\mathbb{Q}_h\nabla u-\nabla u)\cdot\mathbf{n}_3,P_0u-P_bu\rangle_{\partial T}\Big),$$

where P_0u *denotes the value of* P_hu *on the interior of each cell* $T\in\mathcal{T}_h^w$, *and* P_bu *denotes the value of* P_hu *on each edge or face* $e\in\mathcal{E}_h$.

Proof. Consider the eigenvalue problem (2.1), then we have

$$\begin{aligned}b(u,P_hu-u)=&\frac{1}{\lambda}\left(-(\Delta u,P_hu-u)+c^2(u,P_hu-u)_{r^{-2}}\right)\\=&\frac{1}{\lambda}\Big((\nabla u,\nabla(P_hu-u))_{\Omega_s}+c^2(u,P_hu-u)_{r^{-2},\Omega_s}-\langle\nabla u\cdot\mathbf{n}_1,[P_hu-u]\rangle_{\Gamma_R}\\&-\langle\nabla u\cdot\mathbf{n}_2,P_hu-u\rangle_\Gamma-\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,P_0u-u\rangle_{\partial T}\\&+\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla(P_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}c^2(u,P_0u-u)_{r^{-2},T}\Big).\end{aligned}$$

Using the fact that $a_h(P_hu,P_hu)-a_h(u,P_hu)=0$, then we have

$$\begin{aligned}&b(u,P_hu-u)\\=&\frac{1}{\lambda}\Big(\nabla(u-P_hu),\nabla(P_hu-u))_{\Omega_s}+c^2(u-P_hu,P_hu-u)_{r^{-2},\Omega_s}\\&-\langle\nabla u\cdot\mathbf{n}_1,[P_hu-u]\rangle_{\Gamma_R}-\langle\nabla u\cdot\mathbf{n}_2,P_hu-u\rangle_\Gamma-\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,P_0u-u\rangle_{\partial T}\\&+\sum_{T\in\mathcal{T}_h^w}\left((\nabla_w(u-P_hu),\nabla_w(P_hu-u))_T+c^2(u-P_0u,P_0u-u)_{r^{-2},T}\right)\\&+\sum_{T\in\mathcal{T}_h^w}\left((\nabla u,\nabla(P_0u-u))_T-(\nabla_wu,\nabla_w(P_hu-u))_T\right)-s(P_hu,P_hu)\Big)\\=&\frac{1}{\lambda}\Big(-a_h(u-P_hu,u-P_hu)-\langle\nabla u\cdot\mathbf{n}_1,[P_hu-u]\rangle_{\Gamma_R}-\langle\nabla u\cdot\mathbf{n}_2,P_hu-u\rangle_\Gamma\\&-\sum_{T\in\mathcal{T}_h^w}\left(\langle\nabla u\cdot\mathbf{n}_3,P_0u-u\rangle_{\partial T}+(\nabla u,\nabla(P_0u-u))_T-(\nabla_wu,\nabla_w(P_hu-u))_T\right)\Big).\end{aligned}\tag{5.5}$$

It follows from (2.4) that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla(P_0u-u))_T-\sum_{T\in\mathcal{T}_h^w}(\nabla_wu,\nabla_w(P_hu-u))_T\\=&\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla(P_0u-u))_T-\sum_{T\in\mathcal{T}_h^w}(\mathbb{Q}_h\nabla u,\nabla_w(P_hu-u))_T\end{aligned}$$

$$
\begin{aligned}
&=\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla(P_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}(\nabla\cdot\mathbb{Q}_h\nabla u,P_0u-u)_T-\sum_{T\in\mathcal{T}_h^w}\langle\mathbb{Q}_h\nabla u\cdot\mathbf{n}_3,P_bu-u\rangle_{\partial T}\\
&=\sum_{T\in\mathcal{T}_h^w}(\nabla u,\nabla(P_0u-u))_T-\sum_{T\in\mathcal{T}_h^w}(\mathbb{Q}_h\nabla u,\nabla(P_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}\langle\mathbb{Q}_h\nabla u\cdot\mathbf{n}_3,P_0u-P_bu\rangle_{\partial T}\\
&=\sum_{T\in\mathcal{T}_h^w}(\nabla u-\mathbb{Q}_h\nabla u,\nabla(P_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}\langle\mathbb{Q}_h\nabla u\cdot\mathbf{n}_3,P_0u-P_bu\rangle_{\partial T}.
\end{aligned}
\tag{5.6}
$$

In fact,

$$
\langle\nabla u\cdot\mathbf{n}_2,P_hu-u\rangle_\Gamma=-\langle\nabla u\cdot\mathbf{n}_3,P_bu-u\rangle_\Gamma=-\sum_{T\in\mathcal{T}_h^w}\langle\nabla u\cdot\mathbf{n}_3,P_bu-u\rangle_{\partial T}.
\tag{5.7}
$$

Combining (5.5), (5.6) and (5.7), we have

$$
\begin{aligned}
b(u,P_hu-u)=&\frac{1}{\lambda}\Big(-a_h(u-P_hu,u-P_hu)-\langle\nabla u\cdot\mathbf{n}_1,[P_hu-u]\rangle_{\Gamma_R}\\
&+\sum_{T\in\mathcal{T}_h^w}(\nabla u-\mathbb{Q}_h\nabla u,\nabla(P_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}\langle(\mathbb{Q}_h\nabla u-\nabla u)\cdot\mathbf{n}_3,P_0u-P_bu\rangle_{\partial T}\Big)\\
=&\frac{1}{\lambda}\Big(-a_h(u-P_hu,u-P_hu)-\langle\nabla u\cdot\mathbf{n}_1,[P_hu-u]\rangle_{\Gamma_R}\\
&+\sum_{T\in\mathcal{T}_h^w}(\nabla u-\mathbb{Q}_h\nabla u,\nabla(Q_0u-u))_T+\sum_{T\in\mathcal{T}_h^w}\langle(\mathbb{Q}_h\nabla u-\nabla u)\cdot\mathbf{n}_3,P_0u-P_bu\rangle_{\partial T}\Big),
\end{aligned}
$$

where we have used the fact that

$$
\sum_{T\in\mathcal{T}_h^w}(\nabla u-\mathbb{Q}_h\nabla u,\nabla(P_0u-Q_0u))_T=0.
$$

Then we complete the proof. □

Lemma 5.7. *Suppose (λ_j,u_j) is the j-th eigenpair of* (2.2)*, and $(\lambda_{j,h},u_{j,h})$ is the j-th eigenpair of* (2.5)*. Assume $u_j\in\mathscr{H}_0^1(\Omega)$, and $u_j|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge 1$, $u_j|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge 3/2$, $u_j|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge 3/2$, if $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^{3/2}$ and $h_3^{-1}N_3^{4/3}\le Ch_\delta^{-1}N_2$, the following estimate holds*

$$
\left|b(u_{j,h},P_hu_j-u_j)\right|\le Ch_3^{2(\mu_3-1)}N_3^{3-2s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u_j\|_{H^{s_2}(\Omega_2)}+\|u_j\|_{H^{s_3}(\Omega_3)}\right)^2,
$$

where $\mu_3=\min\{s_3,N_3+1\}$.

Proof. Notice that

$$
b(u_{j,h},P_hu_j-u_j)=b(u_{j,h}-u_j+u_j,P_hu_j-u_j)=b(u_{j,h}-u_j,P_hu_j-u_j)+b(u_j,P_hu_j-u_j).
$$

It follows from Lemma 5.3 and the proof of Lemma 5.5 that

$$
b(u_{j,h}-u_j,P_hu_j-u_j)\le C|\!|\!|u_j-u_{j,h}|\!|\!|\,\|u_j-I_hu_j\|_h\le C|\!|\!|u_j-u_{j,h}|\!|\!|\,\|u_j-I_hu_j\|_V.
$$

Then it follows from Lemma 3.6 and Theorem 5.2 that

$$|b(u_{j,h}-u_j,P_hu_j-u_j)|\le Ch_3^{2\mu_3-1}N_3^{5/2-2s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u_j\|_{H^{s_2}(\Omega_2)}+\|u_j\|_{H^{s_3}(\Omega_3)}\right)^2. \quad (5.8)$$

Next, we need to estimate $b(u_j,P_hu_j-u_j)$. Notice that

$$\sum_{T\in\mathcal{T}_h^w} h_T^{-1}N_3^2\|P_0u_j-P_bu_j\|_{\partial T}^2=s(u_j-P_hu_j,u_j-P_hu_j)\le a_h(u_j-P_hu_j,u_j-P_hu_j).$$

We omit details, it follows from Lemmas 3.2, 5.5, 5.6 and [19, Theorem 3.3] that

$$|b(u_j,P_hu_j-u_j)|\le Ch_3^{2(\mu_3-1)}N_3^{3-2s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u_j\|_{H^{s_2}(\Omega_2)}+\|u_j\|_{H^{s_3}(\Omega_3)}\right)^2. \quad (5.9)$$

Based on the above analysis, we have

$$|b(u_{j,h},P_hu_j-u_j)|\le Ch_3^{2(\mu_3-1)}N_3^{3-2s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u_j\|_{H^{s_2}(\Omega_2)}+\|u_j\|_{H^{s_3}(\Omega_3)}\right)^2,$$

which completes the proof. □

We take $v_h=P_hu$ in Lemma 5.1. Finally, from Theorem 5.2 and Lemmas 5.1, 5.2, 5.4, 5.5 and 5.7, we have the following estimate for eigenvalue.

Theorem 5.3. *Suppose λ_j is an eigenvalue of (2.2) with multiplicity m, and $R(E_\mu(K))$ is the corresponding m-dimensional eigenspace. Suppose $\{\lambda_{i,h}\}_{i=1}^m$ are the eigenvalues of (2.5) approximating λ_j, and $u_{i,h}$ is an eigenfunction corresponding to $\lambda_{i,h}$, for $i=1,2,\cdots,m$. For any $u_j\in R(E_\mu(K))$, assume $u_j\in\mathscr{H}_0^1(\Omega)$, and $u_j|_{\Omega_1}\in\mathscr{H}^{s_1}(\Omega_1)$, $s_1\ge 1$, $u_j|_{\Omega_2}\in H^{s_2}(\Omega_2)$, $s_2\ge 3/2$, $u_j|_{\Omega_3}\in H^{s_3}(\Omega_3)$, $s_3\ge 3/2$. For any $i=1,\cdots,m$, if $h_3^{-1}N_3^2\le Ch_\delta^{-1}N_2^{3/2}$ and $h_3^{-1}N_3^{4/3}\le Ch_\delta^{-1}N_2$, the following estimate holds*

$$|\lambda_j-\lambda_{i,h}|\le Ch_3^{2(\mu_3-1)}N_3^{3-2s_3}\left(\|u_j\|_{\mathscr{H}^{s_1}(\Omega_1)}+\|u_j\|_{H^{s_2}(\Omega_2)}+\|u_j\|_{H^{s_3}(\Omega_3)}\right)^2,$$

where $\mu_3=\min\{s_3,N_3+1\}$.

6 Numerical experiments

In this section, we present some numerical results to demonstrate the accuracy of our numerical scheme (2.5). In all examples, we consider the Schrödinger equation (2.1) in $\Omega=[-1,1]^2$. Let Ω_1 be a circle with $R=0.2$ centered at the origin, $\Omega_2=[-0.5,0.5]^2\setminus\Omega_1$, and $\Omega_3=\Omega\setminus[-0.5,0.5]^2$. We set $N_1=N_2=15$ and then vary N_3 to be 1, 2, and 3 to test the convergence orders with respect to h_3. Furthermore, we take N_3 to be 1,2,3,4, and 5 to examine the exponential convergence with respect to N_3 and $\sqrt{DoF}$. Additionally, we set $\rho=1/4$ in (2.5).

Example 6.1 ($c=1$)**.** In the first example, let $c=1$. Since the exact eigenvalues are unknown, we choose $\lambda_1=12.3846240795409$, $\lambda_2=16.0225874297612$, $\lambda_3=16.0225874297612$ and $\lambda_4=22.0450602704655$ as the first four reference values. The reference eigenvalues are evaluated by the mortar spectral finite element method with $R=0.2$ and $N_1=N_2=15$. From Table 1, it is evident that the eigenvalues obtained by the numerical scheme (2.5) serve as the asymptotic lower bounds of the reference values. Furthermore, the observed convergence orders for $N_3=1,2,3$ are 2,4, and 6 respectively, aligning with the theoretical analysis.

Table 1: Errors of the first four eigenvalues and convergence orders with $c=1$.

$1/h_3$	$\lambda_1-\lambda_{1,h}$	order	$\lambda_2-\lambda_{2,h}$	order	$\lambda_3-\lambda_{3,h}$	order	$\lambda_4-\lambda_{4,h}$	order
				$N_3=1$				
8	9.9430E-01		2.0568E+00		1.9406E+00		4.4301E+00	
16	3.1783E-01	1.6454	6.6248E-01	1.6344	6.5281E-01	1.5718	1.4981E+00	1.5642
32	7.4687E-02	2.0893	1.5718E-01	2.0755	1.5501E-01	2.0743	3.5990E-01	2.0575
64	1.8186E-02	2.0380	3.8480E-02	2.0303	3.8130E-02	2.0234	8.9871E-02	2.0017
				$N_3=2$				
8	2.7238E-03		7.6524E-03		6.6679E-03		2.9277E-02	
16	2.4890E-04	3.4520	6.1720E-04	3.6321	5.6751E-04	3.5545	2.0589E-03	3.8298
32	1.4664E-05	4.0852	3.4963E-05	4.1418	3.4672E-05	4.0328	1.2109E-04	4.0877
64	9.3137E-07	3.9767	2.2021E-06	3.9889	2.1960E-06	3.9808	7.4576E-06	4.0212
				$N_3=3$				
8	7.8153E-06		1.8778E-05		1.6245E-05		7.4300E-05	
16	2.3886E-07	5.0320	5.3338E-07	5.1377	5.2310E-07	4.9568	1.9635E-06	5.2419
32	3.4581E-09	6.1101	7.8207E-09	6.0917	7.4351E-09	6.1366	2.7251E-08	6.1710
64	4.9505E-11	6.1263	1.1429E-10	6.0965	1.1037E-10	6.0739	4.1059E-10	6.0525

Example 6.2 ($c=1/2$)**.** In the second example, we take $c=1/2$ and choose $\lambda_1=8.37681498711058$, $\lambda_2=13.35313963139164$, $\lambda_3=13.35313963139164$ and $\lambda_4=20.33106215893244$ in [23] as the first four reference values. From Table 2, we can observe that the eigenvalues obtained by the numerical scheme (2.5) are the asymptotic lower bounds of the reference values, and the corresponding convergence orders for $N_3=1,2,3$ are 2,4, and 6 respectively, which is consistent with the theoretical analysis.

Example 6.3 ($c=2/3$)**.** In the third example, we test $c=2/3$ and choose λ_1=9.65231567885163, $\lambda_2=14.0914338712714$, $\lambda_3=14.0914338712714$ and $\lambda_4=20.7838715370525$ in [23] as the first four reference values. Similar to the results for $c=1$ and $c=1/2$, the cases for $N_3=1,2,3$ in Table 3 all achieve the theoretically predicted optimal convergence orders. This confirms the accuracy of our theoretical analysis and the efficacy of our numerical scheme (2.5).

Table 2: Errors of the first four eigenvalues and convergence orders with $c=1/2$.

$1/h_3$	$\lambda_1-\lambda_{1,h}$	order	$\lambda_2-\lambda_{2,h}$	order	$\lambda_3-\lambda_{3,h}$	order	$\lambda_4-\lambda_{4,h}$	order
				$N_3=1$				
8	4.8037E-01		1.6208E+00		1.5276E+00		4.1322E+00	
16	1.5198E-01	1.6602	5.2217E-01	1.6341	5.1482E-01	1.5691	1.4122E+00	1.5489
32	3.5374E-02	2.1031	1.2285E-01	2.0876	1.2082E-01	2.0913	3.3610E-01	2.0710
64	8.5273E-03	2.0525	2.9775E-02	2.0448	2.9538E-02	2.0322	8.3577E-02	2.0077
				$N_3=2$				
8	1.0893E-03		5.4658E-03		4.8326E-03		2.6618E-02	
16	1.0772E-04	3.3381	4.5715E-04	3.5797	4.2623E-04	3.5031	1.9062E-03	3.8037
32	6.2680E-06	4.1031	2.5756E-05	4.1497	2.5576E-05	4.0588	1.1105E-04	4.1014
64	3.9560E-07	3.9859	1.6204E-06	3.9905	1.6159E-06	3.9844	6.8159E-06	4.0262
				$N_3=3$				
8	2.2060E-06		1.1372E-05		1.0105E-05		6.4820E-05	
16	5.8238E-08	5.2433	2.9796E-07	5.2542	2.9125E-07	5.1167	1.6846E-06	5.2660
32	8.6265E-10	6.0771	4.3611E-09	6.0943	4.1959E-09	6.1171	2.3467E-08	6.1656
64	1.2236E-11	6.1396	6.2727E-11	6.1195	6.1441E-11	6.0937	3.5083E-10	6.0637

Table 3: Errors of the first four eigenvalues and convergence orders with $c=2/3$.

$1/h_3$	$\lambda_1-\lambda_{1,h}$	order	$\lambda_2-\lambda_{2,h}$	order	$\lambda_3-\lambda_{3,h}$	order	$\lambda_4-\lambda_{4,h}$	order
				$N_3=1$				
8	6.3234E-01		1.7396E+00		1.6408E+00		4.2105E+00	
16	2.0111E-01	1.6527	5.6094E-01	1.6329	5.5293E-01	1.5692	1.4352E+00	1.5527
32	4.6906E-02	2.1001	1.3227E-01	2.0844	1.3017E-01	2.0867	3.4241E-01	2.0675
64	1.1340E-02	2.0484	3.2147E-02	2.0407	3.1880E-02	2.0297	8.5245E-02	2.0061
				$N_3=2$				
8	1.5179E-03		6.0075E-03		5.2835E-03		2.7294E-02	
16	1.4605E-04	3.3776	4.9636E-04	3.5973	4.6086E-04	3.5191	1.9440E-03	3.8115
32	8.5349E-06	4.0970	2.8023E-05	4.1467	2.7807E-05	4.0508	1.1357E-04	4.0974
64	5.4101E-07	3.9796	1.7641E-06	3.9896	1.7615E-06	3.9806	6.9777E-06	4.0247
				$N_3=3$				
8	3.5896E-06		1.3084E-05		1.1506E-05		6.7052E-05	
16	1.0056E-07	5.1577	3.5249E-07	5.2141	3.4519E-07	5.0588	1.7522E-06	5.2580
32	1.4749E-09	6.0913	5.1727E-09	6.0905	4.9467E-09	6.1248	2.4376E-08	6.1676
64	2.0552E-11	6.1651	7.4143E-11	6.1245	7.2228E-11	6.0978	3.6489E-10	6.0618

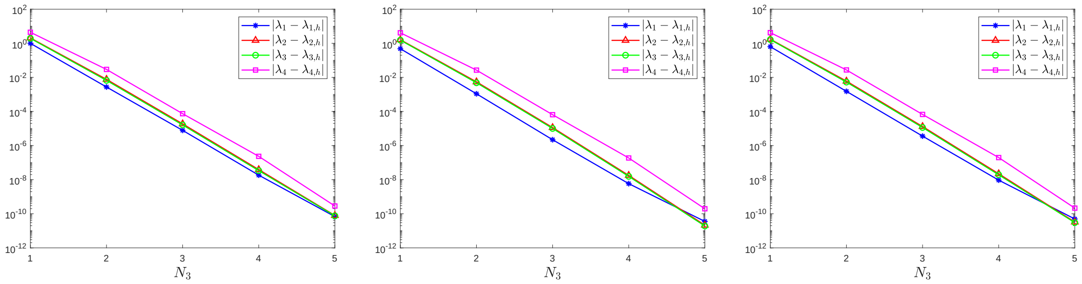

Figure 2: Errors of the first four eigenvalues versus N_3 in semi-log scale for $h_3=1/8$ with $c=1,1/2,2/3$.

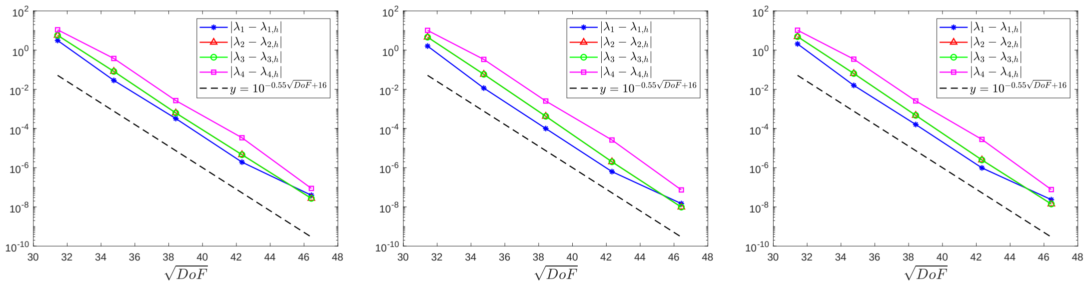

Figure 3: Errors of the first four eigenvalues versus $\sqrt{DoF}$ in semi-log scale with $c=1,1/2,2/3$. The dashed lines are $y=10^{-0.55\sqrt{DoF}+16}$.

Finally, we draw the error graphs for the first four eigenvalues against N_3 on a semi-log scale for $h_3=1/8$, with $c=1,1/2$, and $2/3$ as shown in Fig. 2. Subsequently, with N_1 and N_2 set to 10, we plot the error graphs for the first four eigenvalues versus $\sqrt{DoF}$ on a semi-log scale for c values of $1,1/2$, and $2/3$ in Fig. 3. For $c=1,1/2$, and $2/3$, the errors of eigenvalues converge exponentially with respect to both N_3 and $\sqrt{DoF}$. This observation supports the validity of our theoretical analysis. Although our numerical experiments have obtained the asymptotic lower bound approximations of the exact eigenvalues, providing a rigorous theoretical proof remains challenging and is left for future work.

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