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Numerical Analysis of an Optimal Control Problem Governed by the Stationary Navier-Stokes Equations with Global Velocity-Constrained

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Abstract. A state-constrained optimal control problem governed by the stationary Navier-Stokes equations is studied. Finite element approximation is constructed, the optimal-order *a priori* $\mathbf{H}^1$-norm and $\mathbf{L}^2$-norm error estimates are given, for which the optimal state is a nonsingular solution of the Navier-Stokes equations to the optimal control.

AMS subject classifications: 68U01, 76D05, 90C90, 93C20

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1 Introduction

Fluid flow exists various aspects of human activities, it is an important object of study of mechanics. From 1960s, fluid mechanics began permeating to other disciplines, and they were keeping intersecting with each other, which forms many new cross-disciplinary and interdisciplinary. As the impact of fluid mechanics expanding to many areas, people often encounters such problem in engineering applications: how to control the external conditions of the flow fields such as the temperature, the volume force or the boundary conditions to obtain or get closest to the flow field which has the observed velocity, pressure and boundary. That type of flow control problem becomes a more and more active field, which attracts the interest of researches in the recent years.

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Let us list some theoretical work existing in the literature. Abergel and Temam derive the first optimality conditions and give a gradient algorithm for the control problem of time-dependent Navier-Stokes (N-S) flow in [1]. The state-constrained flow control problems of unsteady N-S flow in the three-dimensional space are studied in [31]. Some theoretical works of the point-wise state-constrained flow control problem are discussed in [11–13], in which the optimality conditions, the regularity of the multiplier and the Lipschitz-stability of disturbance about the constraint set are investigated. In addition, some researchers study the more difficult Cahn-Hilliard-Navier-Stokes system, see [22], etc.

It is obvious that an efficient numerical analysis is quite essential to many applications of flow control. And the finite element method is undoubtedly the most appropriate tool to compute flow control problems. Recently some results of finite element approximation to control problems are developed. Gunzburger, Hou and Svobodny [18–20] study the finite element approximation of stationary N-S flow control problems with some weak control constraints, in which they assume the corresponding linearized system of the first-order optimality conditions defines some isomorphism. Casas and his colleagues discuss the point-wised control constrained stationary N-S flow control problem, and derive the error estimates by using the sufficient optimality conditions in [7]. Chang [8] give the superconvergence of the finite element approximation of the stationary Benard type under the pointwise control constraint. Liu and Yan [24] study the a posteriori error estimates of Stokes flow control problem. On the point of view of numerical algebra, the preconditioner for the N-S flow control model is given in [27]. In addition, a type of topology optimization problems for unsteady N-S flows is studied in [14] and references cited therein. And an approach to shape optimization problem governed by the instationary N-S equations is studied in [4], which is based on transformation to a reference domain with continuous adjoint computations. In addition, Gong and Zhou consider finite element approximations of parabolic control problems and convection-dominated diffusion control problems [16, 32, 33].

Very recently, some people are concerned to study the theory of the state constrained optimal control problems governed by partial differential equations, see, for example, [11, 13], in which they study the state pointwise constrained problem and the mixed control-state constrained one. In fact, one may concentrate on control problems with some weaker constraint conditions on the state in engineering applications. And there have some developments of the numerical analysis for the state integral constrained and the state energy-norm constrained problems, see [6, 25, 26]. But, to the best knowledge of the author, there is no result for the numerical analysis of the state-constrained optimal control problems governed by the stationary N-S equations existing in the literature. This paper is to study the optimal control problem with some global velocity constraints. We pay attention to the numerical analysis of the finite element approximation and concentrate on deriving the optimal-order error estimates of the optimal control and the corresponding states.

The outline of this article is as follows. In Section 2, we state the model problem