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Measures of Asymmetry Dual to Mean Minkowski Measures of Asymmetry for Convex Bodies

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Abstract: We introduce a family of measures (functions) of asymmetry for convex bodies and discuss their properties. It turns out that this family of measures shares many nice properties with the mean Minkowski measures. As the mean Minkowski measures describe the symmetry of lower dimensional sections of a convex body, these new measures describe the symmetry of lower dimensional orthogonal projections.

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1 Introduction

In the recent years, the measures of asymmetry (or symmetry), formulated by Grünbaum[1] for convex bodies (i.e., compact convex sets with nonempty interior in the standard Euclidean spaces) have drawn renewed attentions in convex geometry (see [2]–[10] and the references therein).

Among known measures of asymmetry (or symmetry), the Minkowski measure is probably the most important one, which has various generalizations and extensions (see, e.g., [5]–[7] and [9]). Among these generalizations, Toth[6] introduced a family of measures (functions) of symmetry σ_m, $m \geq 1$, called the mean Minkowski measures of symmetry (see below for definition), and did a systematic study on them in a series of papers (see [6]–[10]). Roughly speaking, for a convex body K, its (mth) mean Minkowski measure of symmetry

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σ_m is a function defined on int$K$, the interior of K, which, when $1 \le m \le n$, provides information of the shapes of m-dimensional sections of K.

As a main result of this paper, we introduce another family of measures (functions) of symmetry σ_m°, $m \ge 1$, defined on int$K$ as well, called the dual mean Minkowski measures of symmetry, which in a sense are the dual concept of the mean Minkowski measures. It turns out, for 2 or 3 dimensional convex bodies, that the dual mean Minkowski measures share almost all nice properties with the mean Minkowski measures and, to some extent, describe the shapes of projections of a convex body to lower dimensions.

2 Preliminaries

Denote by $\mathbf{R}^n$ the n-dimensional standard Euclidean space, and by $\mathcal{K}^n$ the family of all convex bodies in $\mathbf{R}^n$. $\langle \cdot , \cdot \rangle$ denotes the canonical inner product on $\mathbf{R}^n$ and $d(\cdot , \cdot)$ denotes the metric induced by this inner product. For a subset $S \subset \mathbf{R}^n$, conv$S$, cone$S$ denote the convex hull, the convex conical hull of S respectively, and lin$S$ denotes the linear subspace generated by S. $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere and Δ_n (or Δ simply) denotes an n-dimensional simplex (or n-simplex simply), i.e., $\Delta_n = \text{conv}\{v_0, v_1, \cdots, v_n\} \in \mathbf{R}^n$, where v_0, v_1, $\cdots$, v_n, called the vertices of Δ_n, are affinely independent. For other standard notations we refer to [11].

A function $f : \mathbf{R}^n \to \mathbf{R}$ is called affine if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in \mathbf{R}^n$ and $\alpha, \beta \in \mathbf{R}$. It is known that for any affine function f, there are unique $u \in \mathbf{R}^n$ and $b \in \mathbf{R}$ such that $f(\cdot) = \langle u , \cdot \rangle + b$.

Denote by $\mathcal{A}ff(\mathbf{R}^n)$ the family of all affine functions, called the affine dual space of $\mathbf{R}^n$, which is a linear space under the ordinary addition and scalar multiplication of functions and can be identified with $\mathbf{R}^{n+1}$ in a natural way (see [4] and [11]).

We recall the mean Minkowski measures defined by Toth[6]–[7].

Let $K \in \mathcal{K}^n$. The distortion function $\Lambda : \partial K \times \text{int}K \to \mathbf{R}$ is defined by

$$\Lambda(p, x) = \Lambda_K(p, x) = \frac{d(p, x)}{d(p^o, x)}, \qquad p \in \partial K,\ x \in \text{int}K,$$

where p^o denotes the opposite boundary point of p against x.

Let $m \ge 1$ and $x \in \text{int}K$. A multi-set $\{p_0, p_1, \cdots, p_n\}$, where $p_i \in \partial K$ and repetition is allowed, is called an m-configuration with respect to (w.r.t. for brevity) x if $x \in \text{conv}\{p_0, p_1, \cdots, p_n\}$. The set of m-configurations w.r.t. x is denoted by $\mathcal{C}_m(x) = \mathcal{C}_{K,m}(x)$.

Definition 2.1[8] *Let $K \in \mathcal{K}^n$ and $m \ge 1$. We define its (mth) mean Minkowski measure (function) $\sigma_m = \sigma_{K,m} : \text{int}K \to \mathbf{R}$ by*

$$\sigma_m(x) = \inf_{\{p_0, p_1, \cdots, p_m\} \in \mathcal{C}_m(x)} \sum_{i=0}^{m} \frac{1}{\Lambda(p_i, x) + 1}, \qquad x \in \text{int}K.$$

The sequence $\{\sigma_m\}_{m \ge 1}$ has many nice properties, e.g., Toth[7] proved that $\{\sigma_m\}_{m \ge 1}$ is sub-arithmetic (which is a crucial tool in Toth's work), i.e., for any $m, k \ge 1$, $x \in \text{int}K$,