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The Shephard Type Problems for General L_p-Centroid Bodies

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Abstract: In this paper, combining with the L_p-dual geominimal surface area and the general L_p-centroid bodies, we research the Shephard type problems for general L_p-centroid bodies.

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1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbf{R}^n$. Let $\mathcal{K}^n_o$ and $\mathcal{K}^n_c$ respectively denote the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin. Besides, for the set of star bodies (about the origin) and the set of star bodies whose centroids lie at the origin in $\mathbf{R}^n$, we write $\mathcal{S}^n_o$ and $\mathcal{S}^n_c$, respectively. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbf{R}^n$, its volume is written by $\omega_n = V(B)$.

The notion of centroid body was introduced by Petty[1]. In [2], for a compact set K, the centroid body, ΓK, of K is an origin-symmetric convex body whose support function is defined by

$$h(\Gamma K, u) = \frac{1}{V(K)} \int_K |u \cdot x| \mathrm{d}x, \qquad u \in S^{n-1}.$$

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The centroid body is one of the most important notions in the Brunn-Minkowski theory. In recent decades, the centroid bodies have attracted increasing attention (see [2] and [3]).

In 1997, Lutwak and Zhang[4] introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbf{R}^n$ and real number $p \geq 1$, the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h(\Gamma_p K, u)^p &= \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p \mathrm{d}x \\ &= \frac{1}{c_{n,p}(n+p) V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho(K, v)^{n+p} \mathrm{d}S(v), \qquad u \in S^{n-1}. \end{aligned} \tag{1.1}$$

Here

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}},$$

and $\mathrm{d}S(v)$ denotes the standard spherical Lebesgue measure on S^{n-1}. Regarding the investigations of L_p-centroid bodies, we may refer to [5]–[14].

In 2005, Ludwig[15] introduced a function $\varphi_\tau : \mathbf{R} \to [0, +\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t \tag{1.2}$$

with a parameter $\tau \in [-1, 1]$.

Based on L_p-centroid bodies and function (1.2), Feng *et al.*[16] defined a corresponding notion of general L_p-centroid bodies in [16]. For $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p-centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$\begin{aligned} h(\Gamma_p^\tau K, u)^p &= \frac{1}{c_{n,p}(\tau) V(K)} \int_K \varphi_\tau(u \cdot x)^p \mathrm{d}x \\ &= \frac{1}{c_{n,p}(\tau)(n+p) V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho(K, v)^{n+p} \mathrm{d}S(v), \end{aligned} \tag{1.3}$$

where

$$c_{n,p}(\tau) = \frac{1}{2} c_{n,p} [(1+\tau)^p + (1-\tau)^p].$$

The normalization is chosen such that

$$\Gamma_p^\tau B = B, \qquad \tau \in [-1, 1]$$

and

$$\Gamma_p^0 K = \Gamma_p K.$$

For the more investigations of general L_p-centroid bodies, see [16]–[18].

Combined with L_p-mixed volume, Lutwak[19] gave the definition of L_p-geominimal surface area. For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf \left\{ n V_p(K, Q) V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n \right\}.$$

Here $V_p(M, N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}_o^n$. When $p = 1$, $G_1(K)$ is just the classical counterpart which was introduced by Petty[20]. The L_p-geominimal surface area have got many results from these articles (see [21]–[24]).

According to the notion of L_p-geominimal surface area and L_p-dual mixed volume, Wang and Qi[25] introduced the definition of L_p-dual geominimal surface area. For $K \in \mathcal{S}_c^n$, $p \geq 1$, the L_p-dual geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}(K)=\inf\left\{n\widetilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}}:Q\in\mathcal{K}_c^n\right\}. \tag{1.4}$$

Here $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M,N\in\mathcal{S}_o^n$. For the studies of L_p-dual geominimal surface area, some results have been obtained in many articles (see [26]–[32]).

Let $\mathcal{W}_p^n$ denote the set of the polar of all L_p-centroid bodies. Because of the polar of L_p-centroid body is the origin-symmetric, thus

$$\mathcal{W}_p^n\subseteq\mathcal{K}_c^n.$$

For the convenience of our work, we improve the definition (1.4) from $Q\in\mathcal{K}_c^n$ to $Q\in\mathcal{W}_p^n$ as follows:

If $Q\in\mathcal{W}_p^n$ in (1.4), we write $\widetilde{G}^{\circ}_{-p}(K)$ by

$$\omega_n^{-\frac{p}{n}}\widetilde{G}^{\circ}_{-p}(K)=\inf\left\{n\widetilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}}:Q\in\mathcal{W}_p^n\right\}. \tag{1.5}$$

In this paper, associated with the L_p-dual geominimal surface area (1.5), we research the Shephard type problem for the general L_p-centroid bodies. In Section 2, we recall some notations and background material. In Section 3, we give and prove main results.

2 Notation and Background Material

2.1 Support Function, Radial Function and Polar of Convex Bodies

Let $\mathbf{R}$ be the set of real numbers. If $K\in\mathcal{K}^n$, then the support function of K,

$$h_K=h(K,\,\cdot\,):\mathbf{R}^n\to\mathbf{R},$$

is defined by (see [2])

$$h(K,\,\boldsymbol{x})=\max\{\boldsymbol{x}\cdot\boldsymbol{y}:\boldsymbol{y}\in K\},\qquad \boldsymbol{x}\in\mathbf{R}^n,$$

where $\boldsymbol{x}\cdot\boldsymbol{y}$ denotes the standard inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{R}^n$.

If K is a compact star shaped (about the origin) in $\mathbf{R}^n$, then its radial function,

$$\rho_K=\rho(K,\,\cdot\,):\mathbf{R}^n\setminus\{0\}\to[0,\,+\infty),$$

is defined by ([3])

$$\rho(K,\,\boldsymbol{x})=\max\{\lambda\ge 0:\lambda\boldsymbol{x}\in K\},\qquad \boldsymbol{x}\in\mathbf{R}^n\setminus\{0\}.$$

If ρ_K is positive and continuous, K is called a star body (with respect to origin).

If E is a nonempty subset and contains the origin in $\mathbf{R}^n$, then the polar set, E^*, of E is defined by (see [2] and [3])

$$E^*=\{\boldsymbol{x}\in\mathbf{R}^n:\boldsymbol{x}\cdot\boldsymbol{y}\le 1,\ \boldsymbol{y}\in E\}.$$

Meanwhile, it is easy to get that $(K^*)^*=K$ for $K\in\mathcal{K}_o^n$.

From the above definitions, we see that if $K\in\mathcal{K}_o^n$, then ([2] and [3])

$$h(K^*,\,\cdot\,)=\frac{1}{\rho(K,\,\cdot\,)},\qquad \rho(K^*,\,\cdot\,)=\frac{1}{h(K,\,\cdot\,)}. \tag{2.1}$$

Associated with (2.1), if $K,L\in\mathcal{K}_o^n$ and $K\subseteq L$, then $K^*\supseteq L^*$.

2.2 L_p-mixed Volume and L_p-dual Mixed Volume

If $K,\, L \in \mathcal{K}_o^n$, then for $p \ge 1$, the L_p-mixed volume, $V_p(K, L)$, of K and L is given by (see [33])

$$V_p(K,\, L) = \frac{1}{n}\int_{S^{n-1}} h(L,\, u)^p \mathrm{d}S_p(K,\, u). \tag{2.2}$$

Here $S_p(K,\,\cdot\,)$ denotes the L_p-surface area measure of K.

In 1996, Lutwak[19] introduced the notion of L_p-dual mixed volume as follows: If $K, L \in \mathcal{S}_o^n$, $p \ge 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is given by

$$\widetilde{V}_{-p}(K,\, L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,\, u)^{n+p}\rho(L,\, u)^{-p}\mathrm{d}S(u). \tag{2.3}$$

From (2.3) it is easy to see that

$$\widetilde{V}_{-p}(K,\, K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,\, u)^{n}\mathrm{d}S(u). \tag{2.4}$$

The L_p-dual Minkowski's inequality can be stated as follows:

Theorem 2.1[19] *If $K, L \in \mathcal{S}_o^n$, $p \ge 1$, then*

$$\widetilde{V}_{-p}(K,\, L) \ge V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}} \tag{2.5}$$

with equality if and only if K and L are dilates.

2.3 General L_p-projection Body

For $K \in \mathcal{K}_o^n$, $p \ge 1$, the L_p-projection body, $\mathit{\Pi}_p K$, of K is an origin-symmetric convex body whose support function is given by (see [9])

$$h(\mathit{\Pi}_p K,\, u)^p = \frac{1}{n\omega_n c_{n-2,\,p}}\int_{S^{n-1}} |u\cdot v|^p \mathrm{d}S_p(K,\, v), \qquad u, v \in S^{n-1}.$$

In 2005, Ludwig[15] introduced the general L_p-projection body by the function $\varphi_\tau(t)$: For $K \in \mathcal{K}_o^n$, $p \ge 1$, the general L_p-projection body, $\mathit{\Pi}_p^\tau K \in \mathcal{K}_o^n$, of K is given by

$$h(\mathit{\Pi}_p^\tau K,\, u)^p = \alpha_{n,\,p}(\tau)\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p \mathrm{d}S_p(K,\, v), \tag{2.6}$$

where

$$\alpha_{n,\,p}(\tau) = \frac{\alpha_{n,\,p}}{(1+\tau)^p + (1-\tau)^p},$$

with

$$\alpha_{n,\,p} = \frac{1}{c_{n,\,p}(n+p)\omega_n}.$$

The normalization is chosen such that

$$\mathit{\Pi}_p^\tau B = B, \quad \tau \in [-1,\, 1], \qquad \mathit{\Pi}_p^0 K = \mathit{\Pi}_p K.$$

3 Main Theorems and Proofs

Lemma 3.1[16] *If $K, L \in \mathcal{S}_o^n$, $p \ge 1$ and $\tau \in [-1,\, 1]$, then*

$$\frac{\widetilde{V}_{-p}(K,\, \mathit{\Gamma}_p^{\tau,*} L)}{V(K)} = \frac{\widetilde{V}_{-p}(L,\, \mathit{\Gamma}_p^{\tau,*} K)}{V(L)}. \tag{3.1}$$

Theorem 3.1 *For $K,\ L\in\mathcal{S}^n_c$, $p\geq 1$ and $\tau\in[-1,\ 1]$, if $\Gamma^\tau_p K\subseteq\Gamma^\tau_p L$, then*

$$\frac{\widetilde{G}^\circ_{-p}(K)}{V(K)}\leq\frac{\widetilde{G}^\circ_{-p}(L)}{V(L)},\tag{3.2}$$

equality holds when

$$\Gamma^\tau_p K=\Gamma^\tau_p L.$$

Proof. For $K,\ L\in\mathcal{S}^n_c$, $p\geq 1$ and $\tau\in[-1,\ 1]$, if $\Gamma^\tau_p K\subseteq\Gamma^\tau_p L$, then

$$\rho(\Gamma^{\tau,*}_p K,\ \cdot\ )\geq\rho(\Gamma^{\tau,*}_p L,\ \cdot\ )$$

with equality if and only if

$$\Gamma^{\tau,*}_p K=\Gamma^{\tau,*}_p L.\tag{3.3}$$

From (1.5) and (3.1), taking $Q=\Gamma^{\tau,*}_p N$, we get

$$\begin{aligned}\frac{\omega_n^{-\frac{p}{n}}\widetilde{G}^\circ_{-p}(K)}{V(K)}&=\ \inf\left\{n\frac{\widetilde{V}_{-p}(K,\ Q)}{V(K)}V(Q^*)^{-\frac{p}{n}}:Q\in\mathcal{W}^n_p\right\}\\&=\ \inf\left\{n\frac{\widetilde{V}_{-p}(K,\ \Gamma^{\tau,*}_p N)}{V(K)}V(\Gamma^\tau_p N)^{-\frac{p}{n}}:\Gamma^{\tau,*}_p N\in\mathcal{W}^n_p\right\}\\&=\ \inf\left\{n\frac{\widetilde{V}_{-p}(N,\ \Gamma^{\tau,*}_p K)}{V(N)}V(\Gamma^\tau_p N)^{-\frac{p}{n}}:\Gamma^{\tau,*}_p N\in\mathcal{W}^n_p\right\}\\&\leq\ \inf\left\{n\frac{\widetilde{V}_{-p}(N,\ \Gamma^{\tau,*}_p L)}{V(N)}V(\Gamma^\tau_p N)^{-\frac{p}{n}}:\Gamma^{\tau,*}_p N\in\mathcal{W}^n_p\right\}\\&=\ \inf\left\{n\frac{\widetilde{V}_{-p}(L,\ \Gamma^{\tau,*}_p N)}{V(L)}V(\Gamma^\tau_p N)^{-\frac{p}{n}}:\Gamma^{\tau,*}_p N\in\mathcal{W}^n_p\right\}\\&=\frac{\omega_n^{-\frac{p}{n}}\widetilde{G}^\circ_{-p}(L)}{V(L)}.\end{aligned}$$

This gives (3.2), and equality holds when

$$\Gamma^\tau_p K=\Gamma^\tau_p L.$$

Lemma 3.2 *If $K\in\mathcal{K}^n_o$, $L\in\mathcal{S}^n_o$ and $p\geq 1$, then*

$$V_p(K,\ \Gamma^\tau_p L)=\frac{2}{c_{n,p}\alpha_{n,p}(n+p)V(L)}\widetilde{V}_{-p}(L,\ \Pi^{\tau,*}_p K).$$

Proof. By (2.2), (1.3) and (2.6), we have

$$\begin{aligned}V_p(K,\ \Gamma^\tau_p L)&=\frac{1}{n}\int_{S^{n-1}}h(\Gamma^\tau_p L,\ u)^p\mathrm{d}S_p(K,\ u)\\&=\frac{1}{n}\int_{S^{n-1}}\frac{1}{c_{n,p}(\tau)(n+p)V(L)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho(L,\ v)^{n+p}\mathrm{d}S(v)\mathrm{d}S_p(K,u)\\&=\frac{(1+\tau)^p+(1-\tau)^p}{n\alpha_{n,p}c_{n,p}(\tau)(n+p)V(L)}\int_{S^{n-1}}\rho(L,\ v)^{n+p}h(\Pi^\tau_p K,v)^p\mathrm{d}S(v)\\&=\frac{(1+\tau)^p+(1-\tau)^p}{n\alpha_{n,p}c_{n,p}(\tau)(n+p)V(L)}\int_{S^{n-1}}\rho(L,\ v)^{n+p}\rho(\Pi^{\tau,*}_p K,v)^{-p}\mathrm{d}S(v)\\&=\frac{2}{\alpha_{n,p}c_{n,p}(n+p)V(L)}\widetilde{V}_{-p}(L,\ \Pi^{\tau,*}_p K).\end{aligned}$$

This yields (3.3).

Lemma 3.3 *For $K, L \in \mathcal{S}_c^n$ and $p \geq 1$, if $\Gamma_p^\tau K \subseteq \Gamma_p^\tau L$ and L is the polar of general L_p-projection body, then*
$$V(K) \leq V(L) \tag{3.4}$$
with equality if and only if $K = L$.

Proof. For $K, L \in \mathcal{S}_c^n$ and $p \geq 1$, if $\Gamma_p^\tau K \subseteq \Gamma_p^\tau L$, then for any $M \in \mathcal{K}_o^n$, we know
$$V_p(M,\, \Gamma_p^\tau K) \leq V_p(M,\, \Gamma_p^\tau L).$$
This together with (3.3), we get
$$\frac{1}{V(K)}\widetilde{V}_{-p}(K,\, \Pi_p^{\tau,*}M) \leq \frac{1}{V(L)}\widetilde{V}_{-p}(L,\, \Pi_p^{\tau,*}M). \tag{3.5}$$
Since L is the polar of general L_p-projection body, thus taking
$$\Pi_p^{\tau,*}M = L$$
in (3.5), and according to (2.4) and (2.5), we have
$$V(K) \geq \widetilde{V}_{-p}(K, L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}},$$
i.e.,
$$V(K) \leq V(L).$$

According to the equality condition of (2.5), we see that
$$V(K) = V(L)$$
if and only if K and L are dilates and
$$\Gamma_p^\tau K = \Gamma_p^\tau L,$$
these imply
$$K = L.$$
Hence, equality holds in (3.4) if and only if
$$K = L.$$

Theorem 3.2 *For $K,\, L \in \mathcal{S}_c^n$, $p \geq 1$ and $\tau \in [-1,\, 1]$, if $\Gamma_p^\tau K \subseteq \Gamma_p^\tau L$ and L is the polar of general L_p-projection body, then*
$$\widetilde{G}_{-p}^\circ(K) \leq \widetilde{G}_{-p}^\circ(L) \tag{3.6}$$
with equality if and only if
$$K = L.$$

Proof. For $K,\, L \in \mathcal{S}_c^n$, $p \geq 1$ and $\tau \in [-1,\, 1]$, if $\Gamma_p^\tau K \subseteq \Gamma_p^\tau L$, then from Theorem 3.1, we know
$$\frac{\widetilde{G}_{-p}^\circ(K)}{V(K)} \leq \frac{\widetilde{G}_{-p}^\circ(L)}{V(L)},$$
i.e.,
$$\frac{\widetilde{G}_{-p}^\circ(K)}{\widetilde{G}_{-p}^\circ(L)} \leq \frac{V(K)}{V(L)}. \tag{3.7}$$
Since L is the polar of general L_p-projection body, thus from Lemma 3.3 and (3.7), we get
$$\widetilde{G}_{-p}^\circ(K) \leq \widetilde{G}_{-p}^\circ(L).$$

This is just inequality (3.6).

Now, we give the equality condition of inequality (3.6). If

$$\widetilde{G}^{\circ}_{-p}(K) = \widetilde{G}^{\circ}_{-p}(L),$$

then by (3.7) and Lemma 3.3 we obtain

$$1 \leq \frac{V(K)}{V(L)} \leq 1,$$

i.e.,

$$V(K) = V(L).$$

This combining with the equality condition of (3.4) yields $K = L$.

Conversely, if $K = L$, by definition (1.5), then we easily get

$$\widetilde{G}^{\circ}_{-p}(K) = \widetilde{G}^{\circ}_{-p}(L).$$

To sum up, we see that equality holds in (3.6) if and only if

$$K = L.$$

This completes the proof.

Putting $\tau = 0$ in Theorem 3.2, we obtain a positive answer for the Shephard type problem of L_p-centroid body as follows:

Corollary 3.1 *For $K,\, L \in \mathcal{S}^n_c$, $p \geq 1$, if $\Gamma_p K \subseteq \Gamma_p L$ and L is the polar of L_p-projection body, then*

$$\widetilde{G}^{\circ}_{-p}(K) \leq \widetilde{G}^{\circ}_{-p}(L)$$

with equality if and only if

$$K = L.$$

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