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Rota-Baxter Operators on 3-dimensional Lie Algebras and Solutions of the Classical Yang-Baxter Equation

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Abstract: In this paper, we compute Rota-Baxter operators on the 3-dimensional Lie algebra g whose derived algebra's dimension is 2. Furthermore, we give the corresponding solutions of the classical Yang-Baxter equation in the 6-dimensional Lie algebras $g \ltimes_{ad^*} g^*$ and some new structures of left-symmetric algebra induced from g and its Rota-Baxter operators.

Key words: Rota-Baxter operators, 3-dimensional Lie algebra, classical Yang-Baxter equation, left-symmetric algebra
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1 Introduction

According to the Winternitz classification (see [1]), there are six kinds of 3-dimensional Lie algebras up to isomorphism over the complex field $\mathbb{C}$. That is,

$$
\begin{aligned}
&g_1\colon [e_1, e_2] = 0,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0,\\
&g_2\colon [e_1, e_2] = 0,\ [e_1, e_3] = e_3,\ [e_2, e_3] = 0,\\
&g_3\colon [e_1, e_2] = e_3,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0,\\
&g_4\colon [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\\
&g_5\colon [e_1, e_2] = e_1,\ [e_1, e_3] = 0,\ [e_2, e_3] = e_1 + e_3,\\
&g_6\colon [e_1, e_2] = e_1,\ [e_1, e_3] = 0,\ [e_2, e_3] = ke_3 \quad (0 < |k| \leq 1).
\end{aligned}
$$

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We know that g_4 is the famous 3-dimensional simple Lie algebra $sl(2, \mathbb{C})$. The others are nonsimple. In [2], the authors gave all Rota-Baxter operators (of weight zero) on g_4 and the corresponding solutions of the classical Yang-Baxter equation. In [3], Rota-Baxter operators on another 3-dimensional non-simple Lie algebra g_5 were determined, the corresponding solutions of the classical Yang-Baxter equation and some new structures of left symmetric algebra are given. In [4], the authors determine the Rota-Baxter operators on g_2 and g_3. For g_1, it is clear that its Rota-Baxter operators are belong to its endomorphisms. Thus, in order to determine the Rota-Baxter operators on 3-dimensional Lie algebras, we just determine the Rota-Baxter operators on g_6. The aim of this paper is to determine the Rota-Baxter operators (of weight zero) on g_6 and the corresponding solutions of the Yang-Baxter equation. After this, we completely determine all of the Rota-Baxter operators (of weight zero) on all 3-dimensional Lie algebras. From now on, we denote g_6 as g.

A Rota-Baxter operator of weight zero on an associative algebra A is defined to be a linear map $P\colon A \to A$ satisfying

$$P(x)P(y) = P(P(x)y + xP(y)), \qquad x, y \in A. \tag{1.1}$$

Rota-Baxter operators on associative algebras were introduced by G. Baxter to solve an analytic formula in probability (see [5]). It has been related to other areas in Mathematics and Mathematical Physics (see [6]–[9]). A Rota-Baxter operator of weight zero on a Lie algebra $(g, [\,\cdot\,, \cdot\,])$ is a linear operator $P\colon g \to g$ such that

$$[P(x), P(y)] = P([P(x), y] + [x, P(y)]), \qquad x, y \in g. \tag{1.2}$$

In fact, a Rota-Baxter operator is also called the operator form of the classical Yang-Baxter equation (see [10] and [11]). Let g be a Lie algebra and

$$r = \sum_i a_i \otimes b_i \in g \otimes g.$$

Then r is called a solution of the classical Yang-Baxter equation (CYBE) in g if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.3}$$

in $U(g)$, where $U(g)$ is the universal enveloping algebra of g and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

Semenov-Tian-Shansky[12] proved that a Rota-Baxter operator of weight 0 on a Lie algebra is exactly the operator form of the classical Yang-Baxter equation (1.3). On the one hand, Rota-Baxter operators of weight 0 on a Lie algebra g give rise to solutions of CYBE on the double Lie algebra $g \ltimes_{ad^*} g^*$ over the direct sum $g \bigoplus g^*$ of the Lie algebra g and its dual space g^* (see [2], [13]). Moreover, some solutions of CYBE in $g \ltimes_{ad^*} g^*$ Lie algebras through Rota-Baxter operators of any weight on g can be obtained (see [3], [14]). On the other hand, some certain interesting algebraic structures, such as left-symmetric algebras, coming out of the Rota-Baxter operators. In this paper, we determine the Rota-Baxter operators on 3-dimensional Lie algebra g and give a family of solutions of CYBE in $g \ltimes_{ad^*} g^*$. Finally, the induced left-symmetric algebraic structures from the Rota-Baxter operator of weight 0 on a Lie algebra g are obtained.