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The Homotopy Perturbation Renormalization Group Method to Solve the WKB Problem with Turn Points

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Abstract: In this paper, we give the homotopy perturbation renormalization group method, this is a new method for turning point problem. Using this method, the independent variables are introduced by transformation without introducing new related variables and no matching is needed. The WKB approximation method problem can be solved.
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1 Introduction

Perturbation method is widely used in solving nonlinear problems (see [1]), but for some problems the expansions of solutions are inconsistent. In order to overcome inconsistency, scholars have developed many singular perturbation techniques, and WKB method (see [2]–[3]) is one of the effective methods. WKB approximation method theory is a well-known powerful tool for obtaining global approximate solutions of linear differential equations. Many linear problems often solved by boundary layer theory can also be solved by WKB theory. The limitation of the WKB method is that it can only be applied to linear equations. The standard WKB approximation fails near the turning point. By using Langer or Liouville-green transformation, an effective expansion including turning points can be obtained. However, for transformations such as Liouville-green and Langer, interdependent scaling variables are introduced into the transformation process. These variables are not

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easy to select, and also make the calculation cumbersome. In 1998, He[4] combined the perturbation method with the homotopy method and proposed the homotopy perturbation method. The homotopy perturbation method can be used to solve the nonlinear problems effectively (see [5]–[7]). In order to overcome the above difficulties, inspired by the perturbed renormalization group theory (see [8]–[9]) and the homotopy perturbation method, we propose the homotopy perturbation renormalization group method in this paper. We use homotopy renormalization group method by independent variables through transformation to easily solve the problem of WKB approximation without introducing new related variables and matching. Therefore, many WKB approximation method or multi-scale analysis of dynamics quantum problems can also be solved by the homotopy renormalization group method. In this paper, we use homotopy renormalization group method to solve the WKB problem with turning point.

2 The Homotopy Perturbation Renormalization Group Method

In 1915, Gans[10], a physicist, studied the propagation of light in inhomogeneous media and got typical equation from the Maxwell equation. We consider the following typical equation of WKB problem with turning point:

$$\epsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = Q(x)u(x), \tag{2.1}$$

$$u(0) = a, \quad u(+\infty) = 0, \tag{2.2}$$

where ϵ is a small parameter.

We make the following assumptions:

(C1) The equation (2.1) has solutions under the initial condition (2.2);

(C2) There are isolated zeros at $x = 0$ of $Q(x)$, let $Q(x) = x^\alpha \varphi(x)$, where φ is a positive function.

The system (2.1)–(2.2) is studied by using homotopy permutation renormalization group method. We firstly construct uniform effective evolutionary expansion as the solution of equation (2.1). We introduce new independent variables t, by $\mathrm{d}t = \dfrac{1}{\epsilon}\left(\dfrac{Q}{t^\alpha}\right)^{\frac{1}{2}} \mathrm{d}x$, we have

$$t(x) = \left(\frac{2+\alpha}{2\epsilon} \int_0^x \sqrt{Q(v)}\mathrm{d}v\right)^{\frac{2}{2+\alpha}}. \tag{2.3}$$

The equation (2.1) is transformed into

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = t^\alpha u + \epsilon S(t(x)) \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{2.4}$$

where

$$S \equiv \frac{\mathrm{d}\left[\left(\dfrac{t^\alpha}{Q}\right)^{\frac{1}{2}}\right]}{\mathrm{d}x}. \tag{2.5}$$

When $x \to 0$, $t \sim x$, so S is bounded near $x = 1$. To simplify, we consider the case of $\alpha = 1$.