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Least Squares Properties of Generalized Inverses

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Dedicated to Prof. Roger Penrose for his 90th Birthday

Abstract. The aim of this paper is to systematize solutions of some systems of linear equations in terms of generalized inverses. As a significant application of the Moore-Penrose inverse, the best approximation solution to linear matrix equations (i.e. both least squares and the minimal norm) is considered. Also, characterizations of least squares solution and solution of minimum norm are given. Basic properties of the Drazin-inverse solution and the outer-inverse solution are present. Motivated by recent research, important least square properties of composite outer inverses are collected.

AMS subject classifications: 15A09, 15A24, 65F05

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1 Introduction

Roger Penrose is a famous English mathematical physicist, mathematician, philosopher of science. Penrose has made fundamental contributions to physics, ma-

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thematics, and geometry. In the 1960s calculated many of the basic features of black holes. For his results in black holes investigation, he was awarded the 2020 Nobel Laureate in Physics. List of contributions of Roger Penrose: Moore-Penrose pseudoinverse, Twistor theory, Spin network, Abstract index notation, Black hole bomb, Geometry of spacetime, Cosmic censorship, Weyl curvature hypothesis, Penrose interpretation of quantum mechanics, Diosi-Penrose model, Newman-Penrose formalism, Penrose diagram, Penrose inequality, Penrose process, Penrose tiling, Penrose stairs, Penrose graphical notation, Penrose transform, Penrose-Terrell effect, pp-wave spacetime, Schrödinger-Newton equations, Orch-OR/Penrose-Lucas argument, FELIX experiment, Trapped surface, Andromeda paradox, Conformal cyclic cosmology.

We are interested in the mathematical aspect of the Penrose's scientific research, i.e., in the research related with the Moore-Penrose inverse (known also as the pseudoinverse). The main results of this aspect of Penrose's research are surveyed and analyzed. In addition, the impact of these studies on contemporary research in the field of generalized matrix inversions is presented.

The idea for introducing the definition of generalized inverses of matrices arisen from the necessity of finding a solution of a given system of linear equations (SoLE). This problem appears in many scientific and practical disciplines, such as statistics, operational research, physics, economy, electrical engineering, and many others. Generalized inverses provide a simple way for obtaining a solution of the so called ill-conditioned linear problems. Explicitly, generalized inverses of matrices has appeared bit later in 1920 in the paper [36] of the scientist Moore. However, his work was not continued in the next 30 years, first of all because of the way the work was presented and the ambiguous notation. The research on this topic was initiated by the scientist Bjerhammar in 1951. The real evolution in the development of this area has started with the paper [49] published by Penrose in 1955. The general reciprocal, originated by Bjerhammar and rediscovered by Penrose in [48, 49] is known as the Moore-Penrose inverse and reached enormous popularity.

While working on his doctorate, Penrose published articles on semigroups, and on rings of matrices. The paper [49] has been published in 1955 as the main result of this research. In [49], Penrose defined a generalized inverse X of a complex rectangular (or possibly square and singular) matrix A to be the unique solution to the four matrix equations, known as Penrose equations. This result has been widely used as the following statement: for arbitrary $A\in\mathbb{C}^{m\times n}$ (where $\mathbb{C}^{m\times n}$ denotes the set of $m\times n$ complex matrices), there exists the Moore-Penrose (or shortly MP) inverse of A (denoted by $A^{\dagger}$), that is, the unique matrix $X\in\mathbb{C}^{n\times m}$ which satisfies the Penrose equations [49]