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On Jumped Wenger Graphs

Li-Ping Wang[1,2,*], Daqing Wan[3], Weiqiong Wang[4]
and Haiyan Zhou[5]

[1] *State Key Laboratory of Information Security, Institute of Information Engineering, CAS, China.*
[2] *School of Cyber Security, University of Chinese Academy of Sciences, China.*
[3] *Department of Mathematics, University of California, Irvine, CA, USA.*
[4] *School of Science, Chan'an University, Xi'an, China.*
[5] *School of Mathematics, Nanjing Normal University, Nanjing, China.*

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Abstract. We introduce a new infinite class of bipartite graphs, called jumped Wenger graphs, which has a similar structure with Wenger graphs. We give a tight upper bound of the diameter for these graphs and the exact diameter for some special graphs. We also determine the girth of the jumped Wenger graphs.

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1 Introduction

Recently, researchers started to focus on a class of bipartite graphs related to Wenger graphs because of their nice graph theoretic properties [1, 2, 4, 5, 7–10]. Let us to describe these graphs first.

Let $\mathbb{F}_q$ be a finite field of order $q=p^e$, where p is a prime and e is a positive integer. Let m be a positive integer and let $\mathfrak{P}=\mathbb{F}_q^{m+1}$ and $\mathfrak{L}=\mathbb{F}_q^{m+1}$ be two copies of the $(m+1)$-dimensional vector space over $\mathbb{F}_q$, which are called the point set and the line set respectively. Let $\mathfrak{G}=(V,E)$ be the bipartite graph with vertex

*Corresponding author. *Email address:* `wangliping@iie.ac.cn` (L.-P. Wang)

set $V=\mathfrak{P}\cup\mathfrak{L}$ and edge set E, defined as follow. Given the polynomial functions $f_i(x)\in\mathbb{F}_q[x]$, $2\le i\le m+1$, there is an edge from a point $P=(p_1,p_2,\cdots,p_{m+1})\in\mathfrak{P}$ to a line $L=[l_1,l_2,\cdots,l_{m+1}]\in\mathfrak{L}$, denoted by PL, if the following m equalities hold:

$$\begin{aligned}&l_2+p_2=l_1f_2(p_1),\\&l_3+p_3=l_1f_3(p_1),\\&\quad\cdots\cdots\\&l_{m+1}+p_{m+1}=l_1f_{m+1}(p_1).\end{aligned}$$

If $(1,f_2(x),\ldots,f_{m+1}(x))=(1,x,x^2,\ldots,x^m)$, such graphs are called Wenger graphs [2]. If $(1,f_2(x),\ldots,f_{m+1}(x))=(1,x,x^p,\ldots,x^{p^{m-1}})$, this class of graphs is called linearized Wenger graphs [1]. In this paper, we focus on the more general case when

$$\begin{aligned}&\big(1,f_2(x),\ldots,f_{m+1}(x)\big)\\&=\big(1,x,x^2,\ldots,x^{i-1},x^{i+1},\ldots,x^{j-1},x^{j+1},\ldots,x^{m+2}\big),\quad 1\le i<j\le m+2,\end{aligned}$$

which is called the jumped Wenger graphs with jump points at x^i and x^j, denoted by $J_m(q,i,j)$. In particular, if $j=m+2$, that is,

$$\big(1,f_2(x),\ldots,f_{m+1}(x)\big)=\big(1,x,\ldots,x^{i-1},x^{i+1},\ldots,x^m,x^{m+1}\big),$$

the graphs have only one jump point at x^i. If $i=m+1$ and $j=m+2$, the graphs become Wenger graphs, denoted by $W_m(q)$.

The following properties are immediate, see [1].

Proposition 1.1. *The jumped Wenger graph $J_m(q,i,j)$ for $1\le i<j\le m+2$ is q-regular.*

Proposition 1.2. *If $m+2<q$, the jumped Wenger graph $J_m(q,i,j)$, for $1\le i<j\le m+1$, is connected.*

The organization of this article is as follows. In Section 2 we give an upper bound of the diameter of a jumped Wenger graph $J_m(q,i,j)$ for any integers i,j, $1\le i<j\le m+2$ and the exact diameter for some special jumped Wenger graphs, such as $(i,j)=(m,m+2),(m+1,m+2)$ or $(m,m+1)$. In Section 3, we determine the girth of a jumped Wenger graph $J_m(q,i,j)$ for $1\le i<j\le m+2$. Finally, in Section 4 we present our conclusions.

2 The diameter of jumped Wenger graphs

Recall that a sequence of distinct vertices $v_1,\cdots,v_s$ in a simple connected graph $\mathfrak{G}=(V,E)$ defines a path of length $s-1$ if $(v_i,v_{i+1})\in E$ for every i, $1\le i\le s-1$. For