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Optimal Control for Maximum Instantaneous Convergence in Collective Migration Models

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Abstract. This paper studies consensus tracking of collective migration models that involve the alignment force gathering agents and the tracking force matching target. Each agent's dynamics is controlled by the tracking strategy, which establishes a trade-off between the two forces through a convex combination. In order to drive the system to achieve consensus tracking with the maximum instantaneous convergence speed, an optimal control strategy is proposed that the agents whose alignment force is weaker than, or counteracts its tracking force sense only the target, and become leaders, while the others sense only their neighbours, and become followers. Interestingly, there exist some initial frameworks such that the optimal control strategy consists of letting all agents become followers, which is called "inactivation principle" in [Piccoli, Duteil and Scharf, Math. Models Methods Appl. Sci., 26(2), 2016] and means that the leaderless is better than the leader-follower structure for certain conditions. Both asymptotic and finite-time consensus tracking are investigated. Several numerical simulations show the effect of the optimal control strategy.

AMS subject classifications: 93A16, 93C10, 34D20, 34H05, 93D40

Key words: Collective migration models, consensus tracking, maximum instantaneous convergence, tracking strategy.

1 Introduction

Collective behaviour in large groups is ubiquitous in nature and human societies, such as the flocking of birds, schooling of fishes, pedestrian movements on roads, public opinion in society [10, 11, 32, 36, 41, 43], etc. A fascinating feature of such systems is their self-organization ability that can be described as the formation of patterns, which are governed solely by interactions among its individuals. Many mathematical models of

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self-organized actions, realized at different scales: microscopic, mesoscopic, and macroscopic, have been applied to biology, physics, social science and control engineering, see [12, 19, 24, 25, 28–30, 33, 42] and references therein. At microscopic level, an accurate description of the single agent dynamics is often pursued, also the scope of this paper. Such models are called agent-based models, which are also known as individual-based models (IBMs) that are generally large-scale complex systems composed of multiple independent autonomous agents interconnected through a weighting network.

Collective migration of animals, as a kind of collective behaviour, is typically an adaptive response to spatiotemporal variation in resources (seasonal variability and breeding cycle, for instance) [17]. Observations of this phenomenon show that only a subset of individuals use much energy to track the migration route by environment features, and become leaders, while the rest become followers who only align with their neighbours to travel as a group, and save more energy for other tasks such as foraging and warning [26]. The leader-follower structure that animals form in self-organization has great implications in the field of vehicles, robotics and aircraft [3, 38, 46]. More precisely, the collective migration of animals can be abstracted to the collective tracking of multi-agent systems, in which the migration route becomes the tracking target, and interactions among animals become control protocols of agents. Consequently, some questions are naturally raised: How to model the leader-follower structure and how to select a finite set of leaders for achieving optimization under certain conditions?

Collective tracking, from a mathematical point of view, is usually reduced to the consensus tracking, which is said that all agents reach an agreement at the target state. At present, many studies of the consensus tracking problem have been implemented, as in [5, 13, 20, 27, 39, 44, 45, 47]. To the best of our knowledge, most of the existing literature divides consensus tracking into two unrelated parts: consensus and tracking. Consensus, as a widespread pattern that has been studied for decades, is achieved by the alignment among neighbours, and highly dependent on the connectivity of interaction topology [23, 30, 35, 40]. On the other hand, tracking utilizes the idea of pinning control, which is firstly proposed in the control of spatiotemporal chaos, and means that pin a spatiotemporal system in a desired domain by intervene a local of the system [16, 31]. For multi-agent systems, the traditional intervention is adding external forces to a subset of agents to track the desired target state. Such intervention results in the leader-follower structure of multi-agent systems where the leaders refer to the part of agents who master information about the target state (i.e. those agents affected by the intervention), and the rest of the agents in the system is known as followers. Incidentally, the force that describes the alignment among neighbours is called alignment force, and the external force intervened in the system for tracking is called tracking force.

It is worth noting that leaders created by pinning control are influenced by both alignment and tracking forces, which are independent of each other. In other words, the amount of “effort” of leaders put into tracking and alignment is unrelated, where the “effort” can be abstracted as the control gain. However, the capabilities of any kind of agent, such as animal, human, or engineering device, are limited. The more an agent invests in one thing, the less it invests in the others. This indicates that there should be a trade-off between alignment and tracking forces for each agent. Therefore, we want to establish mathematically a trade-off between the two forces in the model. In 2016, Piccoli, Duteil and Scharf [34] studied a collective migration model that provides a solution about our question. For each agent, they introduce a parameter $\alpha_i \in [0,1]$ to establish a trade-off between the two forces through a convex combination. More precisely, the gain of tracking force is α_i, and the gain of alignment force is $1-\alpha_i$. Each agent's dynamics is governed by α_i, which plays on the balance between the two forces. The model can be written as

$$\dot{x}_i(t)=v_i(t), \quad \dot{v}_i(t)=\alpha_i(t)\big(V-v_i(t)\big)+\big(1-\alpha_i(t)\big)\frac{1}{N}\sum_{j=1}^{N}\big(v_j(t)-v_i(t)\big), \quad i=1,\dots,N, \tag{1.1}$$

where $x_i(t)\in\mathbb{R}^d$ and $v_i(t)\in\mathbb{R}^d$ are the position and velocity of the agent $i, V\in\mathbb{R}^d$ is the target velocity, and the control α_i satisfies the constraint $\sum_i \alpha_i(t)\le M, 0<M\le N$. If $\alpha_i(t)>0$, the agent i is controlled and becomes a leader, otherwise a follower. Define the distance from consensus tracking by

$$\mathbb{V}(t)=\frac{1}{N}\sum_{i=1}^{N}\|v_i(t)-V\|^2.$$

They considered three different performance indexes: (i) the instantaneous decrease speed $\dot{\mathbb{V}}(t)$, (ii) the cost at a given final time $\mathbb{V}(T)$ and (iii) the integral cost $\int_0^T \mathbb{V}(t)\mathrm{d}t$, and obtained optimal control strategies in three cases respectively, which all consist of controlling the agents whose velocities are furthest from the target. Specially, for case (ii), there exist initial conditions such that the optimal control strategy is letting the system without control for an initial period of time, which is called "inactivation principle". However, the model (1.1) has the all-to-all interaction network that is idealized. Recently, the Cucker-Smale type interaction, a nonlinear function, was introduced to the alignment force of the model (1.1). The sufficient framework of asymptotic and finite-time stability under Cucker-Smale type interaction was proposed in [6] and [8], respectively. In addition, the relation between the control $\alpha_i(t)$ and the time delay was also investigated [7]. To our knowledge, the issue of optimal strategy of collective migration models under distributed interaction networks has not been studied.

Our main goal in this work is to analyse the trade-off between the alignment force and tracking force with more realistic and complex constraints. Under distributed interaction networks, we solve the optimal control problem that the system achieves consensus tracking with the maximum instantaneous convergence speed. An optimal control strategy, named sparse optimal strategy, is proposed based on the idea of sparse control [4]. The sampling solution is implemented to avoid chattering phenomena. Moreover, all conclusions about asymptotic consensus tracking are extended to the finite-time consensus tracking. In contrast with the existing results of [6–8, 34], three contributions of our paper are highlighted:

(1) We improve the all-to-all interaction network to distributed interaction networks that is more valuable in practical application. In model (1.1), the direction of the mean velocity

$$\overline{v}(t)=\frac{1}{N}\sum_{i=1}^{N}v_i(t)$$

is an invariant of the dynamics, which guarantees the convergence of the system and is the basis of the projection method appeared in [34]. However, it does not hold under distributed interaction networks. To account for the validity of our models, we describe the conditions of system convergence under the constant control, i.e $\alpha_i(t)\equiv\alpha_i(t_0), t\geq t_0$. Taking the connectedness of interactive networks as an entry point, it is demonstrated by the matrix theory and graph theory that the system achieves asymptotic consensus tracking if and only if

$$\sum_{i=1}^{N}\alpha_i(t_0)>0.$$

If the target state is considered as a real agent within the system, then our result is consistent with the classical result, that is, consensus occurs if and only if there exists a spanning tree in the interaction network [35].

(2) We propose an optimal control strategy to drive the system to achieve consensus tracking with the maximum instantaneous convergence speed. An optimization problem is established and proved that its solution exists but is not unique. According to the principle of sparse control [4], we select the optimal control strategy that controls as few individuals as possible and name it sparse optimal strategy. To be specific, the agents whose alignment force is weaker than, or counteracts its tracking force sense only the target, and become leaders, while the others sense only their neighbours, and become followers. Specially, there exist initial frameworks such that the sparse optimal strategy consists of letting all agents become followers, which is called "inactivation principle" in [34] and means that the leaderless is better than the leader-follower structure for certain conditions. In addition, the sampling solution is implemented to avoid chattering phenomena. The control of the system changes (based on the sparse optimal strategy) only at the sampling points, and remains unchanged at the rest time. It is showed that the sampling solution achieves asymptotic consensus tracking, and converges uniformly to the optimal solution when the sampling time tends to zero.

(3) We further develop the finite-time consensus tracking of the system. The finite-time convergence has faster convergence, better disturbance rejection and robustness against uncertainties, and is often required in practice [1, 22, 39]. We extend all conclusions about asymptotic consensus tracking to the finite-time consensus tracking, obtain the corresponding sufficient and necessary conditions and sparse optimal strategy.

The remainder of this paper is organized as follows. We introduce the preliminaries and problem formation in Section 2. In Section 3, we obtain the sufficient and necessary conditions of asymptotic consensus tracking, and propose the sparse optimal strategy. We develop the finite-time consensus tracking of the system in Section 4. Simulations and conclusions are made in Sections 5 and 6 respectively. Finally, some useful definitions and lemmas are listed in the Appendix A.

2 Preliminaries and problem formation

Firstly, we introduce some basic concepts and notations of the graph theory [15]. Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a weighted undirected graph with the set of N vertices $\mathcal{V}=\{1,2,\ldots,N\}$, set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$, and a weighted adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$. The adjacency elements satisfy $a_{ij}=a_{ji}\geq 0, i,j\in\mathcal{V}$, which are associated with the edges of the graph, i.e. $(i,j)\in\mathcal{E}\Leftrightarrow a_{ij}>0$. Assume that there are no self-loops in $\mathcal{G}$, i.e. $a_{ii}=0, i\in\mathcal{V}$. A path from vertex j to vertex i is a sequence of edges $(i,i_1),(i_1,i_2),\cdots,(i_l,j)\in\mathcal{E}$ with distinct vertexes i_k, $k=1,2,\ldots,l$. $\mathcal{G}$ is called connected if there exists a path between any two vertexes. Denote the degree matrix of $\mathcal{G}$ by $\mathcal{D}_{\mathcal{A}}=\operatorname{diag}(d_i)\in\mathbb{R}^{N\times N}$, where $d_i=\sum_{j=1}^{N}a_{ij}, i\in\mathcal{V}$. $\mathcal{G}$ is connected if and only if $d_i>0$ for any $i\in\mathcal{V}$. The Laplacian matrix of $\mathcal{G}$ is derived from the equation $\mathcal{L}_{\mathcal{A}}=\mathcal{D}_{\mathcal{A}}-\mathcal{A}$, where $\mathcal{L}_{\mathcal{A}}=[l_{ij}], l_{ii}=\sum_{j\neq i}^{N}a_{ij}, i\in\mathcal{V}$ and $l_{ij}=-a_{ij}, i\neq j$. The Laplacian matrix $\mathcal{L}_{\mathcal{A}}$ has at least one zero eigenvalue corresponding to a right eigenvector $\mathbf{1}_N=(1,\ldots,1)^\top\in\mathbb{R}^N$. All nonzero eigenvalues of $\mathcal{L}_{\mathcal{A}}$ have positive real parts. Furthermore, $\mathcal{L}_{\mathcal{A}}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ is connected [30,35].

In this context, we consider a collective migration model under distributed interaction networks with the set of agents $\mathcal{V}$. Let a weighted undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ model the interaction topology among agents of the system, which is assumed to be connected. Each vertex $i\in\mathcal{V}$ represents a agent. Agent i can receive information from agent j if and only if $(i,j)\in\mathcal{E}$. Let $x_i(t)\in\mathbb{R}$ be the state of agent i. Suppose that the target state of the system is $x^*(t)\in\mathbb{R}$, satisfying $\dot{x}^*(t)=\psi(t)$, where $\psi(t)$ denotes the intrinsic, a priori known dynamics of the target. Then, the dynamics of each agent is given by

$$\begin{aligned}\dot{x}_i(t)=\dot{x}^*(t)+m_i\bigg[&\alpha_i(t)\big(x^*(t)-x_i(t)\big)\\&+\big(1-\alpha_i(t)\big)\sum_{j=1}^{N}\frac{a_{ij}}{d_i}\big(x_j(t)-x_i(t)\big)\bigg],\quad t\geq t_0,\quad i\in\mathcal{V},\end{aligned}\tag{2.1}$$

where $m_i\in\mathbb{R}_+$ is the gain limitation of agent i, $\alpha_i(t)\in[0,1]$ is the tracking strategy of agent i, a_{ij}/d_i accounts for the influence of agent j on agent i. The term $x^*(t)-x_i(t)$ represents the tracking force, and the term $\sum_{j=1}^{N}(a_{ij}/d_i)(x_j(t)-x_i(t))$ represents the alignment force. $\alpha_i(t)$ establishes a trade-off for agent i between tracking and alignment forces. Agent i is recognized as a leader if $\alpha_i(t)>0$, otherwise as a follower. Define the tracking strategy of the system (2.1) as $\alpha(t)=(\alpha_1(t),\cdots,\alpha_N(t))$. Because of the limited capability of agents or some other factors, the tracking strategy $\alpha(t)$ satisfies $\alpha(t)\in U, t\geq t_0$, where

$$U=\left\{\beta\in\mathbb{R}^N \mid \beta_i\in[0,1], i\in\mathcal{V}, 0\leq\sum_{i=1}^N\beta_i\leq\Sigma(1\leq\Sigma\leq N)\right\}. \tag{2.2}$$

Define the distance of the system (2.1) from consensus tracking by $\mathbb{X}(t)=\sum_{i=1}^N|x_i(t)-x^*(t)|^2$, then the instantaneous convergence speed of the system (2.1) is defined as the instantaneous variation of the deviation $\mathbb{X}(t)$, i.e $\dot{\mathbb{X}}(t)$. Thus, we obtain the following optimal control problem of finding optimal strategy $\alpha(t)$ and corresponding state $x(t)=(x_1(t),\cdots,x_N(t))^\top$ which solve

$$\begin{cases}\min\limits_{\alpha(t)\in U, t\geq t_0} J\big(\alpha(t)\big)=\dot{\mathbb{X}}(t)\\ \text{s.t.}\quad x(t) \text{ satisfies (2.1) and } x(t_0)=x_0\in\mathbb{R}^N.\end{cases} \tag{2.3}$$

Remark 2.1. The instantaneous convergence speed $\dot{\mathbb{X}}(t)$ measures how much the error diminishes per unit time. The cost at a given final time $\mathbb{X}(T)$ refers to the error magnitude at that specific instant T. The integral cost $\int_0^T\mathbb{X}(t)\mathrm{d}t$ accumulates the system error over a given time interval $[0,T]$ and can be interpreted as energy expenditure. In this paper, we only adopt the instantaneous decrease speed $\dot{\mathbb{X}}(t)$, because ensuring consensus tracking is the overriding goal, whereas optimal control design is of secondary importance. Consequently, no final time T is prescribed.

Moreover, this paper also considers a collective migration model with nonlinear couplings

$$\begin{aligned}\dot{x}_i(t)=\dot{x}^*(t)+m_i\bigg[&\alpha_i(t)\mathrm{sig}^\theta\big(x^*(t)-x_i(t)\big)\\ &+\big(1-\alpha_i(t)\big)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(x_j(t)-x_i(t)\big)\bigg],\quad t\geq t_0,\quad i\in\mathcal{V},\end{aligned} \tag{2.4}$$

where $\mathrm{sig}^\theta(r)=\mathrm{sign}(r)|r|^\theta, r\in\mathbb{R}, \theta\in(0,1), \mathrm{sign}(\cdot)$ is the sign function. An optimal control problem same to (2.3) can be obtained. In Section 3, we analysis the asymptotic consensus flocking of the system (2.1) and solve the optimal control problem (2.3). The finite-time consensus tracking of system (2.4) and the corresponding optimal control problem are studied in Section 4. Finally, we present the mathematical definition of asymptotic and finite-time consensus tracking of the system (2.1) and (2.4).

Definition 2.1. *Suppose $x(t)$ is a solution of the system* (2.1) *or* (2.4)*. The system is said to achieve asymptotic consensus tracking if for any initial conditions $x_0\in\mathbb{R}^N$,*

$$\lim_{t\to\infty}|x_i(t)-x^*(t)|=0,\quad i\in\mathcal{V}.$$

The system is said to achieve finite-time consensus tracking if for any initial conditions $x_0\in\mathbb{R}^N$, there exists a positive number T such that

$$\lim_{t\to T}|x_i(t)-x^*(t)|=0\quad \textit{and}\quad x_i(t)=x^*(t),\quad t\geq T,\quad i\in\mathcal{V}.$$

3 Asymptotic consensus tracking

In this section, we want to drive the system (2.1) to achieve asymptotic consensus tracking with the maximum instantaneous convergence speed. Before solving the optimal control problem (2.3), the fundamental question to get answered: Is whether the system (2.1) can achieve asymptotic consensus tracking? If so, what are the requirements? To answer the question, we fist consider the time-invariant strategy and obtain the corresponding sufficient and necessary conditions.

3.1 Asymptotic convergence with the time-invariant strategy

Assume the tracking strategy is time-invariant, i.e. $\alpha(t)\equiv\alpha(t_0), t\geq t_0$. For convenience, $\alpha_i(t_0)$ is abbreviated to α_i. Define $y_i(t)=x_i(t)-x^*(t), i\in\mathcal{V}$, then the system (2.1) becomes

$$\dot{y}_i(t)=m_i\left[-\alpha_i y_i(t)+(1-\alpha_i)\sum_{j=1}^{N}\frac{a_{ij}}{d_i}\big(y_j(t)-y_i(t)\big)\right],\quad t\geq t_0,\quad i\in\mathcal{V}. \tag{3.1}$$

Rewriting the above system in matrix form yields

$$\dot{y}(t)=-\Lambda_N y(t),\quad t\geq t_0, \tag{3.2}$$

where $y(t)=(y_1(t),\cdots,y_N(t))^\top$ and

$$\Lambda_N=\begin{bmatrix} m_1 & -m_1(1-\alpha_1)\dfrac{a_{12}}{d_1} & \cdots & -m_1(1-\alpha_1)\dfrac{a_{1N}}{d_1} \\ -m_2(1-\alpha_2)\dfrac{a_{21}}{d_2} & m_2 & \cdots & -m_2(1-\alpha_2)\dfrac{a_{2N}}{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ -m_N(1-\alpha_N)\dfrac{a_{N1}}{d_N} & -m_N(1-\alpha_N)\dfrac{a_{N2}}{d_N} & \cdots & m_N \end{bmatrix}_{N\times N}. \tag{3.3}$$

Therefore, the system (2.1) achieves asymptotic consensus tracking if and only if $\|y(t)\|\to 0$ as $t\to\infty$. Solving the Eq. (3.2) yields

$$y(t)=e^{-\Lambda_N(t-t_0)}y(t_0),\quad t\geq t_0.$$

In the following, we first study the properties of the matrix Λ_N, which requires the concept of irreducibly diagonally dominant matrix in Appendix A.

Lemma 3.1. *The matrix Λ_N defined in* (3.3) *satisfies the following two properties:*

(1) Λ_N has only real eigenvalues.

(2) All eigenvalues of Λ_N are strictly positive if and only if $\sum_{i=1}^N\alpha_i>0$.

Proof. (1) Define

$$\mathcal{I}_l=\{i\,|\,\alpha_i=1,i\in\mathcal{V}\},\quad |\mathcal{I}_l|=l, \tag{3.4}$$

where $|\cdot|$ denotes the cardinality of a set, $0\le l\le N, l\in\mathbb{Z}$. The following three cases are discussed.

Case I: $l=0$. For any $i,j\in\mathcal{V}, i\neq j$, we have

$$m_i(1-\alpha_i)\frac{a_{ij}}{d_i}=\sqrt{\frac{d_j}{m_j(1-\alpha_j)}}\left(\sqrt{\frac{m_im_j(1-\alpha_i)(1-\alpha_j)}{d_id_j}}a_{ij}\right)\sqrt{\frac{m_i(1-\alpha_i)}{d_i}}.$$

Define

$$\Gamma_N=\operatorname{diag}\left(\sqrt{\frac{m_1(1-\alpha_1)}{d_1}},\cdots,\sqrt{\frac{m_N(1-\alpha_N)}{d_N}}\right).$$

Then $\Lambda_N=\Gamma_N^{-1}S_N\Gamma_N$, where $S_N=[s_{ij}]$ is a symmetric matrix,

$$s_{ij}=\sqrt{\frac{m_im_j(1-\alpha_i)(1-\alpha_j)}{d_id_j}}a_{ij},\quad s_{ii}=m_i,\quad i,j\in\mathcal{V},\quad i\neq j.$$

From matrix theory, there exists an orthogonal matrix O_N diagonalizing the symmetric matrix S_N, i.e. $S_N=O_N^\top J_NO_N, J_N=\operatorname{diag}(\mu_1,\cdots,\mu_N)\in\mathbb{R}^{N\times N}$. Therefore, Λ_N is similar to a diagonal matrix.

Case II: $1\le l\le N-1$. Let $E_{pq}\in\mathbb{R}^{N\times N}$ be an elementary matrix obtained by swapping the p-th row and the q-th row of the identity matrix I_N. It is easy to know that $E_{pq}^\top E_{pq}=I_N$. According to elementary transformation, there exists a invertible matrix F, which is the product of finite number of elementary matrices E_{pq} such that

$$\Lambda_N=F^\top\begin{bmatrix} m_{k_1} & \cdots & 0 & \mathbf{0}_{N-l}^\top \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & m_{k_l} & \mathbf{0}_{N-l}^\top \\ * & \cdots & * & \Lambda_{N-l}\end{bmatrix}F,\quad k_i\in\mathcal{I}_l,\quad i=1,\ldots,l.$$

Define

$$\Gamma_{N-l}=\operatorname{diag}\left(\sqrt{\frac{m_{r_1}(1-\alpha_{r_1})}{d_{r_1}}},\cdots,\sqrt{\frac{m_{r_{N-l}}(1-\alpha_{r_{N-l}})}{d_{r_{N-l}}}}\right),\quad r_j\in\mathcal{V}\backslash\mathcal{I}_l,\quad j=1,\ldots,N-l.$$

By Case I, there exists an orthogonal matrix O_{N-l} such that

$$\Lambda_{N-l}=\Gamma_{N-l}^{-1}O_{N-l}^\top J_{N-l}O_{N-l}\Gamma_{N-l},$$

where $J_{N-l}=\operatorname{diag}(\mu_{r_1},\cdots,\mu_{r_{N-l}})\in\mathbb{R}^{N-l\times N-l}$. Therefore, Λ_N is similar to a lower triangular matrix.

Case III: $l=N$. $\Lambda_N=\mathrm{diag}(m_1,\cdots,m_N)$ is a diagonal matrix.

According to the above discussions, we complete the proof of (1).

(2) **Sufficiency:** If $\sum_{i=1}^N\alpha_i>0$, then Λ_N is diagonally dominant. On the other hand, Λ_N is irreducible, because the weighted undirected graph $\mathcal{G}$ is connected. Thus, Λ_N is a irreducibly diagonally dominant matrix. By Lemmas A.1 and 3.1(1), we show that all eigenvalues of Λ_N are strictly positive.

Necessity: Suppose $\sum_{i=1}^N\alpha_i=0$, then Λ_N becomes the Laplacian matrix of $\mathcal{G}$, which means that Λ_N has at least one zero eigenvalue. So All eigenvalues of Λ_N are strictly positive only if $\sum_{i=1}^N\alpha_i>0$. $\square$

Remark 3.1. Although similar conclusions could be obtained via diagonal dominance and irreducibility, the present approach deliberately dissects the matrix structure o f Λ_N and clarifies how the tracking strategy α_i affects it, which is crucial to the proof of Theorem 3.1.

Based on the above analysis, we describe the conditions of system convergence under the constant control and get the exponential estimation of $\|y(t)\|$. To facilitate our analysis, we introduce the spectral norm of a matrix $B=[b_{ij}]\in\mathbb{R}^{p\times q}$. Define the spectral norm of B as

$$\|B\|:=\sup_{|\nu|\neq 0}\frac{|B\nu|}{|\nu|}=\sup_{|\nu|=1}\sqrt{\nu^\top B^\top B\nu}=\sqrt{\rho(B^\top B)},$$

where $\nu=(\nu_1,\cdots,\nu_q)^\top\in\mathbb{R}^q$, $|\cdot|$ is the modulus of a scalar or vector, $\rho(\cdot)$ is the spectral radius of a matrix, i.e the largest eigenvalue of a matrix. Specially, as for $y(t)\in\mathbb{R}^N$, we deduce from the definition of the spectral norm that $\|y(t)\|=|y(t)|$.

Theorem 3.1. *Assume the tracking strategy is time-invariant. The system* (2.1) *achieves asymptotic consensus tracking if and only if* $\sum_{i=1}^N\alpha_i>0$. *Moreover, if* $\sum_{i=1}^N\alpha_i>0$, *then there exists a positive constant* $C\geq 1$ *depending on* Λ_N *such that*

$$\|y(t)\|\leq Ce^{-\varepsilon(t-t_0)}\|y(t_0)\|,\quad t\geq t_0,$$

where $\varepsilon=\mu_{\min}/2$ *and* $\mu_{\min}>0$ *is the smallest eigenvalue of* Λ_N.

Proof. **Sufficiency:** By (1) of Lemma 3.1, denote all eigenvalue of Λ_N by $\mu_i\in\mathbb{R},i\in\mathcal{V}$. Define the smallest eigenvalue of Λ_N by $\mu_{\min}$. From Lemma 3.1(2), $\mu_{\min}>0$ because of $\sum_{i=1}^N\alpha_i>0$. Let $\mathcal{I}_l$ be the set defined in (3.4). According to the proof of Lemma 3.1(1), we discuss the following three cases.

Case I: $l=0$. There exists a invertible matrix $P\in\mathbb{R}^{N\times N}$ such that $\Lambda_N=P^{-1}J_NP$, $J_N=\mathrm{diag}(\mu_1,\cdots,\mu_N)$. Then by the definition of the spectral norm we obtain

$$\begin{aligned}\|y(t)\|&=\left\|e^{-\Lambda_N(t-t_0)}y(t_0)\right\|\\&\leq\|P^{-1}\|\left\|e^{-J_Nt}\right\|\|P\|\|y(t_0)\|\leq Ce^{-\mu_{\min}(t-t_0)}\|y(t_0)\|,\end{aligned}$$

where $C=\|P^{-1}\|\|P\|$ is a positive constant depending on Λ_N.

Case II: $1\le l\le N-1$. Without loss of generality, let $\mathcal{I}_l=\{1,\dots,l\}$. Then we have

$$\begin{bmatrix}\dot{y}_l(t)\\ \dot{y}_{N-l}(t)\end{bmatrix}=-\begin{bmatrix}M_l & \mathbf{0}_{l\times N-l}\\ \Delta_{N-l\times l} & \Lambda_{N-l}\end{bmatrix}\begin{bmatrix}y_l(t)\\ y_{N-l}(t)\end{bmatrix},$$

where $y_l=(y_1,\cdots,y_l)^\top, y_{N-l}=(y_{l+1},\cdots,y_N)^\top$ and $M_l=\operatorname{diag}(m_1,\cdots,m_l)$. There exists a invertible matrix $Q\in\mathbb{R}^{N-l\times N-l}$ such that $\Lambda_{N-l}=Q^{-1}J_{N-l}Q, J_{N-l}=\operatorname{diag}(\mu_{l+1},\cdots,\mu_N)$. Solving the equation yields

$$\begin{aligned}&y_l(t)=e^{-M_l(t-t_0)}y_l(t_0),\\ &y_{N-l}(t)=e^{-\Lambda_{N-l}(t-t_0)}y_{N-l}(t_0)-\int_{t_0}^t e^{-\Lambda_{N-l}(t-s)}\Delta_{N-l\times l}y_l(s)\mathrm{d}s.\end{aligned}$$

Using the fact $\|y_l(t)\|\le\|y(t)\|$ and $\|y_{N-l}(t)\|\le\|y(t)\|$, we obtain

$$\begin{aligned}\|y_l(t)\|&\le e^{-\mu_{\min}(t-t_0)}\|y(t_0)\|,\\ \|y_{N-l}(t)\|&\le C_1\|e^{-J_{N-l}(t-t_0)}\|\|y_{N-l}(t_0)\|+C_1C_2\int_{t_0}^t\|e^{-J_{N-l}(t-s)}\|\|y_l(s)\|\mathrm{d}s\\ &\le C_1e^{-\mu_{\min}(t-t_0)}\|y(t_0)\|+C_1C_2\int_{t_0}^t e^{-\mu_{\min}(t-s)}e^{-\mu_{\min}(s-t_0)}\mathrm{d}s\|y(t_0)\|\\ &\le\big(C_1+C_1C_2(t-t_0)\big)e^{-\mu_{\min}(t-t_0)}\|y(t_0)\|,\end{aligned}$$

where $C_1=\|Q^{-1}\|\|Q\|$, $C_2=\|\Delta_{N-l\times l}\|$. There exists a positive constant C_3 such that $(C_1+C_1C_2(t-t_0))\le C_3e^{(\mu_{\min}/2)(t-t_0)}$, so we obtain

$$\|y(t)\|\le\|y_l(t)\|+\|y_{N-l}(t)\|\le Ce^{-\frac{\mu_{\min}}{2}(t-t_0)}\|y(t_0)\|,\quad C=\max\{1,C_3\}.$$

Case III: $l=N$. $\Lambda_N=\operatorname{diag}(\mu_1,\cdots,\mu_N)$ is a diagonal matrix, which directly yields

$$\|y(t)\|\le e^{-\mu_{\min}(t-t_0)}\|y(t_0)\|.$$

In conclusion, let $\varepsilon=\mu_{\min}/2$, then there exists a positive constant $C\ge1$ depending on Λ_N such that $\|y(t)\|\le Ce^{-\varepsilon(t-t_0)}\|y(t_0)\|, t\ge t_0$.

Necessity: Assume $\sum_{i=1}^N\alpha_i=0$. Then Λ_N is the Laplacian matrix of the weight undirected graph $\mathcal{G}$. Because $\mathcal{G}$ is connected, Λ_N has exactly one zero eigenvalue and all of the nonzero eigenvalues are positive number. Define

$$\rho=\frac{1}{\sum_{j=1}^N d_j/m_j}\left(\frac{d_1}{m_1},\cdots,\frac{d_N}{m_N}\right)^\top\in\mathbb{R}^N.\tag{3.5}$$

It is easy to verify that $\mathbf{1}_N$ and ρ are the right and left eigenvector corresponding to zero eigenvalue respectively, and satisfy $\rho^\top\mathbf{1}_N=1$. Then there exists a invertible matrix P

such that $\mathbf{1}_N$ is the first column of P, $\rho^\top$ is the first row of P^{-1} and $\Lambda_N = P^{-1} J_N P$, where $J_N = \mathrm{diag}(0,\mu_2,\cdots,\mu_N)$. Hence, we have

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} P^{-1} e^{-J_N t} P y(t_0) = P^{-1}\mathrm{diag}(1,0,\cdots,0) P y(t_0)\\ &= \mathbf{1}_N \rho^\top y(t_0) = \frac{\sum_{i=1}^N (d_i/m_i) y_i(t_0)}{\sum_{j=1}^N d_j/m_j} \mathbf{1}_N. \end{aligned} \tag{3.6}$$

According to the arbitrary of initial conditions, there exist $y_i(t_0), i \in \mathcal{V}$ such that

$$\sum_{i=1}^N \frac{d_i}{m_i} y_i(t_0) \neq 0,$$

which contradicts the fact that the system achieves asymptotic consensus tracking. Therefore, $\sum_{i=1}^N \alpha_i > 0$. $\square$

Remark 3.2. Add a virtual agent indexed by 0 in the system, which satisfies $x_0(t) = x^*(t)$. Then we obtain a new multi-agent system with $N+1$ agents indexed by $\{0,1,2,\dots,N\}$ and the agent 0 receives no information from other agents. The interactive topology of the new system is a weight directed graph of order $N+1$, denoted by $\hat{\mathcal{G}}$. It can be proved that $\sum_{i=1}^N \alpha_i > 0$ means that $\hat{\mathcal{G}}$ has a spanning tree. So our result is consistent with the classical result, that is, consensus occurs if and only if there exists a spanning tree in the interaction network [35].

Remark 3.3. When $\sum_{i=1}^N \alpha_i = 0$, all agents only interact with their neighbours and achieve an agreement at $\rho^\top y(t_0)$, which is completely caused by alignment force. If $\rho^\top y(t_0) = 0$, then the system achieves asymptotic consensus tracking without tracking force. It leads us to wonder whether the effect of alignment force on agents may have the one of tracking force in some cases. In the process of solving the optimal control in the next section, we find that alignment force is equivalent or even better than tracking force in certain situations. Even more, the optimal control may present an uncontrolled state, i.e $\sum_{i=1}^N \alpha_i = 0$.

3.2 Optimal control of asymptotic consensus tracking

Theorem 3.1 accounts for the validity of the system (2.1). Next, we will study the optimal strategy to maximizing instantaneous convergence speed of the system (2.1). Form (3.2), the optimal control problem (2.3) becomes

$$\begin{cases} \min\limits_{\alpha(t)\in U, t\geq t_0} J\big(\alpha(t)\big) = \dot{\mathbb{X}}(t) \\ \text{s.t.} \quad \dot{y}(t) = -\Lambda_N\big(\alpha(t)\big) y(t), \quad t \geq t_0, \end{cases} \tag{3.7}$$

where

$$\mathbb{X}(t) = \sum_{i=1}^N |y_i(t)|^2 = \|y(t)\|^2,$$

$\Lambda_N(\alpha(t))$ defined in (3.3) is a continuous function of $\alpha(t)$. Direct calculation yields

$$\begin{aligned}\dot{\mathbb{X}}(t)&=2y^\top(t)\dot{y}(t)=2\sum_{i=1}^N y_i(t)m_i\left[-\alpha_i(t)y_i(t)+(1-\alpha_i(t))\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))\right]\\&=2\sum_{i=1}^N m_iy_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))-2\sum_{i=1}^N\alpha_i(t)m_i\left[y_i^2(t)+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))\right].\end{aligned}$$

Define the controllable term of agent i by

$$\phi_i(t)=m_i\left[y_i^2(t)+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))\right],\quad i\in\mathcal{V},\tag{3.8}$$

then minimizing $\dot{\mathbb{X}}(t)$ is equivalent to

$$\max_{\alpha(t)\in U,t\geq t_0}\sum_{i=1}^N\alpha_i(t)\phi_i(t).\tag{3.9}$$

Note that the selection of the tracking strategy $\alpha(t)$ depends on the value of $\phi_i(t)$. Thus, we provide the following proposition to minimize $\dot{\mathbb{X}}(t)$ at any moment $t\geq t_0$.

Proposition 3.1. *For any moment $t\geq t_0$, define $R^+(t)=\{i\in\mathcal{V}|\phi_i(t)\geq 0\}$. We select the sparse optimal strategy $\alpha^*(t)=(\alpha_i^*(t),\cdots,\alpha_N^*(t))$ according to the following criterion:*

(1) If $|R^+(t)|\leq\Sigma$, set $\alpha_i^*(t)=1,i\in R^+(t)$ and $\alpha_j^*(t)=0,j\in\mathcal{V}\backslash R^+(t)$.

(2) If $|R^+(t)|>\Sigma$, sorting controllable terms $\{\phi_i(t)\}_{i\in\mathcal{V}}$ yields $\Phi(t)=(\phi_{k_1}(t),\cdots,\phi_{k_N}(t))\in\mathbb{R}^N$ satisfying $\phi_{k_1}(t)\geq\cdots\geq\phi_{k_N}(t)$. Set $\alpha_{k_i}^*(t)=1,i\leq\lfloor\Sigma\rfloor$, $\alpha^*_{k_{\lfloor\Sigma\rfloor+1}}(t)=\Sigma-\lfloor\Sigma\rfloor$ and $\alpha_{k_j}^*(t)=0,\lfloor\Sigma\rfloor+1<j\leq N$, where $\lfloor\Sigma\rfloor$ denotes the floor of Σ.

Remark 3.4. According to the definition of the system (2.1), $-y_i(t)=x^*(t)-x_i(t)$ represents tracking force,

$$\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))=\sum_{j=1}^N\frac{a_{ij}}{d_i}(x_j(t)-x_i(t))$$

represents alignment force. From the definition of $\phi_i(t)$ in (3.8), $\phi_i(t)\geq 0$ is equivalent to

$$y_i^2(t)\geq -y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t)),$$

which means that alignment force is weaker than, or counteracts tracking force. Therefore, when $|R^+(t)|\leq\Sigma$, the rule of sparse optimal strategy can be summarized as the agents whose alignment force is weaker than, or counteracts its tracking force sense only the target, and become leaders, while the others sense only their neighbours, and become followers.

Remark 3.5. The optimal strategy minimizing $\dot{\mathbb{X}}(t)$ is not unique when $|R^+(t)|>\Sigma$ and $\phi_{k_{\lfloor\Sigma\rfloor}}(t)=\phi_{k_{\lfloor\Sigma\rfloor+1}}(t)$. In this case, let

$$R_{\lfloor\Sigma\rfloor}=\{i\in\mathcal{V}\,|\,\phi_i(t)=\phi_{k_{\lfloor\Sigma\rfloor}}(t)\},\quad R^*_{\lfloor\Sigma\rfloor}=\{1,\ldots,\lfloor\Sigma\rfloor\}\backslash R_{\lfloor\Sigma\rfloor}.$$

Then any strategy satisfying $\alpha_i(t)=1, i\in R^*_{\lfloor\Sigma\rfloor}$ and $\sum_{j\in R_{\lfloor\Sigma\rfloor}}\alpha_j(t)=\Sigma-|R^*_{\lfloor\Sigma\rfloor}|$ is an optimal strategy for the optimization problem (3.9). Adhering to the principle of the sparse control that controlling as few agents as possible, we propose sparse optimal strategy $\alpha^*(t)$, which depends on the order of values of $\{\phi_i(t)\}_{i\in\mathcal{V}}$ i.e. $\Phi(t)$. In order to ensure the uniqueness of $\Phi(t)$, $\Phi(t)$ should satisfy that if $\phi_{k_i}(t)=\phi_{k_{i+1}}(t)$, then $k_i<k_{i+1}$, which means that the smaller numbered agent is placed before the larger numbered one when their controllable terms are equal.

The convergence of the system under the sparse optimal strategy $\alpha^*(t)$ is always guaranteed.

Theorem 3.2. *The system* (2.1) *achieves asymptotic consensus tracking under the sparse optimal strategy $\alpha^*(t)$ proposed in Proposition* 3.1.

Proof. Define a Lyapunov function

$$V(y(t))=\sum_{i=1}^N\frac{d_i}{m_i}y_i^2(t).$$

Direct calculation yields

$$\begin{aligned}\dot V(y(t))&=2\sum_{i=1}^N\frac{d_i}{m_i}y_i(t)\dot y_i(t)=2\sum_{i=1}^N\frac{d_i}{m_i}y_i(t)m_i\left[-\alpha_i(t)y_i(t)+(1-\alpha_i(t))\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))\right]\\&=2\sum_{i=1}^N y_i(t)\sum_{j=1}^N a_{ij}(y_j(t)-y_i(t))-2\sum_{i=1}^N\alpha_i(t)d_i\left[y_i^2(t)+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))\right]\\&=-\sum_{i,j=1}^N a_{ij}(y_j(t)-y_i(t))^2-2\sum_{i=1}^N\alpha_i(t)d_i\left[y_i^2(t)+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}(y_j(t)-y_i(t))\right]\\&=-\sum_{i,j=1}^N a_{ij}(y_j(t)-y_i(t))^2-2\sum_{i=1}^N\frac{d_i}{m_i}\alpha_i(t)\phi_i(t),\end{aligned}\tag{3.10}$$

where the third equation is obtained by Lemma A.3. According to Proposition 3.1, we know that the sparse optimal strategy $\alpha^*(t)$ is better than the strategy $\alpha(t)\equiv(1,0,\cdots,0)$, that is,

$$\sum_{i=1}^N\frac{d_i}{m_i}\alpha_i^*(t)\phi_i(t)\geq\frac{d_1}{m_1}\phi_1(t).$$

So we obtain

$$\begin{aligned}\dot{V}\big(y(t)\big) &\le -\sum_{i,j=1}^{N} a_{ij}\big(y_j(t)-y_i(t)\big)^2 - 2d_1 y_1^2(t) - 2y_1(t)\sum_{j=1}^{N} a_{1j}\big(y_j(t)-y_1(t)\big)\\ &= -\sum_{i,j=1}^{N} a_{ij}\big(y_j(t)-y_i(t)\big)^2 - 2y_1(t)\sum_{j=1}^{N} a_{1j}y_j(t)\\ &= -y^\top(t)(2\mathcal{L}_A+B)y(t),\end{aligned}$$

where the last equation is obtained by Lemma A.2, $A=[a_{ij}], B=[b_{ij}], b_{1j}=a_{1j}, b_{i1}=a_{i1}$, $i,j\in\mathcal{V}$ and $b_{ij}=0, i,j\in\mathcal{V}\backslash\{1\}$. Because the weighted undirected graph $\mathcal{G}$ is connected, $2\mathcal{L}_A+B$ is not only a symmetric matrix, but also a irreducibly diagonally dominant matrix. Therefore, by Lemma A.1 we know that $2\mathcal{L}_A+B$ is a positive definite matrix. Then we have

$$\dot{V}\big(y(t)\big) \le -\frac{y^\top(t)(2\mathcal{L}_A+B)y(t)}{V\big(y(t)\big)}V\big(y(t)\big) \le -\frac{\mu_{\min}}{\max_{i\in\mathcal{V}}\{d_i/m_i\}}V\big(y(t)\big),$$

where $\mu_{\min}>0$ is the smallest eigenvalue of $2\mathcal{L}_A+B$. Thus, we conclude that

$$\lim_{t\to\infty}\|y(t)\|=0.$$

The proof is complete. $\square$

In fact, the sparse optimal strategy $\alpha^*(t)$ is essentially a way of selecting leaders that maximizes the instantaneous convergence speed of the system. By applying the sparse optimal strategy, the agents whose neighbours can not provide sufficient help for the target tracking, or even play the opposite role, will no longer interact with their neighbours, and instead devote to the target tracking. Note an interesting phenomenon is that all agents become followers if $R^+(t)=\emptyset$. In this situation, for each agent, the alignment force is stronger than the tracking force, that is, the help offered by its neighbours is better than its individual contribution to the target tracking. The existence of this situation shows that the leaderless is superior to the leader-follower structure for certain conditions. We provide a example to illustrate the existence of this situation below.

Example 3.1. Consider the case of two agents with initial state $y_1(t_0)\ge y_2(t_0)$ and the constraint $\Sigma=2$

$$\begin{cases}\dot{y}_1(t)=-m_1y_1(t)+m_1\big(1-\alpha_1(t)\big)y_2(t),\\ \dot{y}_2(t)=-m_2y_2(t)+m_2\big(1-\alpha_2(t)\big)y_1(t),\end{cases} \quad t\ge t_0.$$

So (3.9) becomes

$$\max_{\alpha(t)\in U, t\ge t_0}\big(\alpha_1(t)m_1+\alpha_2(t)m_2\big)y_1(t)y_2(t).$$

Now let us solve the optimization problem.

Case I: If $y_1(t_0)\geq y_2(t_0)\geq 0$ or $y_2(t_0)\leq y_1(t_0)\leq 0$, then $y_1(t_0)y_2(t_0)\geq 0$. Set $\alpha(t)=(1,1)$ and obtain

$$\dot{y}_1(t)=-m_1y_1(t),\quad \dot{y}_2(t)=-m_2y_2(t).$$

Solving the equations yields

$$y_1(t)=e^{-m_1(t-t_0)}y_1(t_0),\quad y_2(t)=e^{-m_2(t-t_0)}y_2(t_0),$$

which means $y_1(t)y_2(t)\geq 0, t\in[t_0,\infty)$. Therefore, the solution of the optimization problem is $\alpha(t)\equiv(1,1), t\geq t_0$.

Case II: If $y_2(t_0)<0<y_1(t_0)$, then $y_1(t_0)y_2(t_0)<0$. Set $\alpha(t)=(0,0)$ and obtain

$$\dot{y}_1(t)=-m_1\big(y_1(t)-y_2(t)\big),\quad \dot{y}_2(t)=-m_2\big(y_2(t)-y_1(t)\big).$$

By (3.6), we have

$$\lim_{t\to\infty}y_i(t)=\frac{m_2y_1(t_0)+m_1y_2(t_0)}{m_1+m_2},\quad i=1,2.$$

Next, we prove the monotonicity of $y_i(t)$. Suppose there exists a finite time $T<\infty$ such that $y_1(t)>y_2(t), t\in[t_0,T)$ and $y_1(T)=y_2(T)=c$. Then the solution satisfies $y_1(t)>y_2(t), t\in[t_0,T)$ and $y_1(t)=y_2(t)=c, t\in[T,\infty)$, which contradicts the uniqueness of the solution of the above equation. Thus, $y_1(t)>y_2(t), t\in[t_0,\infty)$. Then $y_1(t)(y_2(t))$ is monotone decreasing (increasing) and converges to $(m_2y_1(t_0)+m_1y_2(t_0))/(m_1+m_2)$ as $t\to\infty$.

If $y_1(t_0)m_2+y_2(t_0)m_1=0$, then $y_1(t)y_2(t)<0, t\in[t_0,\infty)$. Therefore, the solution of the optimization problem is $\alpha(t)\equiv(0,0), t\geq t_0$. If $y_1(t_0)m_2+y_2(t_0)m_1<0$, by the monotonicity of $y_i(t)$, there exists a finite time $t_0<T_1<\infty$ such that $y_1(T_1)=0$. Thus, $y_1(t)y_2(t)<0$, $t\in[t_0,T_1)$ and $y_1(t)y_2(t)\geq 0$, $t\in[T_1,\infty)$. Set $\alpha(t)=(0,0), t\in[t_0,T_1)$ and $\alpha(t)\in\{1\}\times[0,1]$, $t\in[T_1,\infty)$, then we have $y_1(t)\equiv 0$ and $y_2(t)=e^{-m_2t}y_2(T_1), t\in[T_1,\infty)$. Therefore, the solution of the optimization problem is $\alpha(t)=(0,0), t\in[t_0,T_1)$ and $\alpha(t)\in\{1\}\times[0,1], t\in[T_1,\infty)$. Similarly, if $y_1(t_0)m_2+y_2(t_0)m_1>0$, the solution of the optimization problem is $\alpha(t)=(0,0), t\in[t_0,T_2)$ and $\alpha(t)\in[0,1]\times\{1\}, t\in[T_2,\infty)$, where T_2 is the finite time satisfying $y_2(t)<0, t\in[t_0,T_2)$ and $y_2(T_2)=0$.

Remark 3.6. In above example, leaderless structures, if present, only appear in an initial period of time except for the special cases $y_1(t_0)m_2+y_2(t_0)m_1=0$. It is following from (3.6) that $\rho^\top y(t_0)=0$ is the necessary condition of $\alpha^*(t)\equiv\mathbf{0}_N$. Therefore, except for the special cases mentioned above, there must be a leadership structure after a leaderless structure.

The optimal sparse strategy $\alpha^*(t)$ solves the optimal problem (3.7) in theory. However, $\alpha^*(t)$, as a function of $y(t)$, might be highly irregular in time, even everywhere discontinuous, which would cause the chattering phenomena. For example, set the constraint $\Sigma=1$, if $y_1(t_0)=y_2(t_0)\neq 0$, then according to Proposition 3.1, the optimal sparse strategy $\alpha^*(t)$ will keep switching back and forth between $(1,0)$ and $(0,1)$. Numerically,

$\alpha^*(t)$ alternates between (1,0) and (0,1) at every simulation time-step, see Fig. 2(c) for reference. Thus, we introduce the notion of sampling solution appeared in [4, 9] to avoid this mess.

Definition 3.1. *Given the sampling time* $\tau>0$, *the sampling solution of the system* (3.2) *under the sparse optimal strategy* $\alpha^*(y(t))$ *is defined as the continuous function* $y_\tau(t):[t_0,\infty)\to\mathbb{R}^N$ *solving recursively for* $\sigma\in\mathbb{N}$,

$$\dot{y}(t)=-\Lambda_N\big(\alpha^*\big(y(t_0+\sigma\tau)\big)\big)y(t),\quad t-t_0\in[\sigma\tau,(\sigma+1)\tau],$$

using as initial state $y(t_0+\sigma\tau)$, *the endpoint of the solution on the preceding interval, and starting with* $y(t_0)$. *The tracking strategy of the sampling solution is called the sampling strategy, which is denoted by*

$$\alpha^*_\tau(t)=\begin{cases}\alpha^*\big(y(t_0+\sigma\tau)\big), & t-t_0\in\big[\sigma\tau,(\sigma+1)\tau\big),\\ \alpha^*\big(y\big(t_0+(\sigma+1)\tau\big)\big), & t-t_0=(\sigma+1)\tau.\end{cases}\tag{3.11}$$

The sampling strategy $\alpha^*_\tau(t)$ is a piecewise continuous function, which means that the system selects the optimal sparse strategy at the sampling point and maintains its value until the next sampling point. In this way, the chattering caused by the optimal sparse strategy can be effectively controlled by the selection of the sampling time τ at the expense of convergence speed. Next, we show that the sampling solution $y_\tau(t)$ achieves asymptotic consensus tracking.

Theorem 3.3. *For any* $0<\tau<\infty$, *the system* (2.1) *achieves asymptotic consensus tracking under the strategy* $\alpha^*_\tau(t)$ *defined in* (3.11).

Proof. Consider the Lyapunov function $V(y(t))=\sum_{i=1}^N(d_i/m_i)y_i^2(t)$. From (3.10) we have

$$\begin{aligned}\dot{V}\big(y(t)\big)&=-\sum_{i,j=1}^N a_{ij}\big(y_j(t)-y_i(t)\big)^2+\sum_{i=1}^N\alpha_i(t)\left[-2d_iy_i^2(t)-2\sum_{j=1}^N a_{ij}y_i(t)\big(y_j(t)-y_i(t)\big)\right]\\&\le-\sum_{i,j=1}^N a_{ij}\big(y_j(t)-y_i(t)\big)^2+\sum_{i=1}^N\alpha_i(t)\left[-2d_iy_i^2(t)+\sum_{j=1}^N a_{ij}y_i^2(t)+\sum_{j=1}^N a_{ij}\big(y_j(t)-y_i(t)\big)^2\right]\\&=-\sum_{i,j=1}^N a_{ij}\big(y_j(t)-y_i(t)\big)^2+\sum_{i=1}^N\alpha_i(t)\left[-d_iy_i^2(t)+\sum_{j=1}^N a_{ij}\big(y_j(t)-y_i(t)\big)^2\right]\\&=-\sum_{i=1}^N\alpha_i(t)d_iy_i^2(t)-\sum_{i=1}^N\big(1-\alpha_i(t)\big)\sum_{j=1}^N a_{ij}\big(y_j(t)-y_i(t)\big)^2\le0.\end{aligned}\tag{3.12}$$

For any $\sigma\in\mathbb{N}$, the following two cases are discussed under the strategy $\alpha^*_\tau(t)$.

Case I: $\alpha^*_\tau(t)=\alpha^*(\sigma\tau)\neq\mathbf{0}_N, t-t_0\in[\sigma\tau,(\sigma+1)\tau)$. By the definition of $\mathcal{I}_l$ in (3.4), without loss of generality, let $\mathcal{I}_l=\{1,\ldots,l\}$. Then we obtain

$$\begin{aligned}\dot{V}\big(y(t)\big)\leq&-\sum_{i=1}^{l}\alpha_i^*(\sigma\tau)d_iy_i^2(t)-\sum_{i=1}^{l}\big(1-\alpha_i^*(\sigma\tau)\big)\sum_{j=1}^{N}a_{ij}\big(y_j(t)-y_i(t)\big)^2\\&-\sum_{i=l+1}^{N}\sum_{j=1}^{N}a_{ij}\big(y_j(t)-y_i(t)\big)^2\\\leq&-\sum_{i=1}^{l}\alpha_i^*(\sigma\tau)d_iy_i^2(t)-\sum_{i=l+1}^{N}\sum_{j=1}^{N}a_{ij}\big(y_j(t)-y_i(t)\big)^2,\quad t-t_0\in\big[\sigma\tau,(\sigma+1)\tau\big).\end{aligned}$$

Thus, $\dot{V}(y(t))=0$ if and only if $y_k(t)=0, k\in\mathcal{I}_l$ and $y_j(t)=y_i(t), j\in\mathcal{V}, i\in\mathcal{V}\backslash\mathcal{I}_l$. Because $\mathcal{G}$ is connected, there exist indexes $p\in\mathcal{I}_l$ and $q\in\mathcal{V}\backslash\mathcal{I}_l$ such that $a_{qp}>0$. Then $\dot{V}(y(t))=0$ if and only if $y(t)=\mathbf{0}_N$.

Case II: $\alpha_\tau^*(t)=\alpha^*(\sigma\tau)=\mathbf{0}_N, t-t_0\in[\sigma\tau,(\sigma+1)\tau)$. Then we have

$$\dot{V}\big(y(t)\big)=-\sum_{i,j=1}^{N}a_{ij}\big(y_j(t)-y_i(t)\big)^2,\quad t-t_0\in\big[\sigma\tau,(\sigma+1)\tau\big).$$

So $\dot{V}(y(t))=0$ if and only if there is a constant c such that $y(t)=c\mathbf{1}_N$. On the other hand, according to Proposition 3.1, $\alpha^*(\sigma\tau)=\mathbf{0}_N$ if and only if $\phi_i(\sigma\tau)<0, i\in\mathcal{V}$. By the definition of $\phi_i(t)$ in (3.8), we obtain $y(\sigma\tau)\neq c\mathbf{1}_N$ for any $c\in\mathbb{R}$. From (3.6) we know that $\lim_{t\to\infty}y(t)=\mathbf{1}_N\rho^\top y(\sigma\tau)$ if $\alpha_\tau^*(t)\equiv\mathbf{0}_N$, $t-t_0\in[\sigma\tau,\infty)$. Therefore, we conclude that $y(t)\neq c\mathbf{1}_N$, for any $c\in\mathbb{R}, t-t_0\in[\sigma\tau,(\sigma+1)\tau)$, which indicates that $\dot{V}(y(t))<0, t-t_0\in[\sigma\tau,(\sigma+1)\tau)$.

In conclusion, we show that the Lyapunov function $V(y(t))$ satisfies $V(y(t))>0$ and $\dot{V}(y(t))<0, y(t)\in\mathbb{R}^N\backslash\{\mathbf{0}_N\}$ under the strategy $\alpha_\tau^*(t)$. Thus, it is directly from Lemma A.6 that $\|y(t)\|\to 0$ as $t\to\infty$ under the strategy $\alpha_\tau^*(t)$. ☐

The above theorem indicates that the convergence of the sampling solution $y_\tau(t)$ is independent of the sampling time τ. Actually, each process of selecting the optimal strategy according to Proposition 3.1 can be regarded as a switch. Compared to the optimal solution, the number of switches of the sample solution is countable, which restrains the chattering. In addition, the number of switches is not positively correlated with the convergence speed. From simulations in Section 5 we will see that the sampling solution also shows faster convergence speed than the solution under constant control when the sampling time is appropriate. Finally, we study the relationship between the sampling solution and the optimal solution.

Theorem 3.4. *Up to subsequences, the sampling solution $y_\tau(t)$ converges uniformly to an absolutely continuous function $y(t)$ as τ tends to zero. Moreover, $y(t)$ is the optimal solution of the system* (3.2) *under the sparse optimal strategy $\alpha^*(t)$.*

Proof. Let $\tau=1/n, y_\tau(t)=y_n(t)$ and $\alpha_\tau^*(t)=\alpha_n^*(t)$. From (3.12) we have

$$\|y_n(t)\|\leq\max_{i\in\mathcal{V}}\left\{\frac{m_i}{d_i}\right\}V\big(y_n(t)\big)\leq\max_{i\in\mathcal{V}}\left\{\frac{m_i}{d_i}\right\}V\big(y(t_0)\big)=:C_1.$$

So the sequence of continuous functions $\{y_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded by the positive constant C_1. On the other hand, by the Definition 3.1, for $t\in[t_0+\sigma/n,t_0+(\sigma+1)/n)$ we have

$$y_n(t)=y_n(t_0)-\int_{t_0}^{t}\Lambda_N\big(\alpha_n^*(r)\big)y_n(r)\mathrm{d}r. \tag{3.13}$$

Define $\mathcal{K}=\{\alpha^*(t),t\in[t_0,\infty)\}\subseteq\mathbb{R}^N$. According to Proposition 3.1, $|\mathcal{K}|<\infty$ because of $N<\infty$. So there exists a positive number C_2 such that $\|\Lambda_N(\alpha^*(t))\|\le C_2$, $t\in[t_0,\infty)$. Then we obtain

$$\|y_n(t)-y_n(s)\|\le\int_s^t\|\Lambda_N\big(\alpha_n^*(r)\big)y_n(r)\|\mathrm{d}r\le C_1C_2(t-s),$$

which means that the sequence $\{y_n(t)\}_{n\in\mathbb{N}}$ is also equicontinuous. By Ascoli-Arzelá theorem, up to subsequences, $y_n(t)$ converges uniformly to an absolutely continuous function $y(t)$ as n tends to infinity. Then let $n\to\infty$, (3.13) becomes

$$y(t)=y(t_0)-\int_{t_0}^{t}\Lambda_N\big(\alpha^*(r)\big)y(r)\mathrm{d}r.$$

So $y(t)$ is the optimal solution of the system (3.2) under the sparse optimal strategy $\alpha^*(t)$. The proof is complete. ☐

Remark 3.7. Note that the sparse optimal strategy $\alpha^*(t)$ and the sampling strategy $\alpha_\tau^*(t)$ might be discontinuous, so the corresponding solutions $x(t)$ of the system (2.1) should be interpreted in the Filippov sense [14] or Caratheodory sense, e.g. [2]. The existence of sampling solution will always hold because the right-hand side of system (2.1) under the sampling strategy $\alpha_\tau^*(t)$ is bounded and piecewise continuous. However, the existence of optimal solution can not be guaranteed.

Remark 3.8. As shown in Theorems 3.3 and 3.4, the sampling time τ will have direct influences on the sampling solution. In Section 5, Fig. 5(b) shows that the sampling time is roughly proportional to the convergence time. How to choose the sampling time is essentially a trade-off between the convergence speed and the frequency of switching. A possible way to choose the sampling time is to select the smallest possible sampling time within the allowable switching frequency of the system to achieve the fastest convergence.

4 Finite-time consensus tracking

The finite-time convergence has faster convergence, better disturbance rejection and robustness against uncertainties, and is often required in practice. In this section, we will study the finite-time consensus tracking of the system (2.4) through the Lyapunov function method, and obtain the corresponding sufficient and necessary conditions and sparse optimal strategy. Define $y_i(t)=x_i(t)-x^*(t),i\in\mathcal{V}$, then the system (2.4) becomes

$$\dot y_i(t)=m_i\left[-\alpha_i(t)\mathrm{sig}^\theta\big(y_i(t)\big)+\big(1-\alpha_i(t)\big)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right],\quad t\ge t_0,\quad i\in\mathcal{V}. \tag{4.1}$$

Similarly, we first show that the system (2.4) is effective by considering time-invariant strategy.

4.1 Finite-time convergence with the time-invariant strategy

Assume the tracking strategy is time-invariant. Define $\Omega=\mathrm{diag}(\omega_1,\cdots,\omega_N)$ and a Lyapunov function

$$V_\Omega\big(y(t)\big)=y^\top(t)\Omega y(t)=\sum_{i=1}^N\omega_i y_i^2(t),\quad \omega_i=\begin{cases}\dfrac{d_i}{m_i}, & i\in\mathcal{I}_l,\\[2mm] \dfrac{d_i}{m_i(1-\alpha_i)}, & i\in\mathcal{V}\backslash\mathcal{I}_l,\end{cases} \tag{4.2}$$

where $\mathcal{I}_l$ is defined in (3.4). In the following, we obtain sufficient and necessary conditions for finite-time consensus tracking.

Theorem 4.1. *Assume the tracking strategy is time-invariant. The system* (2.4) *achieves finite-time consensus tracking if and only if* $\sum_{i=1}^N\alpha_i>0$. *Moreover, if* $\sum_{i=1}^N\alpha_i>0$, *the settling time* T *satisfies*

$$T\le\begin{cases} t_0+\dfrac{2^{(1-\theta)/2}\omega_{\max}^{(\theta+1)/2}}{(1-\theta)\lambda_{\min}^{(\theta+1)/2}}V_\Omega^{\frac{2-\theta}{2}}\big(y(t_0)\big), & 0\le l\le N-1,\\[3mm] t_0+\max_{i\in\mathcal{V}}\left\{\dfrac{y_i^{1-\theta}(t_0)}{m_i(1-\theta)}\right\}, & l=N,\end{cases}$$

where $\omega_{\max}=\max_{i\in\mathcal{V}}\{\omega_i\},\lambda_{\min}$ *is the smallest eigenvalue of* $\mathcal{L}_{\mathcal{B}}+C$,

$$\mathcal{B}=\left[b_{ij}^{\frac{2}{\theta+1}}\right],\quad b_{pq}=\frac{1-\theta}{\theta+1}a_{pq},\quad b_{qp}=\frac{1-\theta}{\theta+1}a_{qp},\quad p\in\mathcal{I}_l,\quad q\in\mathcal{V},\quad b_{ij}=a_{ij},\quad i,j\in\mathcal{V}\backslash\mathcal{I}_l,$$

$$C=\mathrm{diag}(c_1,\cdots,c_N),\quad c_i=\left(\frac{d_i\theta}{\theta+1}\right)^{\frac{2}{\theta+1}},\quad i\in\mathcal{I}_l,\quad c_j=\left(\frac{d_j\alpha_j}{1-\alpha_j}\right)^{\frac{2}{\theta+1}},\quad j\in\mathcal{V}\backslash\mathcal{I}_l.$$

Proof. **Sufficiency:** Based on the definition of $\mathcal{I}_l$ in (3.4), we discuss the following three cases.

Case I: $l=0$. From (4.1) and the Lyapunov function (4.2), we have

$$\begin{aligned}\dot V_\Omega\big(y(t)\big)&=2\sum_{i=1}^N\omega_i y_i(t)\dot y_i(t)\\&=2\sum_{i=1}^N\omega_i y_i(t)m_i\left[-\alpha_i\mathrm{sig}^\theta\big(y_i(t)\big)+(1-\alpha_i)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right]\end{aligned}$$

$$
\begin{aligned}
&=-2\sum_{i=1}^{N}\frac{d_i\alpha_i}{1-\alpha_i}|y_i(t)|^{\theta+1}+2\sum_{i,j=1}^{N}y_i(t)a_{ij}\mathrm{sig}^{\theta}\big(y_j(t)-y_i(t)\big)\\
&=-2\sum_{i=1}^{N}\frac{d_i\alpha_i}{1-\alpha_i}|y_i(t)|^{\theta+1}-\sum_{i,j=1}^{N}a_{ij}|y_j(t)-y_i(t)|^{\theta+1},
\end{aligned}
$$

where the last equation is obtained by Lemma A.3. According to Lemmas A.4 and A.2, we obtain

$$
\begin{aligned}
\dot{V}_{\Omega}\big(y(t)\big)&\le-\left[2\sum_{i=1}^{N}\left(\frac{d_i\alpha_i}{1-\alpha_i}\right)^{\frac{2}{\theta+1}}|y_i(t)|^2+\sum_{i,j=1}^{N}a_{ij}^{\frac{2}{\theta+1}}|y_j(t)-y_i(t)|^2\right]^{\frac{\theta+1}{2}}\\
&=-\big[2y^{\top}(t)(\mathcal{L}_{\mathcal{B}}+C)y(t)\big]^{\frac{\theta+1}{2}},
\end{aligned}
$$

where

$$
\mathcal{B}=\left[a_{ij}^{\frac{2}{\theta+1}}\right],\quad C=\mathrm{diag}\left(\left(\frac{d_1\alpha_1}{1-\alpha_1}\right)^{\frac{2}{\theta+1}},\dots,\left(\frac{d_N\alpha_N}{1-\alpha_N}\right)^{\frac{2}{\theta+1}}\right).
$$

Because the weighted undirected graph $\mathcal{G}$ is connected, $\mathcal{L}_{\mathcal{B}}+C$ is not only a symmetric matrix, but also a irreducibly diagonally dominant matrix. Therefore, by Lemma A.1 we know that $\mathcal{L}_{\mathcal{B}}+C$ is a positive definite matrix. Then, we obtain

$$
\begin{aligned}
\dot{V}_{\Omega}\big(y(t)\big)&\le-\left[\frac{2y^{\top}(t)(\mathcal{L}_{\mathcal{B}}+C)y(t)}{V_{\Omega}\big(y(t)\big)}\right]^{\frac{\theta+1}{2}}V_{\Omega}^{\frac{\theta+1}{2}}\big(y(t)\big)\\
&\le-\left(\frac{2\lambda_{\min}}{\omega_{\max}}\right)^{\frac{\theta+1}{2}}V_{\Omega}^{\frac{\theta+1}{2}}\big(y(t)\big),\quad t\ge t_0,
\end{aligned}
$$

where $\lambda_{\min}$ is the smallest eigenvalue of $\mathcal{L}_{\mathcal{B}}+C$, $\omega_{\max}=\max_{i\in\mathcal{V}}\{\omega_i\}$. From Lemma A.7, we conclude that $V_{\Omega}(y(t))$ converges to zero in a finite time, and the settling time is estimated by

$$
T_{\mathrm{I}}\le t_0+\frac{V_{\Omega}^{(1-\theta)/2}\big(y(t_0)\big)}{\big((1-\theta)/2\big)(2\lambda_{\min}/\omega_{\max})^{(\theta+1)/2}}=t_0+\frac{2^{(1-\theta)/2}\omega_{\max}^{(\theta+1)/2}}{(1-\theta)\lambda_{\min}^{(\theta+1)/2}}V_{\Omega}^{\frac{1-\theta}{2}}\big(y(t_0)\big).
$$

Case II: $1\le l\le N-1$. In this case, we have

$$
\dot{y}_i(t)=-m_i\mathrm{sig}^{\theta}\big(y_i(t)\big),\quad i\in\mathcal{I}_l. \tag{4.3}
$$

Without loss of generality, let $\mathcal{I}_l=\{1,\dots,l\}$. For any $i\in\mathcal{V}\backslash\mathcal{I}_l, y_i(t)$ satisfies the system (4.1). By (4.1), (4.3) and the Lyapunov function (4.2), we obtain

$$
\begin{aligned}
\dot{V}_{\Omega}\big(y(t)\big)&=2\sum_{i=1}^{N}\omega_i y_i(t)\dot{y}_i(t)\\
&=-2\sum_{i=1}^{l}\omega_i y_i(t)m_i\mathrm{sig}^{\theta}\big(y_i(t)\big)-2\sum_{i=l+1}^{N}\omega_i y_i(t)m_i\alpha_i\mathrm{sig}^{\theta}\big(y_i(t)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{i=l+1}^{N}\omega_i y_i(t)m_i(1-\alpha_i)\sum_{j=1}^{N}\frac{a_{ij}}{d_i}\mathrm{sig}^{\theta}\big(y_j(t)-y_i(t)\big)\\
=&-2\sum_{i=1}^{l}d_i|y_i(t)|^{\theta+1}-2\sum_{i=l+1}^{N}\frac{d_i\alpha_i}{1-\alpha_i}|y_i(t)|^{\theta+1}\\
&+2\sum_{i,j=1}^{N}y_i(t)a_{ij}\mathrm{sig}^{\theta}\big(y_j(t)-y_i(t)\big)-2\sum_{i=1}^{l}y_i(t)\sum_{j=1}^{N}a_{ij}\mathrm{sig}^{\theta}\big(y_j(t)-y_i(t)\big)\\
=&-2\sum_{i=1}^{l}d_i|y_i(t)|^{\theta+1}-2\sum_{i=l+1}^{N}\frac{d_i\alpha_i}{1-\alpha_i}|y_i(t)|^{\theta+1}-\sum_{i,j=1}^{N}a_{ij}|y_j(t)-y_i(t)|^{\theta+1}\\
&-2\sum_{i=1}^{l}y_i(t)\sum_{j=1}^{N}a_{ij}\mathrm{sig}^{\theta}\big(y_j(t)-y_i(t)\big).
\end{aligned}
$$

Using Young's inequality in Lemma A.5 yields

$$
y_i(t)\mathrm{sig}^{\theta}\big(y_j(t)-y_i(t)\big)\le\frac{1}{\theta+1}|y_i(t)|^{\theta+1}+\frac{\theta}{\theta+1}|y_j(t)-y_i(t)|^{\theta+1},\quad\forall i,j\in\mathcal{V}.
$$

So we obtain

$$
\begin{aligned}
\dot{V}_{\Omega}\big(y(t)\big)\le&-2\sum_{i=1}^{l}d_i|y_i(t)|^{\theta+1}-2\sum_{i=l+1}^{N}\frac{d_i\alpha_i}{1-\alpha_i}|y_i(t)|^{\theta+1}-\sum_{i,j=1}^{N}a_{ij}|y_j(t)-y_i(t)|^{\theta+1}\\
&+\frac{2}{\theta+1}\sum_{i=1}^{l}d_i|y_i(t)|^{\theta+1}+\frac{2\theta}{\theta+1}\sum_{i=1}^{l}\sum_{j=1}^{N}a_{ij}|y_j(t)-y_i(t)|^{\theta+1}\\
\le&-\frac{2\theta}{\theta+1}\sum_{i=1}^{l}d_i|y_i(t)|^{\theta+1}-2\sum_{i=l+1}^{N}\frac{d_i\alpha_i}{1-\alpha_i}|y_i(t)|^{\theta+1}-\sum_{i,j=1}^{N}b_{ij}|y_j(t)-y_i(t)|^{\theta+1}\\
\le&-\Bigg[2\sum_{i=1}^{l}\left(\frac{d_i\theta}{\theta+1}\right)^{\frac{2}{\theta+1}}|y_i(t)|^2+2\sum_{i=l+1}^{N}\left(\frac{d_i\alpha_i}{1-\alpha_i}\right)^{\frac{2}{\theta+1}}|y_i(t)|^2\\
&\quad+\sum_{i,j=1}^{N}b_{ij}^{\frac{2}{\theta+1}}|y_j(t)-y_i(t)|^2\Bigg]^{\frac{\theta+1}{2}}\\
=&-\big[2y^{\top}(t)(\mathcal{L}_{\mathcal{B}}+C)y(t)\big]^{\frac{\theta+1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{B}=\Big[b_{ij}^{\frac{2}{\theta+1}}\Big],\quad b_{pq}=\frac{1-\theta}{\theta+1}a_{pq},\quad b_{qp}=\frac{1-\theta}{\theta+1}a_{qp},\quad p\in\mathcal{I}_l,q\in\mathcal{V},\quad b_{ij}=a_{ij},\quad i,j\in\mathcal{V}\backslash\mathcal{I}_l,\\
&C=\mathrm{diag}(c_1,\cdots,c_N),\quad c_i=\left(\frac{d_i\theta}{\theta+1}\right)^{\frac{2}{\theta+1}},\quad i\in\mathcal{I}_l,\quad c_j=\left(\frac{d_j\alpha_j}{1-\alpha_j}\right)^{\frac{2}{\theta+1}},\quad j\in\mathcal{V}\backslash\mathcal{I}_l.
\end{aligned}
$$

Similar to Case I, we can prove that $\mathcal{L}_{\mathcal{B}}+C$ is a positive definite matrix, and

$$\dot{V}_{\Omega}\big(y(t)\big)\le-\big[2y^{\top}(t)(\mathcal{L}_{\mathcal{B}}+C)y(t)\big]^{\frac{\theta+1}{2}}\le-\left(\frac{2\lambda_{\min}}{\omega_{\max}}\right)^{\frac{\theta+1}{2}}V_{\Omega}^{\frac{\theta+1}{2}}\big(y(t)\big),\quad t\ge t_0.$$

From Lemma A.7, we conclude that $V_{\Omega}(y(t))$ converges to zero in a finite time, and the settling time is estimated by

$$T_{\rm II}\le t_0+\frac{V_{\Omega}^{(1-\theta)/2}\big(y(t_0)\big)}{\big((1-\theta)/2\big)(2\lambda_{\min}/\omega_{\max})^{(\theta+1)/2}}=t_0+\frac{2^{(1-\theta)/2}\omega_{\max}^{(\theta+1)/2}}{(1-\theta)\lambda_{\min}^{(\theta+1)/2}}V_{\Omega}^{\frac{1-\theta}{2}}\big(y(t_0)\big).$$

Case III: $l=N$. Solving (4.3) yields

$$y_i^2(t)=\begin{cases}\big[y_i^{1-\theta}(t_0)-m_i(1-\theta)(t-t_0)\big]^{\frac{2}{1-\theta}}, & t_0\le t\le t_i,\\ 0, & t>t_i,\end{cases}$$

where $t_i=t_0+y_i^{1-\theta}(t_0)/(m_i(1-\theta))$. It is directly from above equations that the system (4.1) achieves finite-time consensus tracking, and the settling time

$$T_{\rm III}=\max_{i\in\mathcal{V}}\left\{t_0+\frac{y_i^{1-\theta}(t_0)}{m_i(1-\theta)}\right\}.$$

Necessity: Assume $\sum_{i=1}^N\alpha_i=0$. Then the system (4.1) becomes

$$\dot{y}_i(t)=m_i\sum_{j=1}^N\frac{a_{ij}}{d_i}{\rm sig}^{\theta}\big(y_j(t)-y_i(t)\big).$$

Define

$$\overline{y}(t)=\frac{\sum_{i=1}^N(d_i/m_i)y_i(t)}{\sum_{j=1}^N d_j/m_j},$$

we have

$$\begin{aligned}\dot{\overline{y}}(t)&=\frac{\sum_{i=1}^N(d_i/m_i)\dot{y}_i(t)}{\sum_{j=1}^N d_j/m_j}=\frac{\sum_{i=1}^N(d_i/m_i)m_i\sum_{j=1}^N(a_{ij}/d_i){\rm sig}^{\theta}\big(y_j(t)-y_i(t)\big)}{\sum_{j=1}^N d_j/m_j}\\&=\frac{\sum_{i,j=1}^N a_{ij}{\rm sig}^{\theta}\big(y_j(t)-y_i(t)\big)}{\sum_{j=1}^N d_j/m_j}=0.\end{aligned}$$

Define $z_i(t)=y_i(t)-\overline{y}(t)$ and $V(z(t))=\sum_{i=1}^N(d_i/m_i)z_i^2(t)$. Then

$$\begin{aligned}\dot{V}\big(z(t)\big)&=2\sum_{i=1}^N\frac{d_i}{m_i}z_i(t)\dot{z}_i(t)=2\sum_{i=1}^N\frac{d_i}{m_i}z_i(t)m_i\sum_{j=1}^N\frac{a_{ij}}{d_i}{\rm sig}^{\theta}\big(y_j(t)-y_i(t)\big)\\&=2\sum_{i,j=1}^N z_i(t)a_{ij}{\rm sig}^{\theta}\big(z_j(t)-z_i(t)\big)=-\sum_{i,j=1}^N a_{ij}|z_j(t)-z_i(t)|^{\theta+1},\end{aligned}$$

where the last equation is obtained by Lemma A.3. According to the first inequality of Lemma A.4, let $r=\theta+1<s=2$, we obtain

$$\dot{V}\big(z(t)\big)\leq -\left[\sum_{i,j=1}^{N} a_{ij}^{\frac{2}{\theta+1}}|z_j(t)-z_i(t)|^2\right]^{\frac{\theta+1}{2}}.$$

Define $\mathcal{B}=[b_{ij}]\in\mathbb{R}^{N\times N}, b_{ij}=a_{ij}^{2/(\theta+1)}$. Notice that

$$\xi^\top z(t)=\sum_{i=1}^{N}\frac{d_i}{m_i}z_i(t)=\sum_{i=1}^{N}\frac{d_i}{m_i}\big(y_i(t)-\overline{y}(t)\big)=0,\quad \xi=\left(\frac{d_1}{m_1},\cdots,\frac{d_N}{m_N}\right)^\top.$$

Using Lemma A.2 yields

$$\begin{aligned}\dot{V}\big(z(t)\big)&\leq -\big[2z^\top(t)\mathcal{L}_{\mathcal{B}}z(t)\big]^{\frac{\theta+1}{2}}=-\left[\frac{2z^\top(t)\mathcal{L}_{\mathcal{B}}z(t)}{V\big(z(t)\big)}\right]^{\frac{\theta+1}{2}}V^{\frac{\theta+1}{2}}\big(z(t)\big)\\&\leq -\left(\frac{2c_0}{\xi_{\max}}\right)^{\frac{\theta+1}{2}}V^{\frac{\theta+1}{2}}\big(z(t)\big),\end{aligned}$$

where $c_0=\min\{(z^T\mathcal{L}_{\mathcal{B}}z)/(z^Tz):\xi^Tz=0\}>0$ because the set is closed. Therefore, by Lemma A.7, we have

$$\begin{aligned}&y_i(t)=\overline{y}(t)=\overline{y}(t_0)=\frac{\sum_{i=1}^{N}(d_i/m_i)y_i(t_0)}{\sum_{j=1}^{N}d_j/m_j},\quad t\geq T_z,\\&T_z\leq t_0+\frac{V^{(1-\theta)/2}\big(z(t_0)\big)}{(2c_0/\xi_{\max})^{(\theta+1)/2}(1-\theta)/2}.\end{aligned}\tag{4.4}$$

However, by the arbitrary of initial conditions, there exists $y(t_0)$ such that $\overline{y}(t_0)\neq 0$, which contradicts the fact that the system achieves finite-time consensus tracking. Therefore, $\sum_{i=1}^{N}\alpha_i>0$. □

Remark 4.1. Comparison Theorems 3.1 and 4.1 shows that the conditions for the system (2.4) achieving finite-time consensus tracking are the same as those for the system (2.1) achieving asymptotic consensus tracking. Unfortunately, the estimates of settling time obtained in Theorem 4.1 do not really reflect the convergence speed of the system (4.1). Thus, we still take the instantaneous convergence speed $\dot{\mathbb{X}}(t)$ as the performance index to solve the optimal control of the system (4.1).

4.2 Optimal control of finite-time consensus tracking

Theorem 4.1 accounts for the validity of the system (2.4). Next, we will study the following optimal control problem:

$$\begin{cases}\displaystyle\min_{\alpha(t)\in U,t\geq t_0} J\big(\alpha(t)\big)=\dot{\mathbb{X}}(t)\\ \text{s.t.}\quad y(t)\text{ satisfies (4.1).}\end{cases}\tag{4.5}$$

Direct calculation yields

$$
\begin{aligned}
\dot{\mathbb{X}}(t)=2y^\top(t)\dot{y}(t)&=2\sum_{i=1}^N y_i(t)m_i\left[-\alpha_i(t)\mathrm{sig}^\theta\big(y_i(t)\big)+\big(1-\alpha_i(t)\big)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right]\\
&=2\sum_{i=1}^N m_iy_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\\
&\quad -2\sum_{i=1}^N\alpha_i(t)m_i\left[|y_i(t)|^{\theta+1}+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right].
\end{aligned}
$$

Similar to (3.8), define the controllable term of agent i by

$$
\phi_i(t)=m_i\left[|y_i(t)|^{\theta+1}+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right],\quad i\in\mathcal{V}, \tag{4.6}
$$

then minimizing $\dot{\mathbb{X}}(t)$ is equivalent to (3.9). Therefore, the sparse optimal strategy $\alpha^*(t)$ selected by Proposition 3.1 is also an optimal solution of the optimal control problem (4.5). The explanations in Remarks 3.4 and 3.5 are valid here as well. Next, we show that the convergence of the system (4.1) under the sparse optimal strategy $\alpha^*(t)$ is always guaranteed.

Theorem 4.2. *The system* (2.4) *achieves finite-time consensus tracking under the sparse optimal strategy $\alpha^*(t)$ proposed in Proposition* 3.1.

Proof. Consider the Lyapunov function $V(y(t))=\sum_{i=1}^N(d_i/m_i)y_i^2(t)$. Direct calculation yields

$$
\begin{aligned}
\dot{V}\big(y(t)\big)&=2\sum_{i=1}^N\frac{d_i}{m_i}y_i(t)\dot{y}_i(t)\\
&=2\sum_{i=1}^N\frac{d_i}{m_i}y_i(t)m_i\left[-\alpha_i(t)\mathrm{sig}^\theta\big(y_i(t)\big)+\big(1-\alpha_i(t)\big)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right]\\
&=2\sum_{i=1}^N y_i(t)\sum_{j=1}^N a_{ij}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\\
&\quad -2\sum_{i=1}^N\alpha_i(t)d_i\left[|y_i(t)|^{\theta+1}+y_i(t)\sum_{j=1}^N\frac{a_{ij}}{d_i}\mathrm{sig}^\theta\big(y_j(t)-y_i(t)\big)\right]\\
&=-\sum_{i,j=1}^N a_{ij}|y_j(t)-y_i(t)|^{\theta+1}-2\sum_{i=1}^N\frac{d_i}{m_i}\alpha_i(t)\phi_i(t).
\end{aligned} \tag{4.7}
$$

Applying the sparse optimal strategy $\alpha^*(t)$ proposed in Proposition 3.1 yields

$$
\dot{V}\big(y(t)\big)\le-\sum_{i,j=1}^N a_{ij}|y_j(t)-y_i(t)|^{\theta+1}-2\frac{d_1}{m_1}\phi_1(t)
$$

$$\begin{aligned}
&= -\sum_{i,j=1}^{N} a_{ij}|y_j(t)-y_i(t)|^{\theta+1} - 2d_1|y_1(t)|^{\theta+1} - 2y_1(t)\sum_{j=1}^{N} a_{1j}\mathrm{sig}^{\theta}\big(y_j(t)-y_1(t)\big) \\
&\le -\sum_{i,j=1}^{N} a_{ij}|y_j(t)-y_i(t)|^{\theta+1} - \frac{2\theta}{\theta+1}d_1|y_1(t)|^{\theta+1} + \frac{2\theta}{\theta+1}\sum_{j=1}^{N} a_{1j}|y_j(t)-y_1(t)|^{\theta+1} \\
&\le -\sum_{i,j=1}^{N} h_{ij}|y_j(t)-y_i(t)|^{\theta+1} - \frac{2\theta}{\theta+1}d_1|y_1(t)|^{\theta+1} \\
&\le -\left[\sum_{i,j=1}^{N} h_{ij}^{\frac{2}{\theta+1}}|y_j(t)-y_i(t)|^{2} - \left(\frac{2\theta}{\theta+1}d_1\right)^{\frac{2}{\theta+1}}|y_1(t)|^{2}\right]^{\frac{\theta+1}{2}} \\
&= -\left[y^{\top}(t)(2\mathcal{L}_H+S)y(t)\right]^{\frac{\theta+1}{2}},
\end{aligned}$$

where the second inequality is obtained by Lemma A.5, the last inequality is obtained by Lemma A.4, the last equation is obtained by Lemma A.2 and

$$\begin{aligned}
&H=\left[h_{ij}^{\frac{2}{\theta+1}}\right], \quad h_{1k}=\frac{1-\theta}{\theta+1}a_{1k}, \quad h_{k1}=\frac{1-\theta}{\theta+1}a_{k1}, \quad k\in\mathcal{V}, \\
&h_{ij}=a_{ij}, \quad i,j\in\mathcal{V}\backslash\{1\}, \quad S=\mathrm{diag}\left(\left(\frac{2\theta}{\theta+1}d_1\right)^{\frac{2}{\theta+1}},0,\dots,0\right).
\end{aligned}$$

Because the weighted undirected graph $\mathcal{G}$ is connected, $2\mathcal{L}_H+S$ is not only a symmetric matrix, but also a irreducibly diagonally dominant matrix. Therefore, by Lemma A.1 we know that $2\mathcal{L}_H+S$ is a positive definite matrix. Then we have

$$\begin{aligned}
\dot{V}\big(y(t)\big) &\le -\left[\frac{y^{\top}(t)(2\mathcal{L}_H+S)y(t)}{V\big(y(t)\big)}\right]^{\frac{\theta+1}{2}} V^{\frac{\theta+1}{2}}\big(y(t)\big) \\
&\le -\left[\frac{\nu_{\min}}{\max_{i\in\mathcal{V}}\{d_i/m_i\}}\right]^{\frac{\theta+1}{2}} V^{\frac{\theta+1}{2}}\big(y(t)\big),
\end{aligned}$$

where $\nu_{\min}>0$ is the smallest eigenvalue of $2\mathcal{L}_H+S$. Thus, by Lemma A.7 we conclude that the system achieves finite-time convergence. ☐

In addition, the situation of leaderless also exists and is illustrated by an example below.

Example 4.1. Consider the case of two agents with $m_1=m_2=1$, initial state $y_1(t_0)\ge y_2(t_0)$ and the constraint $\Sigma=2$

$$\begin{cases}
\dot{y}_1(t)=-\alpha_1(t)\mathrm{sig}^{\theta}\big(y_1(t)\big)+\big(1-\alpha_1(t)\big)\mathrm{sig}^{\theta}\big(y_2(t)-y_1(t)\big), \\
\dot{y}_2(t)=-\alpha_2(t)\mathrm{sig}^{\theta}\big(y_2(t)\big)+\big(1-\alpha_2(t)\big)\mathrm{sig}^{\theta}\big(y_1(t)-y_2(t)\big).
\end{cases}$$

So (3.9) becomes $\max \alpha_1(t)\phi_1(t)+\alpha_2(t)\phi_2(t)$, where

$$\begin{aligned}\phi_1(t)=|y_1(t)|^{\theta+1}+y_1(t)\mathrm{sig}^\theta\big(y_2(t)-y_1(t)\big),\\ \phi_2(t)=|y_2(t)|^{\theta+1}+y_2(t)\mathrm{sig}^\theta\big(y_1(t)-y_2(t)\big).\end{aligned}$$

Now let us solve the optimization problem.

Case I: If $y_1(t_0)\geq y_2(t_0)\geq 0$, then we have

$$\begin{aligned}\phi_1(t_0)=|y_1(t_0)|\big(|y_1(t_0)|^\theta-|y_1(t_0)-y_2(t_0)|^\theta\big)\geq 0,\\ \phi_2(t_0)=|y_2(t_0)|\big(|y_2(t_0)|^\theta+|y_1(t_0)-y_2(t_0)|^\theta\big)\geq 0.\end{aligned}$$

Set $\alpha(t)=(1,1)$ and obtain $\dot{y}_i(t)=-\mathrm{sig}^\theta(y_i(t)),i=1,2$. Solving equations yields

$$\begin{aligned}&y_i(t)=\left[|y_i(t_0)|^{1-\theta}-\frac{1-\theta}{2}t\right]^{\frac{1}{1-\theta}}, && t_0\leq t\leq\frac{2|y_i(t_0)|^{1-\theta}}{1-\theta}:=t_i,\\ &y_i(t)\equiv 0, && t>t_i.\end{aligned}$$

Because $t_1\geq t_2, y_1(t)\geq y_2(t), t_0\leq t\leq t_1$. Therefore, the solution of the optimization problem is $\alpha(t)=(1,1), t\in[t_0,t_1]$.

Case II: If $0\geq y_1(t_0)\geq y_2(t_0)$, then we have

$$\begin{aligned}\phi_1(t_0)=|y_1(t_0)|\big(|y_1(t_0)|^\theta+|y_1(t_0)-y_2(t_0)|^\theta\big)\geq 0,\\ \phi_2(t_0)=|y_2(t_0)|\big(|y_2(t_0)|^\theta-|y_1(t_0)-y_2(t_0)|^\theta\big)\geq 0.\end{aligned}$$

Similar to Case I, we know that the solution of the optimization problem is $\alpha(t)=(1,1)$, $t\in[t_0,t_2]$.

Case III: If $y_2(t_0)<0<y_1(t_0)$, then we have

$$\begin{aligned}\phi_1(t_0)=|y_1(t_0)|\big(|y_1(t_0)|^\theta-|y_1(t_0)-y_2(t_0)|^\theta\big)<0,\\ \phi_2(t_0)=|y_2(t_0)|\big(|y_2(t_0)|^\theta-|y_1(t_0)-y_2(t_0)|^\theta\big)<0.\end{aligned}$$

Set $\alpha(t)=(0,0)$ and obtain

$$\dot{y}_1(t)=\mathrm{sig}^\theta\big(y_2(t)-y_1(t)\big),\quad \dot{y}_2(t)=\mathrm{sig}^\theta\big(y_1(t)-y_2(t)\big).$$

By (4.4), we have

$$\lim_{t\to T_z}y_i(t)=\frac{y_1(t_0)+y_2(t_0)}{2},\quad i=1,2.$$

Next, we prove the monotonicity of $y_i(t)$. Suppose there exists a finite time $T<\infty$ such that $y_1(t)>y_2(t), t\in[t_0,T)$ and $y_1(T)=y_2(T)=c$. Then $c=(y_1(t_0)+y_2(t_0))/2$.

Thus, $T\geq T_z$. Then $y_1(t)$ $(y_2(t))$ is monotone decreasing (increasing) and converges to $(y_1(t_0)+y_2(t_0))/2$ as $t\to T_z$.

If $y_1(t_0)+y_2(t_0)=0$, then the solution of the optimization problem is $\alpha(t)=(0,0)$, $t\in[t_0,T_z]$. If $y_1(t_0)+y_2(t_0)<0$, by the monotonicity of $y_i(t)$, there exists a finite time $t_{11}<T_z$ such that $y_1(t)>0,t\in[t_0,t_{11})$ and $y_1(t_{11})=0$. Thus, $\phi_1(t)<0,\phi_2(t)<0,t\in[t_0,t_{11})$ and $\phi_1(t)\geq 0,\phi_2(t)\geq 0,t\in[t_{11},T_z]$. Set $\alpha(t)=(0,0),t\in[t_0,t_{11})$ and $\alpha(t)=(1,1),t\in[t_{11},\infty)$, then we have $y_1(t)\equiv 0$ and

$$y_2(t)=\left[|y_2(t_{11})|^{1-\theta}-\frac{1-\theta}{2}t\right]^{\frac{1}{1-\theta}},\quad t\in[t_{11},t_{12}],$$

where $t_{12}=t_{11}+2|y_2(t_{11})|^{1-\theta}/(1-\theta)$. Therefore, the solution of the optimization problem is $\alpha(t)=(0,0),t\in[t_0,t_{11})$ and $\alpha(t)=(1,1),t\in[t_{11},t_{12}]$. Similarly, if $y_1(t_0)+y_2(t_0)>0$, the solution of the optimization problem is $\alpha(t)=(0,0),t\in[t_0,t_{22})$ and $\alpha(t)=(1,1),t\in[t_{22},t_{21}]$, where $t_{22}<T_z$ satisfies $y_2(t)<0$, $t\in[t_0,t_{22})$ and $y_2(t_{22})=0,t_{21}=t_{22}+2|y_1(t_{22})|^{1-\theta}/(1-\theta)$.

The sampling solution is also used here to avoid the chattering phenomena. We show that the sampling solution converges in the following.

Theorem 4.3. *For any* $0<\tau<\infty$, *the system* (2.4) *achieves finite-time consensus tracking under the strategy* $\alpha^*_\tau(t)$ *defined in* (3.11).

Proof. By (4.7), Lemmas A.5, A.4 and A.2 we have

$$\begin{aligned}
\dot{V}(y(t))\leq&-\sum_{i,j=1}^N a_{ij}|y_j(t)-y_i(t)|^{\theta+1}\\
&+2\sum_{i=1}^N\alpha_i(t)\left[-d_i|y_i(t)|^{\theta+1}+\frac{1}{\theta+1}d_i|y_i(t)|^{\theta+1}+\frac{\theta}{\theta+1}\sum_{j=1}^N a_{ij}|y_j(t)-y_i(t)|^{\theta+1}\right]\\
=&-\sum_{i,j=1}^N a_{ij}|y_j(t)-y_i(t)|^{\theta+1}-\sum_{i=1}^N\alpha_i(t)\frac{2\theta}{\theta+1}d_i|y_i(t)|^{\theta+1}\\
&+\sum_{i,j=1}^N\alpha_i(t)\frac{2\theta}{\theta+1}a_{ij}|y_j(t)-y_i(t)|^{\theta+1}\\
\leq&-\sum_{i,j=1}^N h_{ij}|y_j(t)-y_i(t)|^{\theta+1}-\sum_{i=1}^N\alpha_i(t)\frac{2\theta}{\theta+1}d_i|y_i(t)|^{\theta+1}\\
\leq&-\left[\sum_{i,j=1}^N h_{ij}^{\frac{2}{\theta+1}}|y_j(t)-y_i(t)|^2+\sum_{i=1}^N\left(\alpha_i(t)\frac{2\theta}{\theta+1}d_i\right)^{\frac{2}{\theta+1}}|y_i(t)|^2\right]^{\frac{\theta+1}{2}}\\
=&-\left[y^\top(t)\left(2\mathcal{L}_H+S(\alpha(t))\right)y(t)\right]^{\frac{\theta+1}{2}},
\end{aligned}\tag{4.8}$$

where

$$H=[h_{ij}^{\frac{2}{\theta+1}}],\quad h_{ij}=\frac{1-\theta}{\theta+1}a_{ij},\quad i,j\in\mathcal{V},$$

$$S(\alpha(t))=\mathrm{diag}\left(\left(\alpha_1(t)\frac{2\theta}{\theta+1}d_1\right)^{\frac{2}{\theta+1}},\cdots,\left(\alpha_N(t)\frac{2\theta}{\theta+1}d_N\right)^{\frac{2}{\theta+1}}\right).$$

For any $\sigma\in\mathbb{N}$, if $\alpha_\tau^*(t)=\alpha^*(\sigma\tau)\neq\mathbf{0}_N,t-t_0\in[\sigma\tau,(\sigma+1)\tau)$, then $2\mathcal{L}_H+S(\alpha_\tau^*(t))$ is not only a symmetric matrix, but also a irreducibly diagonally dominant matrix. Define the smallest eigenvalue of $2\mathcal{L}_H+S(\alpha(t))$ by $\nu_{\min}(\alpha(t))$. So, by Lemma A.1, we have $\nu_{\min}(\alpha_\tau^*(t))>0$, $t-t_0\in[\sigma\tau,(\sigma+1)\tau)$. Because there is a limited number of the strategy $\alpha^*(\sigma\tau)$ to be selected, there exists a positive number ν^* such that $\nu_{\min}(\alpha^*(\sigma\tau))\geq\nu^*>0$ for any $\sigma\in\mathbb{N}$ and $\alpha^*(\sigma\tau)\neq\mathbf{0}_N$. Thus, we obtain

$$\begin{aligned}\dot V(y(t))&\leq-\left[\frac{y^\top(t)(2\mathcal{L}_H+S(\alpha(t)))y(t)}{V(y(t))}\right]^{\frac{\theta+1}{2}}V^{\frac{\theta+1}{2}}(y(t))\\&\leq-\left[\frac{\nu^*}{\max_{i\in\mathcal{V}}\{d_i/m_i\}}\right]^{\frac{\theta+1}{2}}V^{\frac{\theta+1}{2}}(y(t)),\quad t-t_0\in[\sigma\tau,(\sigma+1)\tau).\end{aligned}\tag{4.9}$$

On the other hand, if $\alpha_\tau^*(t)=\alpha^*(\sigma\tau)=\mathbf{0}_N,t-t_0\in[\sigma\tau,(\sigma+1)\tau)$, then

$$\dot V(y(t))\leq-\left[y^\top(t)(2\mathcal{L}_H)y(t)\right]^{\frac{\theta+1}{2}}\leq 0,\quad t-t_0\in[\sigma\tau,(\sigma+1)\tau).$$

For any $k\in\mathbb{N}$, we claim that there exists a natural number $\sigma_k\geq k$ such that $\alpha_\tau^*(t)=\alpha^*(\sigma_k\tau)\neq\mathbf{0}_N,t-t_0\in[\sigma_k\tau,(\sigma_1+1)\tau)$. If not, there is a natural number $\sigma\in\mathbb{N}$ such that $\alpha_\tau^*(t)=\mathbf{0}_N$, $t-t_0\in[\sigma\tau,\infty)$, which contradicts (4.4) and Proposition 3.1. Then, there is an infinite number of intervals such that (4.9) holds, and $V(y(t))$ does not increase in the remaining intervals. So we conclude that the sampling solution achieves finite-time convergence. ☐

Remark 4.2. Theorems 4.2 and 4.3 indicate that the sparse optimal strategy in Proposition 3.1 and the sampling solution are also applicable to finite-time consensus tracking. Similar to Theorem 3.4, we can also show that the sampling solution converges to the optimal solution as the sampling time tends to zero.

5 Simulations

In this section, several numerical simulations are provided to observe the effects of the sampling solution. We focus on the influence of initial value $x(t_0)$, control constraint $\sum_{i=1}^N\alpha_i(t)\leq\Sigma$ and sampling time τ on the system respectively. Here, only the finite-time convergence results are presented, because the simulations corresponding to asymptotic and finite-time convergence differ only in their convergence rates under identical initial conditions. Consider the system (2.4) and $N=8$. The target state of the system is $x^*(t)=2+\sin(t)$. To easily observe the different representations of the sampling solution when other factors are changed, let the initial time $t_0=0$ and the gain limitation

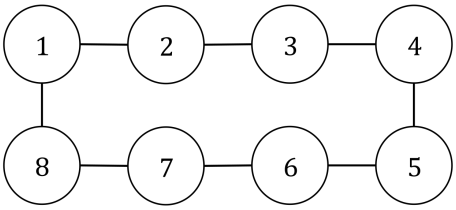


Figure 1: A weighted ring network $\mathcal{G}$, where weights of all edge are 1.

$m_i = 1, i \in \mathcal{V}$. In all the simulations, we take the Eulerian difference methods with the time step of $\mathrm{d}t = 10^{-3}$. The system is determined to achieve finite-time convergence if $\mathbb{X}(t) < 10^{-3}, t \geq T$, and T is the settling time. Consider a weighted ring network in Fig. 1, the control constraint $\Sigma = 2$.

Simulation 1: Set the sampling time equal to the simulation time-step ($\tau = dt$) to emulate the case of finite-time convergence without sampling. Taking the initial value

$$x(t_0) = (3,4,5,6,7,8,9,10)^\top$$

yields numerical results shown in Fig. 2, which show that the sampling solution achieves finite-time convergence. However, Fig. 2(c) appears to show three or four leaders simultaneously, which seems to violate the constraint $\Sigma = 2$. This illusion arises because the optimal sparse strategy $\alpha^*(t)$ rapidly switching among these agents. The sampling solution was introduced precisely to resolve this issue.

Simulation 2: Taking sampling time $\tau = 1$ and the initial value

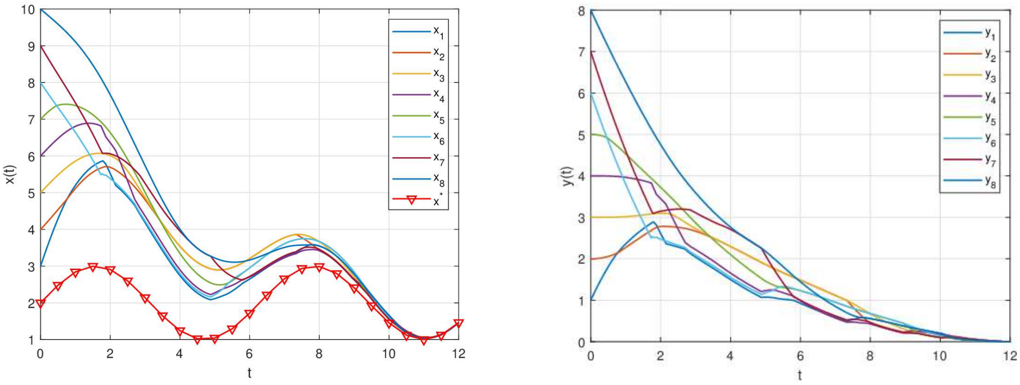
$$x(t_0) = (3,4,5,6,7,8,9,10)^\top$$

yields numerical results shown in Fig. 3, which show that the sampling solution achieves finite-time convergence. There is no leaderless structure throughout the process. Comparing with constant controls $\alpha_{sq}(t) \equiv (1,1,0,0,0,0,0,0)$ and $\alpha_{tr}(t) \equiv \alpha^*(t_0)$, the sampling solution updates the strategy value at each sampling point to achieve faster convergence, where $\alpha_{tr}(t)$ means that apply optimal strategy at the beginning and never change it again. Moreover, the settling times of the three cases are 31.957,27.174 and 14.061, which shows that the strategy $\alpha_{tr}(t)$ is slightly better than $\alpha_{sq}(t)$, and the sampling solution is significantly better than $\alpha_{tr}(t)$ and $\alpha_{sq}(t)$. The results indicate the necessity of timely updating the strategy.

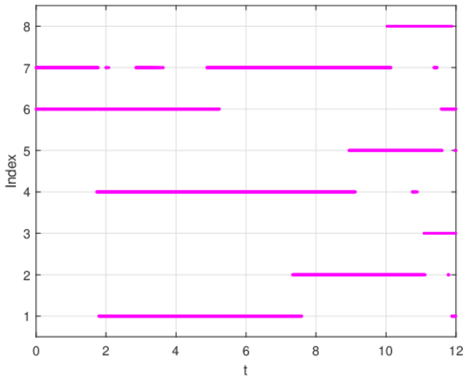
Simulation 3: Changing the initial value in Simulation 2 to

$$x(t_0) = (4,-2,5,-3,6,-4,8,-6)^\top$$

and keeping other parameters yield numerical results shown in Fig. 4. The sampling solution achieves finite-time convergence. A leaderless structure appears in the first interval, a single leader structure appears in the second interval, and a two leaders structure



(a) Agents' state: $x(t)$ and target state: $x^*(t)$ (b) Agents' deviations: $y(t)$



(c) Controlled agents

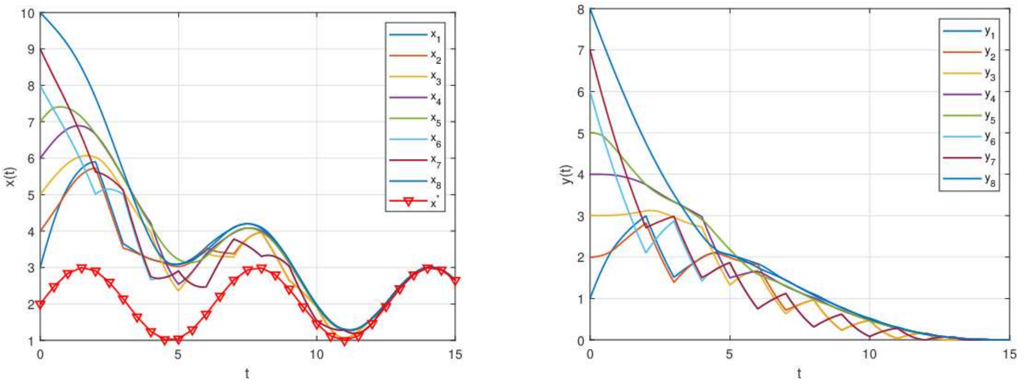
Figure 2: The factors are $\mathcal{G}, \Sigma = 2, \tau = dt, x(t_0) = (3,4,5,6,7,8,9,10)^\top$. (a) shows trajectories of agents and the target, (b) shows the trajectories of agents' deviations, (c) shows controlled agents that selected by the sparse optimal strategy. The optimal sparse strategy $\alpha^*(t)$ rapidly switching among three or four agents, which means $\alpha^*(t)$ is highly irregular.

appears in the rest interval. The system can achieve finite-time consensus tracking under $\alpha_{sq}(t)$, but can not tracking target under $\alpha_{tr}(t)$. The settling times of $\alpha_{sq}(t)$ and the sampling time are 13.518 and 7.020. The results indicate the sparse optimal strategy is highly dependent on the initial value $x(t_0)$.

Simulation 4: Consider the weighted ring network $\mathcal{G}$ and initial value

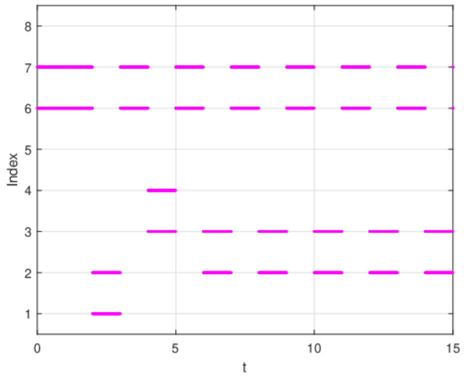
$$x(t_0) = (3,4,5,6,7,8,9,10)^\top.$$

Taking the control constraint $\Sigma = 1:0.1:N$ and keeping the sampling time $\tau = 1$ yield numerical result (a) in Fig. 5, which shows that the time T is roughly inversely proportional to the control constraint Σ. Taking the sampling time $\tau = 0.01:0.01:1$ and keeping the control constraint $\Sigma = 2$ yield numerical result in Fig. 5(b), which shows that the settling time T as a whole has an upward trend when the sampling time τ increases.

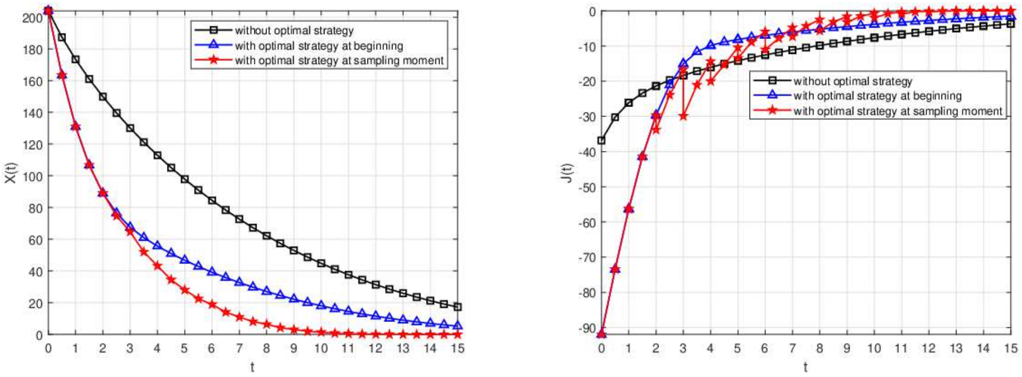


(a) Agents' state: $x(t)$ and target state: $x^*(t)$

(b) Agents' deviations: $y(t)$



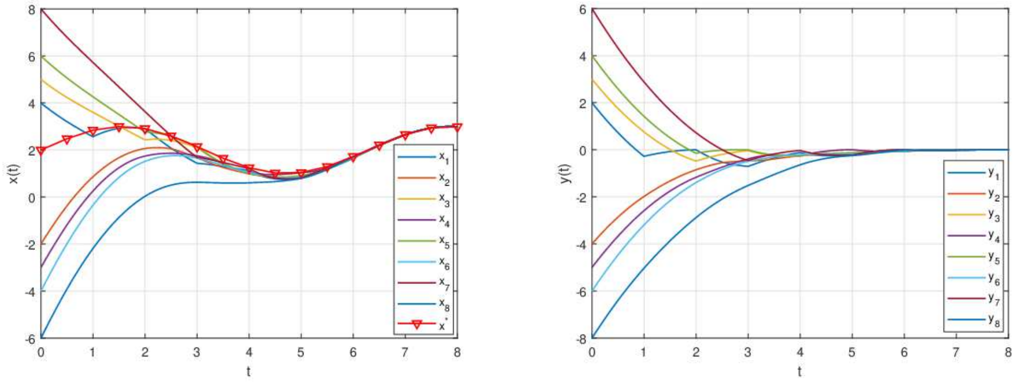
(c) Controlled agents



(d) The deviation of the system: $\mathbb{X}(t)$

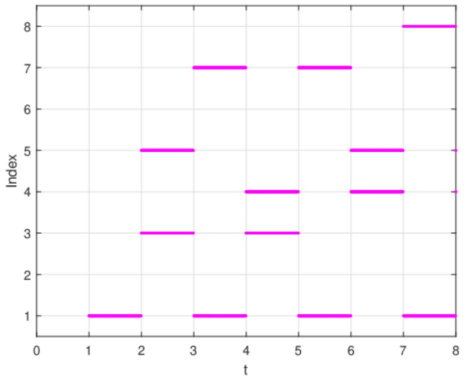
(e) The instantaneous convergence speed of the system: $\dot{\mathbb{X}}(t)$

Figure 3: The factors are $\mathcal{G},\Sigma=2,\tau=1,x(t_0)=(3,4,5,6,7,8,9,10)^\top$. (a) shows trajectories of agents and the target, (b) shows the trajectories of agents' deviations, (c) shows controlled agents that selected by the sparse optimal strategy. (d) shows the deviation of the system, (e) shows the instantaneous convergence speed, (d) and (e) form a comparison of $\alpha_{sq}(t)\equiv(1,1,0,0,0,0,0,0),\alpha_{tr}(t)\equiv\alpha^*(t_0)$ and the sampling solution, where the black square, blue upper triangle and red star correspond to $\alpha_{sq}(t)$, $\alpha_{tr}(t)$ and the sampling solution respectively. The settling times of the three cases are 31.957,27.174 and 14.061.

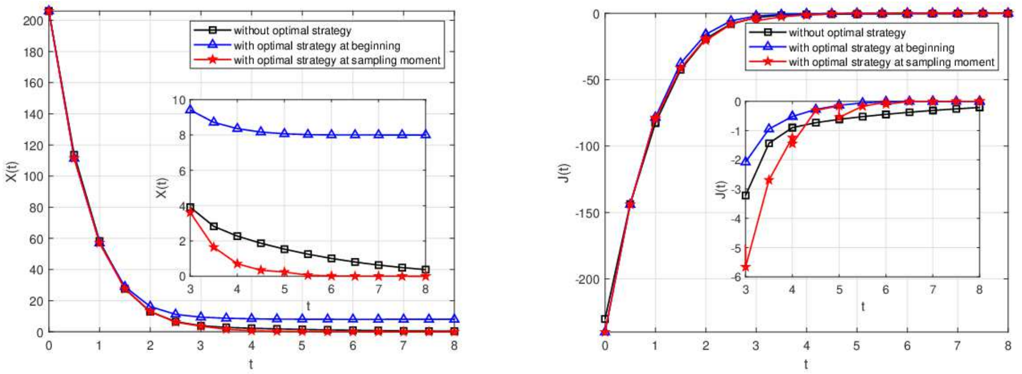


(a) Agents' state: $x(t)$

(b) Agents' deviations: $y(t)$



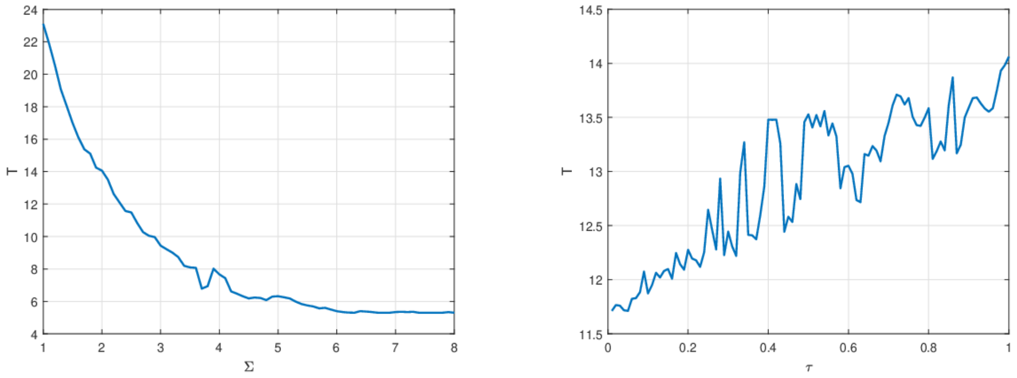
(c) Controlled agents



(d) The deviation of the system: $\mathbb{X}(t)$

(e) The instantaneous convergence speed of the system: $\dot{\mathbb{X}}(t)$

Figure 4: The factors are $\mathcal{G},\Sigma=2,\tau=1,x(t_0)=(4,-2,5,-3,6,-4,8,-6)^\top$. (a) shows trajectories of agents and the target, (b) shows the trajectories of agents' deviations,, (c) shows controlled agents that selected by the sparse optimal strategy, (d) shows the deviation of the system, (e) shows the instantaneous convergence speed, (d) and (e) form a comparison of $\alpha_{sq}(t),\alpha_{tr}(t)$ and the sampling solution, where the black square, blue upper triangle and red star correspond to $\alpha_{sq}(t),\alpha_{tr}(t)$ and the sampling solution respectively. The settling times of the three cases are $13.518,\infty$ and 7.020.



(a) The relation between T and Σ

(b) The relation between T and τ

Figure 5: The factors are $\mathcal{G}, x(t_0) = (3,4,5,6,7,8,9,10)^\top$. (a) shows the settling time T roughly decrease as the control constraint $\Sigma = 1{:}0.1{:}N$, (b) shows the settling time T as a whole has an upward trend when the sampling time $\tau = 0.01{:}0.01{:}1$.

6 Conclusions

This paper studies collective migration modeled by a first-order multi-agent system involving alignment and tracking forces. Each agent's dynamics is controlled by a parameter α_i that establishes a trade-off between the two forces through a convex combination, which we think is closer to the real operating mechanism of agents. We establish an optimization problem with the maximum convergence speed as the performance index, and use the concept of sparse control to select an optimal solution that is named sparse optimal strategy. The sparse optimal strategy is highly dependent on the interaction network and initial value, and is suitable for both asymptotic and finite-time consensus tracking. The sampling solution is implemented to avoid chattering phenomena that is caused by the heavily irregular of sparse optimal strategy in time. We show that the sampling solution achieves asymptotic or finite-time convergence for any sampling time, and converges to the optimal solution as the sampling time tends to zero. Compared with traditional protocol such as pinning control, the protocol with trade-off between the two forces fits the realistic constraint that agents' capability is limited, and demonstrates the intelligence of agents. In the future work, we will improve the sampling solution with periodic sampling into the event-triggered sampling solution to make our control scheme more general and flexible in practice.

Appendix A.

The following results are useful in this paper.

Definition A.1. *The matrix $B = [b_{ij}] \in \mathbb{R}^{p\times p}$ is said a irreducibly diagonally dominant matrix if*

(i) B is irreducible;

(ii) *B is diagonally dominant, that is,* $|b_{ii}| \geq \sum_{j=1,j\neq i}^{p} |b_{ij}|$, $i=1,\ldots,p$ *and there exists at least one value of i such that* $|b_{ii}| > \sum_{j=1,j\neq i}^{p} |b_{ij}|$.

Lemma A.1 (Taussky's Theorem, [21]). *Let* $B=[b_{ij}]\in\mathbb{R}^{p\times p}$ *be a irreducibly diagonally dominant matrix. If B has only real eigenvalues, and all main diagonal entries of B are strictly positive, then all eigenvalues of B are strictly positive.*

Lemma A.2 ([30]). *Assume* $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ *is a weighted undirected graph of order N whose Laplacian matrix is* $\mathcal{L}_{\mathcal{A}}$. *Then for any* $x\in\mathbb{R}^N$,

$$2x^\top \mathcal{L}_{\mathcal{A}} x = \sum_{i,j=1}^{N} a_{ij}(x_j - x_i)^2, \quad \min_{x\perp \mathbf{1}_N, x\neq 0} \frac{x^\top \mathcal{L}_{\mathcal{A}} x}{x^\top x} = \lambda_2(\mathcal{L}_{\mathcal{A}}),$$

where $\lambda_2(\mathcal{L}_{\mathcal{A}})$ *is the second smallest eigenvalue of* $\mathcal{L}_{\mathcal{A}}$.

Lemma A.3 ([22]). *Suppose the function* $\phi:\mathbb{R}^2\to\mathbb{R}$ *satisfies* $\phi(x_i,x_j) = -\phi(x_j,x_i)$ *for any* $x\in\mathbb{R}^N$. *Then for any weighted undirected graph* $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ *of order N and a group of numbers* $s_1,\cdots,s_N$,

$$\sum_{i,j=1}^{N} a_{ij} s_i \phi(x_j,x_i) = -\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(s_j - s_i)\phi(x_j,x_i).$$

Lemma A.4 ([18]). *Let* $z\in\mathbb{R}^n$ *and* $0<r<s$. *Then the following norm equivalence property holds:*

$$\left(\sum_{i=1}^{n}|z_i|^s\right)^{\frac{1}{s}} \leq \left(\sum_{i=1}^{n}|z_i|^r\right)^{\frac{1}{r}}, \quad \left(\frac{1}{n}\sum_{i=1}^{n}|z_i|^s\right)^{\frac{1}{s}} \geq \left(\frac{1}{n}\sum_{i=1}^{n}|z_i|^r\right)^{\frac{1}{r}}.$$

Lemma A.5 (Young's Inequality, [18]). *Let p,q be positive real numbers satisfying* $1/p+1/q=1$. *Then for any* $a,b>0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and equality holds if and only if $a^p = b^q$.

Lemma A.6 ([37]). *Suppose there exists a continuous function* $V(x): D\to\mathbb{R}$ *defined on a neighbourhood* $D\subseteq\mathbb{R}^N$ *of the origin such that*

$$V(0)=0 \quad \text{and} \quad V(x)>0, \quad \dot{V}(x)<0, \quad x\in D\backslash\{\mathbf{0}_N\},$$

then the origin is asymptotically stable, i.e. $\lim_{t\to\infty}\|x(t)\|=0$.

Lemma A.7 ([1]). *Suppose there exists a continuous function* $V(x): D\to\mathbb{R}$ *defined on a neighbourhood* $D\subseteq\mathbb{R}^N$ *of the origin, real numbers* $c>0$ *and* $p\in(0,1)$ *such that*

$$V(0)=0 \quad \text{and} \quad V(x)>0, \quad \dot{V}(x)\leq -cV^p(x), \quad x\in D\backslash\{\mathbf{0}_N\},$$

then $V(x)\equiv 0$, $t\geq T$ *and the settling time T can be bounded from above as*

$$T\leq \frac{V^{1-p}(x_0)}{c(1-p)}.$$

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