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Multi-Camera Robot-World Hand-Eye Calibration by Solving Multi-Unit Dual Quaternion Equations

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Abstract. In this paper, we study the solution to the multi-camera robot-world hand-eye calibration problem by employing dual quaternions to represent transformation matrices. This approach yields a system of multi-unit dual quaternion equations of the form $\mathtt{a}_d \check{z}_d = (-1)^{\sigma_d} \odot \check{x}\mathtt{b}, d = 1,\ldots,p$. We propose a novel formulation for the subspace constrained least squares solution to $\mathtt{a}_d \check{z}_d = \check{x}\mathtt{b}$ to avoid discussing the unknown signs $(-1)^{\sigma_d}$ and derive the closed-form expression for the solution. We prove that when the transformation matrix equation associated with the multi-camera robot-world hand-eye calibration admits a solution, the corresponding unit dual quaternion obtained from this matrix equation constitutes a subspace constrained least squares solution for the system of multi-unit dual quaternion vector equations. We present an algorithm for multi-camera robot-world hand-eye calibration, using the derived closed-form subspace constrained least squares solution to the multi-unit dual quaternion equations. We introduce a correction strategy to handle real-world data scenarios where the basic assumption may not hold. Experimental results demonstrate that the proposed subspace constrained least squares solutions exhibit competitive performance compared to state-of-the-art methods in multi-camera robot-world hand-eye calibration.

AMS subject classifications: 68W05, 90C26, 16W55

Key words: Multi-camera robot-world hand-eye calibration, dual quaternion, multi-unit dual quaternion vector equations, subspace constrained least squares solution.

1 Introduction

The main aim of this paper is to study the robot-world hand-eye (multi-camera) calibration problem [20,22] and to demonstrate that it can be formulated as follows:

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$$A_d^{(i)} X = Z_d B^{(i)}, \quad i=1,\ldots,n, \quad d=1,\ldots,p, \tag{1.1}$$

where $A_d^{(i)} \in \mathbb{R}^{4\times 4}$ represents the transformation from the world coordinate frame (calibration board) to the d-th camera coordinate frame, which can be calculated using a camera calibration method [24]; and $B^{(i)} \in \mathbb{R}^{4\times 4}$ denotes the transformation from the robot-base frame to the hand coordinate frame, which can be obtain from the forward kinematics of the robot; n denotes the number of distinct robot poses and p denotes the number of cameras. X and Z_d ($d=1,\ldots,p$) denote unknown transformation from the robot-base coordinate frame to the world coordinate frame and the transformation from the hand coordinate frame to the d-th camera coordinate frame, respectively. Multi-camera calibration is the process of determining the transformation matrices X and $\{Z_d\}_{d=1}^p$ by solving Eq. (1.1) constructed from multiple robot poses, where the transformation matrix takes the form of $\left[\begin{smallmatrix} R & t \\ 0^\top & 1 \end{smallmatrix}\right]$, $R \in \mathbb{R}^{3\times 3}$ and $t \in \mathbb{R}^{3\times 1}$ denote the rotation and translation part of a rigid transformation, respectively. Fig. 1 displays an example for a robot manipulator with three cameras attached to its end effector. In [20], multi-cost functions are introduced based on different metrics and Levenberg-Marquardt method [13] was employed to address the corresponding nonlinear least squares. Wang *et al.* [22] employed the Kronecker product to formulate the transformation equation, yielding a closed-form solution for the vectorized rotation matrices and translation components. Subsequently, normalization is applied to enforce orthogonality of the rotation components.

Unit dual quaternions [3], which have been used in computer graphics [7, 17] and robotics [2, 5, 10, 21, 23, 25], offer a compact and computationally efficient representation for rigid transformations (using 8 float values compared to 12 in transformation matrices) and have been shown to be the most effective method represent rotation and translation [5, 8]. Zhu and Ng [25] introduced a closed-form solution to Eq. (1.1) for the case of $p=1$, employing the unit dual quaternion representation of the transformation matrix.

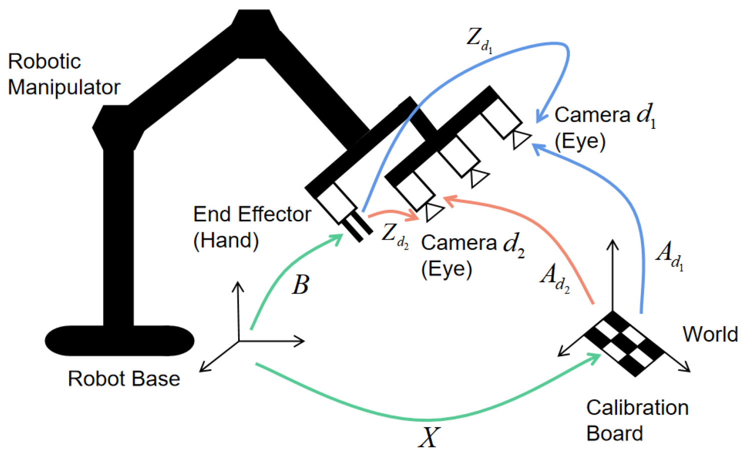

Figure 1: Rigid transformation in a robot manipulator with three cameras attached to its end effector.

This specific instance of the equation corresponds to the robot-world hand-eye calibration problem. The closed-form solution is characterized by the singular value decomposition (SVD) of a 4-by-4 matrix, thereby exhibiting exceptionally low computational complexity. In this paper, we study the closed-form solution to Eq. (1.1) for $p\geq 2$ by using the unit dual quaternion representation of the transformation matrix. Let $\mathtt{b}$ and $\{\mathtt{a}_d\}_{d=1}^p$ be n-dimensional dual quaternion vectors, with each element of $\mathtt{b}$ and $\mathtt{a}_d$ being a unit dual quaternion corresponding to $B^{(i)}$ and $A_d^{(i)}$, $\check{x}$ and $\{\check{z}_d\}_{d=1}^p$ be unknown unit dual quaternions corresponding to X and $\{Z_d\}_{d=1}^p$. Eq. (1.1) can be expressed as multi-unit dual quaternion vector equations

$$\mathtt{a}_d\check{z}_d=(-1)^{\sigma_d}\odot\check{x}\mathtt{b},\quad d=1,\dots,p, \tag{1.2}$$

where $(-1)^{\sigma_d}\in\mathbb{R}^n$ is an unknown vector with elements equal ± 1 (that is, $\sigma_d\in\mathbb{R}^n$ is an unknown vector with each element σ_d^i equals to 0 or 1), and $\odot$ denotes element-wise multiplication. The unknown sign $\{\{(-1)^{\sigma_d^i}\}_{i=1}^n\}_{d=1}^p$, comes from the antipodal property of classical quaternions (that is, both unit dual quaternions $\check{q}$ and $-\check{q}$ represent the same rigid transformation), results in a total of $np(np-1)/2$ possibilities for Eq. (1.2), which introduces significant difficulties in solving the transformation matrix equation using dual quaternions because the value of $\{\{(-1)^{\sigma_d^i}\}_{d=1}^p\}_{i=1}^n$ is unknown before obtaining matrices X and $\{Z_d\}_{d=1}^p$ and Eq. (1.2) is nonlinear. To avoid the discussion of the unknown sign $\{\{(-1)^{\sigma_d^i}\}_{i=1}^n\}_{d=1}^p$, in this paper we address the multi-unit dual quaternion vector equations

$$\mathtt{a}_d\check{z}_d=\check{x}\mathtt{b},\quad d=1,\dots,p \tag{1.3}$$

directly. We define the "subspace constrained least squares solution" of Eq. (1.3) by studying properties satisfied by the solutions of the matrix equation (1.1), where the "subspace" corresponds to the singular vector space of some relevant matrix. We provide the closed-form expression to our defined subspace constrained least squares solution by analyzing the optimality conditions of the corresponding problem and apply it on the multi-camera robot-world hand-eye calibration problem. Numerical comparisons with state-of-the-art algorithms demonstrate that the subspace constrained least squares solution effectively computes the transformation matrices X and $\{Z_d\}_{d=1}^p$ in Eq. (1.1). Without loss of generality, we assume $p\geq 2$ in this paper. It is worth mentioning that the subspace constrained least squares solution we proposed in this paper can be regarded as a definition of the solution to the unit dual quaternion vector equations (1.3) within the framework of rigid body transformation.

The remainder of this paper is organized as follows. In Section 2, we present a brief review of dual quaternions and explore the relationship between unit dual quaternions and rigid transformations. The subspace constrained least squares solution for Eq. (1.1) was discussed in Section 3, which includes the closed-form solution of multi-unit dual quaternion equations as well as the algorithm to solve the multi-camera robot-world hand-eye calibration. Numerical experiments are given in Section 4 and some concluding remarks are presented in Section 5.

2 Mathematical settings

Dual quaternions ($\mathbb{DQ}$) are a combination of the dual number ($\mathbb{D}$) theory [4, 19] and quaternions ($\mathbb{Q}$) [1], providing a novel mathematical framework with a unique set of properties. In recent years, the mathematical properties of dual quaternions have attracted widespread attention, including the properties of dual quaternion vectors [16], the singular values of dual quaternion matrices [11], the eigenvalues of dual quaternion Hermitian matrices [12], standard dual quaternion optimization [16], and the unit dual quaternion vector equations [25].

A quaternion $q\in\mathbb{Q}$ is given by $q\triangleq(q_0,q_1,q_2,q_3)=q_0+q_1\mathtt{i}+q_2\mathtt{j}+q_3\mathtt{k}$, where $\mathtt{i},\mathtt{j}$ and $\mathtt{k}$ are three imaginary units of quaternions that satisfy

$$\mathtt{i}^2=\mathtt{j}^2=\mathtt{k}^2=\mathtt{ijk}=-1,\quad \mathtt{ij}=-\mathtt{ji}=\mathtt{k},\quad \mathtt{jk}=-\mathtt{kj}=\mathtt{i},\quad \mathtt{ki}=-\mathtt{ik}=\mathtt{j}.$$

Denote $\overrightarrow{q}'=(q_1,q_2,q_3)$, q is also written as $q=(q_0,\overrightarrow{q}')$, where q_0 and $\overrightarrow{q}'$ are named as the scalar and vector parts of q, respectively. q is called a unit quaternion if $|q|^2=qq^*=1$, where $q^*=q_0-q_1\mathtt{i}-q_2\mathtt{j}-q_3\mathtt{k}$ is the conjugate of q. We use $\vec{q}$ to denote the column vector with the entries (q_0,q_1,q_2,q_3). The scalar part of $q\in\mathbb{Q}$ is also indicated as $\mathrm{Sc}(q)$, where $\mathrm{Sc}(q)=(q+q^*)/2=q_0$. Let $\boldsymbol{x}=(x_1,\dots,x_n)^\top$ be a vector of quaternions of dimensions n (denoted $\mathbb{Q}^n$), that is, $x_i\in\mathbb{Q}$, $i=1,\dots,n$. The 2-norm [16] of $\boldsymbol{x}$ is defined as

$$\|\boldsymbol{x}\|_2=\sqrt{\sum_{i=1}^n|x_i|^2}=\sqrt{\sum_{i=1}^n\|\vec{x}_i\|_2^2}.$$

The unit quaternion can be used to represent a rotation [9, 18]. Let $R=(r_{ij})\in\mathbb{R}^{3\times3}$ be the matrix representation of a rotation in the three-dimensional space. Then the corresponding unit quaternion is given by

$$q=q_0+q_1\mathtt{i}+q_2\mathtt{j}+q_3\mathtt{k}, \tag{2.1}$$

where

$$\begin{aligned} q_0&=\frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}, & q_1&=\frac{1}{4q_0}(r_{32}-r_{23}),\\ q_2&=\frac{1}{4q_0}(r_{13}-r_{31}), & q_3&=\frac{1}{4q_0}(r_{21}-r_{12}). \end{aligned}$$

Conversely, let $q=q_0+q_1\mathtt{i}+q_2\mathtt{j}+q_3\mathtt{k}$ be a unit quaternion. The corresponding rotation matrix R is given by

$$R=\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2)\\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1)\\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}. \tag{2.2}$$

The rotation operation $y=Rx$, where $x=(x_1,x_2,x_3)^\top$ and $y=(y_1,y_2,y_3)^\top$, can be equitably represented by the dual quaternion operation $(0,y_1,y_2,y_3)=q(0,x_1,x_2,x_3)q^*$. Note that q

and $-q$ both correspond to R, where q is calculated using Eq. (2.1). The non-uniqueness of the mapping from rotation matrix to unit quaternion introduces sign ambiguity in representing rotation via quaternion. Taking the rotation matrix equation $R_A R_X = R_Y R_B$ as an example. Let a,x,y, and b be unit quaternions corresponding to R_A, R_X, R_Y, and R_B calculated according to Eq. (2.1). Then we have $ax=(-1)^\sigma yb$, where $\sigma=0$ or 1 is ambiguous prior to complete knowledge of R_A, R_X, R_Y, and R_B.

We restate some properties of the quaternions that will be used in the following formulations. The statements of Proposition 2.1 can be found in the classical literature on quaternions.

Proposition 2.1. *For any $p=(p_0,p_1,p_2,p_3)\in\mathbb{Q}, q=(q_0,q_1,q_2,q_3)\in\mathbb{Q}$, the following statements hold:*

(i) $|p|^2=\|\vec{p}\|_2^2$;

(ii) $\mathrm{Sc}(p^*q)=\mathrm{Sc}(pq^*)=\mathrm{Sc}(q^*p)=\mathrm{Sc}(qp^*)=\vec{p}^\top\vec{q}$;

(iii) $\mathrm{Sc}(\alpha p+\beta q)=\alpha\mathrm{Sc}(p)+\beta\mathrm{Sc}(q)$ *for any* $\alpha,\beta\in\mathbb{R}$.

Define

$$\mathbf{M}(q)=\begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}, \quad \mathbf{W}(q)=\begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix}.$$

We have

(iv) $\mathbf{M}(q)^\top\mathbf{M}(q)=\mathbf{M}(q)\mathbf{M}(q)^\top=\mathbf{W}(q)^\top\mathbf{W}(q)=\mathbf{W}(q)\mathbf{W}(q)^\top=\|\vec{q}\|_2^2 I_4$;

(v) $qp=\mathbf{M}(q)\vec{p}=\mathbf{W}(p)\vec{q}$.

$\mathbf{M}(q)$ and $\mathbf{W}(q)$ are known as the matrix representation of q. The following two lemmas hold.

Lemma 2.1 ([25], Lemma 1)**.** *For any $p,q\in\mathbb{Q}$, if $\vec{p}^\top\vec{q}=0$, then*

$$\mathbf{M}(p)^\top\mathbf{M}(q)+\mathbf{M}(q)^\top\mathbf{M}(p)=\mathbf{W}(p)^\top\mathbf{W}(q)+\mathbf{W}(q)^\top\mathbf{W}(p)=\mathbf{0}.$$

Lemma 2.2 ([25], Lemma 2)**.** *For any unit quaternions a,b,c, and d,*

$$\mathbf{W}(c)^\top\mathbf{M}(a)\mathbf{M}(b)^\top\mathbf{W}(d)+\mathbf{W}(d)^\top\mathbf{M}(b)\mathbf{M}(a)^\top\mathbf{W}(c)=2I_4$$

if and only if $a=b$ and $c=d$ or $a=-b$ and $c=-d$.

2.1 Dual quaternion operations and properties

A dual quaternion $\check{z}\in\mathbb{DQ}$ is given by $\check{z}=z_{st}+z_{\mathcal{I}}\epsilon\in\mathbb{DQ}$, where $z_{st},z_{\mathcal{I}}\in\mathbb{Q}$ are the standard part (also named as the real part) and the infinitesimal part (also named as the dual part) of $\check{z}$, respectively, and ϵ is the infinitesimal unit satisfying $\epsilon^2=0$. The magnitude of $\check{z}\in\mathbb{DQ}$ is a dual number, which is given by

$$|\check{z}|=\begin{cases}|z_{st}|+\dfrac{\mathrm{Sc}(z_{st}^*z_{\mathcal{I}})}{2|z_{st}|}\epsilon, & \text{if} \quad z_{st}\neq 0,\\ |z_{\mathcal{I}}|\epsilon, & \text{otherwise.}\end{cases} \tag{2.3}$$

$\check{z}\in\mathbb{DQ}$ is called a unit dual quaternion if $|\check{z}|=1$. From (2.3), for any $\check{z}\in\mathbb{DQ},|\check{z}|=1$ if and only if $z_{st}^*z_{st}=1$ and $z_{st}^*z_{\mathcal{I}}+z_{\mathcal{I}}^*z_{st}=0$, or equally, $|\check{z}|=1$ if and only if $\|\overrightarrow{z_{st}}\|=1$ and $\overrightarrow{z_{st}}^{\top}\overrightarrow{z_{\mathcal{I}}}=0$. Let $\boldsymbol{a}=(\check{a}^{(1)},\ldots,\check{a}^{(n)})^{\top}$ be a n-dimensional dual quaternion vector (denoted as $\mathbb{DQ}^n$), that is, $\check{a}^{(i)}=a_{st}^{(i)}+a_{\mathcal{I}}^{(i)}\epsilon\in\mathbb{DQ}$, $i=1,\ldots,n$. The 2-norm [16] of $\boldsymbol{a}$ is a dual number, which is given by

$$\|\boldsymbol{a}\|_2=\begin{cases}\|\boldsymbol{a}_{st}\|_2+\dfrac{\sum_{i=1}^n\mathrm{Sc}\big(\big(a_{st}^{(i)}\big)^*a_{\mathcal{I}}^{(i)}\big)}{2\|\boldsymbol{a}_{st}\|_2}\epsilon, & \text{if} \quad \boldsymbol{a}_{st}\neq 0,\\ \|\boldsymbol{a}_{\mathcal{I}}\|_2\epsilon, & \text{otherwise.}\end{cases} \tag{2.4}$$

According to (2.4), $\|\boldsymbol{a}\|_2^2=0$ if and only if $\boldsymbol{a}_{st}=0$.

The unit dual quaternion can be used to represent rigid transformation [5]. Let (R,t) denote the rotation matrix and translation vector of a rigid transformation. Then the corresponding unit dual quaternion is given by

$$\check{q}=q_{st}+q_{\mathcal{I}}\epsilon=q_{st}+\frac{1}{2}(0,\overrightarrow{t}')q_{st}\epsilon,$$

where q_{st} is the unit quaternion with respect to R obtained via (2.1). Let $\check{a}$ and $\check{b}$ be unit dual quaternions corresponding to lines ℓ_a and ℓ_b, where ℓ_a is transformed with (R,t) into ℓ_b. Then we have $\check{a}=\check{q}\check{b}\check{q}^*$. Given a unit dual quaternion $q_{st}+q_{\mathcal{I}}\epsilon$, the associated rigid transformation (R,t) is derived as follows: R is obtained from q via (2.2) and $(0,\overrightarrow{t}')=2q_{\mathcal{I}}q_{st}^*$.

The inverse of a transformation matrix remains a transformation matrix. Therefore, Eq. (1.1) is equivalent to

$$\big(A_d^{(i)}\big)^{-1}Z_d=X(B^{(i)})^{-1}, \quad i=1,\ldots,n, \quad d=1,\ldots,p.$$

Denote $\hat{A}_d^{(i)}\triangleq(A_d^{(i)})^{-1}$ and $\hat{B}^{(i)}=(B^{(i)})^{-1},i=1,\ldots,n,d=1,\ldots,p$. Problem (1.1) can be rewritten as

$$\hat{A}_d^{(i)}Z_d=X\hat{B}^{(i)}, \quad i=1,\ldots,n, \quad d=1,\ldots,p. \tag{2.5}$$

Let

$$\check{x} = x_{st} + x_{\mathcal{I}}\epsilon, \qquad \check{z}_d = z_{st}^d + z_{\mathcal{I}}^d\epsilon,$$
$$\check{a}_d^{(i)} = (a_d^{(i)})_{st} + (a_d^{(i)})_{\mathcal{I}}\epsilon, \quad \check{b}^{(i)} = b_{st}^{(i)} + b_{\mathcal{I}}^{(i)}\epsilon$$

be unit dual quaternions corresponding to $X, Z_d, \hat{A}_d^{(i)}$, and $\hat{B}^{(i)}, i=1,\dots,n, d=1,\dots,p$, respectively. By using the relationship between unit dual quaternions and line transformations, we have

$$\check{a}_d^{(i)}\check{z}_d = (-1)^{\sigma_d^i}\check{x}\check{b}^{(i)}, \quad d=1,\dots,p, \quad i=1,\dots,n,$$

where $\{\{(-1)^{\sigma_d^i}\}_{d=1}^p\}_{i=1}^n$, $\sigma_d^i = 0$ or 1 refers to one of the possible signs, which comes from the non-uniqueness when mapping a rotation matrix to dual quaternion. Denote $\mathtt{a}_d = (\check{a}_d^{(1)}, \check{a}_d^{(2)}, \dots, \check{a}_d^{(n)})^\top, d = 1,\dots,p$, $\mathtt{b} = (\check{b}^{(1)}, \check{b}^{(2)}, \dots, \check{b}^{(n)})^\top$, $\mathtt{b}_{st} = (b_{st}^{(1)}, \dots, b_{st}^{(n)})^\top$, and $\sigma_d = (\sigma_d^1, \dots, \sigma_d^n)^\top$. The above equation is exactly Eq. (1.2). Denote

$$A_d^{(i)} = \begin{bmatrix} R_{A_d^{(i)}} & t_{A_d^{(i)}} \\ 0^\top & 1 \end{bmatrix}, \quad B^{(i)} = \begin{bmatrix} R_{B^{(i)}} & t_{B^{(i)}} \\ 0^\top & 1 \end{bmatrix}, \quad i=1,\dots,n, \quad d=1,\dots,p.$$

We make the following assumption on Eq. (1.1).

Assumption 2.1. Suppose there exist rotation matrices $\{R_{Z_d}\}_{d=1}^p$ and R_X such that

$$R_{A_d^{(i)}}^\top R_{Z_d} = R_X R_{B^{(i)}}^\top, \quad i=1,\dots,n, \quad d=1,\dots,p.$$

Assumption 2.1 is equivalent to assume that there exist unit quaternions $\{z_{st}^d\}_{d=1}^p$ and x_{st} such that

$$(a_d^{(i)})_{st} z_{st}^d = (-1)^{\sigma_d^i} x_{st} b_{st}^{(i)}, \quad i=1,\dots,n \ (\geq 2), \quad d=1,\dots,p \ (\geq 2)$$

for some $\sigma_d^i = 0$ or 1. Define

$$M_d^1 := \begin{bmatrix} M\big((a_d^{(1)})_{st}\big) \\ \vdots \\ M\big((a_d^{(n)})_{st}\big) \end{bmatrix}, \quad W_1 := \begin{bmatrix} W\big(b_{st}^{(1)}\big) \\ \vdots \\ W\big(b_{st}^{(n)}\big) \end{bmatrix}, \quad K_{11}^d := \big(M_d^1\big)^\top W_1, \quad d=1,\dots,p.$$

The following lemma demonstrate that Assumption 2.1 can be verified by checking whether matrices $\{K_{11}^d\}_{d=1}^p$ share at least one common right singular vector.

Lemma 2.3. *Suppose Assumption* 2.1 *holds. There is at least a common right singular vector to* $\{K_{11}^d\}_{d=1}^p$.

Proof. Under Assumption 2.1, there exist unit quaternions $\check{x}$ and $\check{z}_d, d=1,\ldots,p$ such that

$$\big(a_d^{(i)}\big)_{st} z_{st}^d = (-1)^{\sigma_d^i} x_{st} b_{st}^{(i)}, \quad i=1,\ldots,n\ (\geq 2), \quad d=1,\ldots,p\ (\geq 2) \tag{2.6}$$

for some $\sigma_d^i = 0$ or 1.

From (2.6), $\overrightarrow{x_{st}}$ and $\overrightarrow{z_{st}^d}, d=1,\ldots,p$, satisfy

$$M\big(\big(a_d^{(i)}\big)_{st}\big)\overrightarrow{z_{st}^d} = (-1)^{\sigma_d^i} W\big(b_{st}^{(i)}\big)\overrightarrow{x_{st}}, \quad i=1,\ldots,n, \quad d=1,\ldots,p,$$

which implies that

$$\begin{aligned} K_{11}^d \overrightarrow{x_{st}} &= \sum_{i=1}^n M\big(\big(a_d^{(i)}\big)_{st}\big)^\top W\big(b_{st}^{(i)}\big)\overrightarrow{x_{st}} = \left(n - 2\sum_{i=1}^n \sigma_d^i\right)\overrightarrow{z_{st}^d} \\ &= \operatorname{sign}\left(n - 2\sum_{i=1}^n \sigma_d^i\right) \cdot \left(n - 2\sum_{i=1}^n \sigma_d^i\right) \cdot \operatorname{sign}\left(n - 2\sum_{i=1}^n \sigma_d^i\right)\overrightarrow{z_{st}^d}. \end{aligned}$$

Hence, $\overrightarrow{x_{st}}$ is a common right singular vector to $\{K_{11}^d\}_{d=1}^p$. □

3 The solution method

In this section, we present the closed-from expression of the subspace constrained least squares solution to the multi-unit dual quaternion equations (1.3), and then apply this solution on multi-camera robot-world hand-eye calibration.

3.1 The subspace constrained least squares solution

Define the dual quaternion vector $g(\check{x},\mathtt{z}) := (g_1(\check{x},\check{z}_1),\ldots,g_p(\check{x},\check{z}_p))^\top$, where

$$g_d(\check{x},\check{z}_d) = \mathtt{a}_d \check{z}_d - \check{x}\mathtt{b}, \quad d=1,\ldots,p.$$

We have

$$\begin{aligned} &\|g(\check{x},\mathtt{z})\|_2^2 = \sum_{d=1}^p \|g_d(\check{x},\check{z}_d)\|_2^2, \\ &g_{st}(\check{x},\mathtt{z}) = \big((g_1)_{st}(\check{x},\check{z}_1)^\top,\ldots,(g_p)_{st}(\check{x},\check{z}_p)\big)^\top, \\ &g_{\mathcal{I}}(\check{x},\mathtt{z}) = \big((g_1)_{\mathcal{I}}(\check{x},\check{z}_1)^\top,\ldots,(g_p)_{\mathcal{I}}(\check{x},\check{z}_p)\big)^\top, \end{aligned}$$

and

$$\left.\begin{aligned} &\big(g_d^{(i)}\big)_{st}(\check{x},\check{z}_d) = \big(a_d^{(i)}\big)_{st} z_{st}^d - x_{st} b_{st}^{(i)}, \\ &\big(g_d^{(i)}\big)_{\mathcal{I}}(\check{x},\check{z}_d) = \big(\check{a}_d^{(i)}\big)_{st} z_{\mathcal{I}}^d + \big(\check{a}_d^{(i)}\big)_{\mathcal{I}} z_{st}^d - x_{st} b_{\mathcal{I}}^{(i)} - x_{\mathcal{I}} b_{st}^{(i)}, \end{aligned}\right. \quad i=1,\ldots,n.$$

Define

$$M_d^2 := \begin{bmatrix} M\big((a_d^{(1)})_{\mathcal{I}}\big) \\ \vdots \\ M\big((a_d^{(n)})_{\mathcal{I}}\big) \end{bmatrix}, \quad W_2 := \begin{bmatrix} W\big(b_{\mathcal{I}}^{(1)}\big) \\ \vdots \\ W\big(b_{\mathcal{I}}^{(n)}\big) \end{bmatrix}, \quad K_{12}^d := \big(M_d^1\big)^\top W_2, \quad K_{21}^d := \big(M_d^2\big)^\top W_1, \quad d=1,\dots,p.$$

The following statement holds.

Lemma 3.1. *Suppose Assumption* 2.1 *holds. Let* $\check{x}, \{\check{z}_d\}_{d=1}^p, \{\{\check{a}_d^{(i)}\}_{i=1}^n\}_{d=1}^p$, *and* $\{\check{b}^{(i)}\}_{i=1}^n$ *be unit dual quaternions. Then*

(i)

$$\|(g_d)_{st}(\check{x},\check{z}_d)\|_2^2 = 2n - 2\overrightarrow{z_{st}^d}^\top K_{11}^d \overrightarrow{x_{st}}, \quad d=1,\dots,p. \tag{3.1}$$

(ii) $\sigma_{\max}(K_{11}^d) \le n$ *for all* $d=1,\dots,p$, *where* $\sigma_{\max}(K_{11}^d)$ *denotes the maximal singular value of* K_{11}^d.

(iii) When $\sigma_{\max}(K_{11}^d) = n$ *for all* $d=1,\dots,p$, *there is at least a common right singular vector to* $\{K_{11}^d\}_{d=1}^p$ *corresponding to singular value* n.

(iv) Define

$$\Omega_{st} = \left\{ (v,u_1,\dots,u_p) \in \mathbb{Q}^{p+1} \;\middle|\; \begin{array}{l} \vec{v} \text{ is a unit common right singular vector to } \{K_{11}^d\}_{d=1}^p \\ \text{corresponding to singular value } n;\ (\vec{v},\vec{u}_d) \text{ is a unit} \\ \text{singular vector pair of } K_{11}^d,\ d=1,\dots,p \end{array} \right\}.$$

$\Omega_{st} \neq \emptyset$ *and* $\|g_{st}(\check{x},\mathbf{z})\|_2^2 = 0$ *if and only if* $\sigma_{\max}(K_{11}^d) = n$ *for all* $d=1,\dots,p$ *and* $(x_{st}, z_{st}^1, \dots, z_{st}^p) \in \Omega_{st}$.

(v)

$$\begin{aligned} &\mathbf{Sc}\big(g_{st}^*(\check{x},\mathbf{z}) g_{\mathcal{I}}(\check{x},\mathbf{z})\big) \\ &= -\sum_{d=1}^p \Big(\overrightarrow{z_{st}^d}\Big)^\top \big(K_{12}^d + K_{21}^d\big)\overrightarrow{x_{st}} - \sum_{d=1}^p \Big[\Big(\overrightarrow{z_{st}^d}\Big)^\top K_{11}^d \overrightarrow{x_{\mathcal{I}}} + \overrightarrow{x_{st}}^\top \big(K_{11}^d\big)^\top \overrightarrow{z_{\mathcal{I}}^d}\Big]. \end{aligned} \tag{3.2}$$

(vi) Define

$$\Omega_{st}^j = \left\{ (v,u_1,\dots,u_p) \in \mathbb{Q}^{p+1} \;\middle|\; \begin{array}{l} \vec{v} \text{ is a unit common right singular vector to } \{K_{11}^d\}_{d=1}^p \\ \text{corresponding to the } j\text{-th mutually different singular} \\ \text{value; } (\vec{v},\vec{u}_d) \text{ is a unit singular vector pair of} \\ K_{11}^d,\ d=1,\dots,p \end{array} \right\},$$

where $j=1,\ldots,l$, l is the maximal number of distinct singular values of $\{K_{11}^d\}_{d=1}^p$ associated with its common right singular vectors. We have $1\le l\le 4$. If $(x_{st},z_{st}^1,\ldots,z_{st}^p)\in\Omega_{st}^j$ for any $j=1,\ldots,l$, then

$$\mathbf{Sc}\big(g_{st}^*(\check{x},\mathbf{z})g_{\mathcal{I}}(\check{x},\mathbf{z})\big)=-\sum_{d=1}^p\left(\overrightarrow{z_{st}^d}\right)^\top\left(K_{12}^d+K_{21}^d\right)\overrightarrow{x_{st}}. \tag{3.3}$$

Proof. (i) Notice that $\{\{\check{a}_d^{(i)}\}_{i=1}^n\}_{d=1}^p$ and $\{\check{b}^{(i)}\}_{i=1}^n$ are unit dual quaternions, from Proposition 2.1, we have $(M_d^1)^\top M_d^1=W_1^\top W_1=nI_4$. The statement holds by noting that

$$\begin{aligned}\|(g_d)_{st}(\check{x},\check{z}_d)\|_2^2&=\sum_{i=1}^n\big|\big(g_d^{(i)}\big)_{st}(\check{x},\check{z}_d)\big|^2\\&=\sum_{i=1}^n\Big\|M\big(\big(a_d^{(i)}\big)_{st}\big)\overrightarrow{z_{st}^d}-W\big(b_{st}^{(i)}\big)\overrightarrow{x_{st}}\Big\|_2^2.\end{aligned}$$

Statement (ii) is valid in light of $\|(g_d)_{st}(\check{x},\check{z}_d)\|_2^2\ge 0$ and (3.1).

(iii) It follows from (3.1) that $\overrightarrow{z_{st}^d}^\top K_{11}^d\overrightarrow{x_{st}}=n$ for all $d=1,\ldots,p$, which implies that $\overrightarrow{x_{st}}$ is a common right singular vector corresponding to the singular value n for each $d=1,\ldots,p$.

Statement (iv) follows from statements (i) and (iii).

(v) Since $\check{x},\{\check{z}_d\}_{d=1}^p,\{\{\check{a}_d^{(i)}\}_{i=1}^n\}_{d=1}^p$, and $\{\check{b}^{(i)}\}_{i=1}^n$ are unit dual quaternions, we have

$$\left.\begin{aligned}&\mathrm{Sc}\Big(\big(\big(\check{a}_d^{(i)}\big)_{st}z_{st}^d\big)^*\big(\check{a}_d^{(i)}\big)_{st}z_{\mathcal{I}}^d\Big)=\mathrm{Sc}\Big(\big(b_{st}^{(i)}x_{st}\big)^*b_{st}^{(i)}x_{\mathcal{I}}\Big)=0,\\&\mathrm{Sc}\Big(\big(\big(\check{a}_d^{(i)}\big)_{st}z_{st}^d\big)^*\big(\check{a}_d^{(i)}\big)_{\mathcal{I}}z_{st}^d\Big)=\mathrm{Sc}\Big(\big(b_{st}^{(i)}x_{st}\big)^*b_{\mathcal{I}}^{(i)}x_{st}\Big)=0,\end{aligned}\right.\quad i=1,\ldots,n,\quad d=1,\ldots,p.$$

Therefore,

$$\begin{aligned}&\mathbf{Sc}\big(g_{st}^*(\check{x},\mathbf{z})g_{\mathcal{I}}(\check{x},\mathbf{z})\big)\\&=\sum_{d=1}^p\sum_{i=1}^n\mathbf{Sc}\big(\big((g_d)_{st}^{(i)}\big)^*(\check{x},\check{z}_d)(g_d)_{\mathcal{I}}^{(i)}(\check{x},\check{z}_d)\big)\\&=\sum_{d=1}^p\sum_{i=1}^n-\Big[\mathbf{Sc}\Big(\big(\big(\check{a}_d^{(i)}\big)_{st}z_{st}^d\big)^*\big(x_{st}b_I^{(i)}+x_Ib_{st}^{(i)}\big)\Big)\\&\qquad\qquad+\mathbf{Sc}\Big(\big(x_{st}b_{st}^{(i)}\big)^*\big(\big(\check{a}_d^{(i)}\big)_{st}z_I^d+\big(\check{a}_d^{(i)}\big)_Iz_{st}^d\big)\Big)\Big]\\&=-\sum_{d=1}^p\big(\overrightarrow{z_{st}^d}\big)^\top\big(K_{12}^d+K_{21}^d\big)\overrightarrow{x_{st}}-\sum_{d=1}^p\Big[\Big(\overrightarrow{z_{st}^d}\Big)^\top K_{11}^d\overrightarrow{x_{\mathcal{I}}}+\overrightarrow{x_{st}}^\top\big(K_{11}^d\big)^\top\overrightarrow{z_{\mathcal{I}}^d}\Big].\end{aligned}$$

(vi) From Lemma 2.3, we have $l\ge 1$.

Let $\tilde{\sigma}_d\ge 0$ be the singular value corresponding to $(\overrightarrow{x_{st}},\overrightarrow{z_{st}^d}),d=1,\ldots,p$. Then, according to Lemma 2.3, $\overrightarrow{x_{st}},\overrightarrow{z_{st}^d}$, and $\tilde{\sigma}_d,d=1,\ldots,p$, are well defined and $K_{11}^d\overrightarrow{x_{st}}=\tilde{\sigma}_d\overrightarrow{z_{st}^d}$ and $(\overrightarrow{z_{st}^d})^\top K_{11}^d=$

$\tilde{\sigma}_d \overrightarrow{x_{st}}^\top, d=1,\ldots,p$, which yields

$$\begin{aligned} &\left(\overrightarrow{z_{st}^d}\right)^\top K_{11}^d \overrightarrow{x_{\mathcal{I}}} = \tilde{\sigma}_d \overrightarrow{x_{st}}^\top \overrightarrow{x_{\mathcal{I}}} = 0, \\ &\overrightarrow{x_{st}}^\top \left(K_{11}^d\right)^\top \overrightarrow{z_{\mathcal{I}}^d} = \tilde{\sigma}_d \left(\overrightarrow{z_{st}^d}\right)^\top \overrightarrow{z_{\mathcal{I}}^d} = 0, \end{aligned} \qquad d=1,\ldots,p.$$

Combing with (3.2), the desired result holds. □

The following lemma characterizes the properties of the solution to Eq. (1.1), providing crucial theoretical justification for the subsequent definition of the subspace constrained least squares solution to the multi-unit dual quaternion vector equations (1.3).

Lemma 3.2. *Let $X^\star, \{Z_d^\star\}_{d=1}^p$ be the solution of Eq.* (1.1) *and $\check{x}^\star$ and $\{\check{z}_d^\star\}_{d=1}^p$ be unit dual quaternions corresponding to $X^\star$ and $\{Z_d^\star\}_{d=1}^p$, respectively. Then*

$$\check{a}_d^{(i)} \check{z}_d^\star = (-1)^{\sigma_d^i} \check{x}^\star \check{b}^{(i)}, \quad d=1,\ldots,p, \quad i=1,\ldots,n \tag{3.4}$$

for some $\{\{(-1)^{\sigma_d^i}\}_{d=1}^p\}_{i=1}^n$, where $\sigma_d^i = 0$ or 1. Denote

$$\begin{aligned} J_+^d &= \{i \in \{1,\ldots,n\} \,|\, \sigma_d^i = 0\}, \\ J_-^d &= \{i \in \{1,\ldots,n\} \,|\, \sigma_d^i = 1\}, \end{aligned} \qquad d=1,\ldots,p.$$

Then

$$\begin{aligned} &K_{11}^d \overrightarrow{x_{st}^\star} = (|J_+^d| - |J_-^d|) \overrightarrow{z_{st}^{d\star}}, \\ &\left(K_{11}^d\right)^\top \overrightarrow{z_{st}^{d\star}} = (|J_+^d| - |J_-^d|) \overrightarrow{x_{st}^\star}, \end{aligned} \qquad d=1,\ldots,p, \tag{3.5}$$

where $|J_+^d|$ and $|J_-^d|$ denote the cardinality of J_+^d and J_-^d, respectively. Moreover,

$$\mathrm{Sc}\left(g_{st}^*(\check{x}^\star, \mathbf{z}^\star) g_{\mathcal{I}}(\check{x}^\star, \mathbf{z}^\star)\right) = 0.$$

Proof. Notice that $X^\star, \{Z_d^\star\}_{d=1}^p$ is also the solution of Eq. (2.5). Eq. (3.4) comes from the relationship between unit dual quaternions and line transformations.

Define

$$\begin{aligned} K_+^d &:= \sum_{i \in J_+^d} \mathbf{M}\left(\left(a_d^{(1)}\right)_{st}\right)^\top \mathbf{W}\left(b_{st}^{(i)}\right), \\ K_-^d &:= -\sum_{i \in J_-^d} \mathbf{M}\left(\left(a_d^{(1)}\right)_{st}\right)^\top \mathbf{W}\left(b_{st}^{(i)}\right), \end{aligned} \qquad d=1,\ldots,p.$$

Notice that for each $d \in \{1,\ldots,p\}$, we have

$$\begin{aligned} M\left(\left(a_d^{(i)}\right)_{st}\right) \overrightarrow{z_{st}^{d\star}} &= W\left(b_{st}^{(i)}\right) \overrightarrow{x_{st}^\star}, \quad \forall i \in J_+^d, \\ -M\left(\left(a_d^{(i)}\right)_{st}\right) \overrightarrow{z_{st}^{d\star}} &= W\left(b_{st}^{(i)}\right) \overrightarrow{x_{st}^\star}, \quad \forall i \in J_-^d, \end{aligned} \tag{3.6}$$

which yields

$$K_+^d \overrightarrow{x_{st}^\star} = |J_+^d| \overrightarrow{z_{st}^{d\star}}, \quad (K_+^d)^\top \overrightarrow{z_{st}^{d\star}} = |J_+^d| \overrightarrow{x_{st}^\star},$$
$$K_-^d \overrightarrow{x_{st}^\star} = |J_-^d| \overrightarrow{z_{st}^{d\star}}, \quad (K_-^d)^\top \overrightarrow{z_{st}^{d\star}} = |J_-^d| \overrightarrow{x_{st}^\star}.$$

The desired result (3.5) follows from the fact that $K_{11}^d = K_+^d - K_-^d$.

Notice that

$$\mathbf{Sc}\big(\big((g_d)_{st}^{(i)}\big)^*(\breve{x}^\star,\breve{z}_d^\star)(g_d)_{\mathcal{I}}^{(i)}(\breve{x}^\star,\breve{z}_d^\star)\big) = 0, \quad \forall i \in J_+^d, \quad d = 1,\dots,p.$$

We have

$$\begin{aligned}
&\mathbf{Sc}\big(g_{st}^*(\breve{x}^\star,\mathbf{z}^\star)g_{\mathcal{I}}(\breve{x}^\star,\mathbf{z}^\star)\big)\\
=&\sum_{d=1}^{p}\sum_{i\in J_-^d}\mathbf{Sc}\big(\big((g_d)_{st}^{(i)}\big)^*(\breve{x}^\star,\breve{z}_d^\star)(g_d)_{\mathcal{I}}^{(i)}(\breve{x}^\star,\breve{z}_d^\star)\big)\\
=&\sum_{d=1}^{p}\sum_{i\in J_-^d}\Big[M\big((\breve{a}_d^{(i)})_{st}\big)\overrightarrow{z_{st}^{d\star}} - W\big(b_{st}^{(i)}\big)\overrightarrow{x_{st}^\star}\Big]^\top\\
&\quad\times\Big[M\big((\breve{a}_d^{(i)})_{st}\big)\overrightarrow{z_{\mathcal{I}}^{d\star}} + M\big((\breve{a}_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^{d\star}} - W\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}^\star} - W\big(b_{st}^{(i)}\big)\overrightarrow{x_{\mathcal{I}}^\star}\Big].
\end{aligned}$$

Combing (3.6) with $\overrightarrow{z_{st}^{d\star}}^\top \overrightarrow{z_{\mathcal{I}}^{d\star}} = \overrightarrow{x_{st}^\star}^\top \overrightarrow{x_{\mathcal{I}}^\star} = 0$, it follows that

$$\Big[M\big((\breve{a}_d^{(i)})_{st}\big)\overrightarrow{z_{st}^{d\star}} - W\big(b_{st}^{(i)}\big)\overrightarrow{x_{st}^\star}\Big]^\top\Big[M\big((\breve{a}_d^{(i)})_{st}\big)\overrightarrow{z_{\mathcal{I}}^{d\star}} - W\big(b_{st}^{(i)}\big)\overrightarrow{x_{\mathcal{I}}^\star}\Big] = 0, \quad \forall i \in J_-^d, \quad d = 1,\dots,p.$$

Using (3.6) again, for any $i \in J_-^d, d = 1,\dots,p$, we have

$$\begin{aligned}
&\Big[M\big((\breve{a}_d^{(i)})_{st}\big)\overrightarrow{z_{st}^{d\star}} - W\Big(b_{st}^{(i)}\Big)\overrightarrow{x_{st}^\star}\Big]^\top\Big[M\big((\breve{a}_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^{d\star}} - W\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}^\star}\Big]\\
=&\Big[\overrightarrow{z_{st}^{d\star}}^\top M\big((\breve{a}_d^{(i)})_{st}\big)^\top - \overrightarrow{x_{st}^\star}^\top W\big(b_{st}^{(i)}\big)^\top\Big]\Big[M\big((\breve{a}_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^{d\star}} - W\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}^\star}\Big]\\
=&\overrightarrow{z_{st}^{d\star}}^\top M\big((\breve{a}_d^{(i)})_{st}\big)^\top M\big((\breve{a}_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^{d\star}} - \overrightarrow{z_{st}^{d\star}}^\top M\big((\breve{a}_d^{(i)})_{st}\big)^\top W\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}^\star}\\
&\quad - \overrightarrow{x_{st}^\star}^\top W\big(b_{st}^{(i)}\big)^\top M\big((\breve{a}_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^{d\star}} + \overrightarrow{x_{st}^\star}^\top W\big(b_{st}^{(i)}\big)^\top W\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}^\star}\\
=&2\overrightarrow{z_{st}^{d\star}}^\top M\big((\breve{a}_d^{(i)})_{st}\big)^\top M\big((\breve{a}_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^{d\star}} + 2\overrightarrow{x_{st}^\star}^\top W\big(b_{st}^{(i)}\big)^\top W\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}^\star} = 0,
\end{aligned}$$

where the last inequality holds since both $M((\breve{a}_d^{(i)})_{st})^\top M((\breve{a}_d^{(i)})_{\mathcal{I}})$ and $W(b_{st}^{(i)})^\top W(b_{\mathcal{I}}^{(i)})$ are skew-symmetric according to Lemma 2.1. Therefore, the statement holds. □

Notice that in the proof of Lemma 3.2, only the property is used that the unit dual quaternions $\breve{x}^\star$ and $\breve{z}_d{}^\star$ satisfy $\overrightarrow{z_{st}^{d\star}}^\top \overrightarrow{z_{\mathcal{I}}^{d\star}} = \overrightarrow{x_{st}^\star}^\top \overrightarrow{x_{\mathcal{I}}^\star} = 0$, which is independent of the norm of $x_{\mathcal{I}}^{\star}$ and $z_{\mathcal{I}}^{d^{\star}}$. Hence, under the constraints $|\check{x}|=|\check{z}^1|=\cdots=|\check{z}^p|=1$, it suffices to assume that the equation of rotation matrices has a solution as in Assumption 2.1.

From (3.5), solution $\overrightarrow{x_{st}^{\star}}$ is a common right singular vector to $\{K_{11}^d\}_{d=1}^p$, $(\overrightarrow{z_{st}^{d\star}},\overrightarrow{x_{st}^{\star}})$ or $(-\overrightarrow{z_{st}^{d\star}},\overrightarrow{x_{st}^{\star}})$ constitutes a pair of singular vectors for $K_{11}^d, d=1,\ldots,p$. According to Lemma 3.2, we define the subspace constrained least squares solution to Eq. (1.1) as follows.

Definition 3.1. *We name $(X_{\star},Z_{1\star},\ldots,Z_{p\star})$ as a subspace constrained least squares solution of Eq.* (1.1) *if the corresponding unit dual quaternions $(\check{x}_{\star},\check{z}_{1\star},\ldots,\check{z}_{p\star})$ is a subspace constrained least squares solution of equation*

$$\mathtt{a}_d\check{z}_d=\check{x}\mathtt{b},\quad d=1,\ldots,p,$$

that is, it satisfies

$$\|g_{\mathcal{I}}(\check{x}_{\star},\mathtt{z}_{\star})\|_2^2=\min\left\{\|g_{\mathcal{I}}(\check{x},\mathtt{z})\|_2^2:(x_{st},z_{st}^1,\ldots,z_{st}^p)\in\Omega_{st}^{j_{\star}},|\check{x}|=|\check{z}_1|=\cdots=|\check{z}_p|=1\right\},\quad(3.7)$$

where

$$j_{\star}=\arg\min_{1\le j\le l}\left\{\left|\mathrm{Sc}\left(g_{st}^{*}(\check{x}_{j_{\Diamond}},\mathtt{z}_{j_{\Diamond}})g_{\mathcal{I}}(\check{x}_{j_{\Diamond}},\mathtt{z}_{j_{\Diamond}})\right)\right|\right\},\quad \mathtt{z}_{j_{\Diamond}}=\left(\check{z}_1^{j_{\Diamond}},\ldots,\check{z}_p^{j_{\Diamond}}\right),\quad j=1,\ldots,l$$

is the solution of the problem

$$\begin{aligned}&\min_{\check{x},\mathtt{z}}\ \mathrm{Sc}\left(g_{st}^{*}(\check{x},\mathtt{z})g_{\mathcal{I}}(\check{x},\mathtt{z})\right)\\&\text{s.t.}\quad\left(x_{st},z_{st}^1,\ldots,z_{st}^p\right)\in\Omega_{st}^{j},\quad|\check{x}|=|\check{z}_1|=\cdots=|\check{z}_p|=1.\end{aligned}\tag{3.8}$$

From Lemma 3.1(iv), $\mathrm{Sc}(g_{st}^{*}(\check{x},\mathtt{z})g_{\mathcal{I}}(\check{x},\mathtt{z}))=0$ for any $(x_{st},z_{st}^1,\ldots,z_{st}^p)\in\Omega_{st}$. Hence, if $\Omega_{st}\neq\emptyset$, then the subspace constrained least squares solution of Eq. (1.3) reduces to the solution of the problem

$$\begin{aligned}&\min\ \|g_{\mathcal{I}}(\check{x},\mathtt{z})\|_2^2\\&\text{s.t.}\quad(x_{st},z_{st}^1,\ldots,z_{st}^p)\in\Omega_{st},\quad|\check{x}|=|\check{z}_1|=\cdots=|\check{z}_p|=1.\end{aligned}\tag{3.9}$$

Next, we study the solutions satisfying Definition 3.1 by considering two cases:

(1) $\sigma_{\max}(K_{11}^d)=n$ for all $d=1,\ldots,p$,

(2) $\sigma_{\max}(K_{11}^d)\neq n$ for some $d\in\{1,\ldots,p\}$.

3.1.1 Case 1. $\sigma_{\max}(K_{11}^d)=n$ for all $d=1,\ldots,p$

Assumption 3.1. Suppose that there exists $\hat{d}\in\{1,\ldots,p\}$, for any $i\neq j$, $i,j=1,\ldots,n$ (≥ 2), $R_{A_{\hat{d}}^{(i)}}=\pm R_{A_{\hat{d}}^{(j)}}$ and $R_{B^{(i)}}=\pm R_{B^{(j)}}$ can not hold simultaneously.

Assumption 3.1 is equivalent to assume that there exists $\hat{d}\in\{1,\ldots,p\}$, for any $i\neq j$, $i,j=1,\ldots,n$ (≥ 2), $(\check{a}_{\hat{d}}^{(i)})_{st}=\pm(\check{a}_{\hat{d}}^{(j)})_{st}$ and $b_{st}^{(i)}=\pm b_{st}^{(j)}$ do not hold simultaneously. Under Assumption 3.1, recall Lemma 2.2, for any $i\neq j, i,j=1,\ldots,n$ (≥ 2),

$$\begin{aligned}&\mathbf{W}\big(b_{st}^{(i)}\big)^\top\mathbf{M}\big(\big(\check{a}_{\hat{d}}^{(i)}\big)_{st}\big)\mathbf{M}\big(\big(\check{a}_{\hat{d}}^{(j)}\big)_{st}\big)^\top\mathbf{W}\big(b_{st}^{(j)}\big)\\&\quad+\mathbf{W}\big(b_{st}^{(j)}\big)^\top\mathbf{M}\big(\big(\check{a}_{\hat{d}}^{(j)}\big)_{st}\big)\mathbf{M}\big(\big(\check{a}_{\hat{d}}^{(i)}\big)_{st}\big)^\top\mathbf{W}\big(b_{st}^{(i)}\big)\neq 2I_4.\end{aligned}$$

Define

$$S_{21}:=W_2^\top W_1,\quad S_{12}^d:=\big(M_d^1\big)^\top M_d^2,\quad S_{22}^d:=\big(M_d^2\big)^\top W_2,\quad d=1,\ldots,p.$$

The following property holds when $\sigma_{\max}(K_{11}^d)=n$ for all $d=1,\ldots,p$.

Lemma 3.3. *Suppose Assumptions* 2.1 *and* 3.1 *hold. Let*

$$K_{11}^d=U_d\Sigma_dV_d^\top=\big[Q_d^1\ U_d^2\big]\begin{bmatrix}nI_{\bar{k}} & \\ & \hat{\sigma}_d\end{bmatrix}\big[Q_d^2\ V_d^2\big]^\top,\quad d=1,\ldots,p$$

be the SVD of $\mathbf{K}_{11}^d, Q_d^1\in\mathbb{R}^{4\times\bar{k}}$ *and* $Q_d^2\in\mathbb{R}^{4\times\bar{k}}$ *be the matrix whose columns are unit singular vectors of* K_{11}^d *corresponding to* n, $\bar{k}=\min_{1\leq d\leq p}\{k_d\}$, *where* k_d *is the number of singular values of* K_{11}^d *that equal to* n, *and* $\hat{\sigma}_d$ *be the diagonal matrix generated by* $\hat{\sigma}_d\in\mathbb{R}^{4-\bar{k}}$, $\hat{\sigma}_d$ *contains the remaining* $4-\bar{k}$ *singular values of* K_{11}^d. *Then we have* $\bar{k}\leq 2$.

Denote

$$\begin{aligned}\Delta_1^d&\triangleq\big(\big(Q_d^1\big)^\top K_{21}^d-\big(Q_d^2\big)^\top S_{21}\big)Q_d^2,\\ \Delta_2^d&\triangleq\big(\big(Q_d^1\big)^\top\big(S_{12}^d\big)^\top-\big(Q_d^2\big)^\top(K_{12}^d)^\top\big)Q_d^1,\end{aligned}\qquad d=1,\ldots,p.$$

For $d=1,\ldots,p$, *we have*

$$Q_2^d\equiv Q,\tag{3.10a}$$

$$\begin{aligned}&\big(Q_d^1\big)^\top K_{11}^dQ=nI_k,\quad \big(Q_d^1\big)^\top K_{11}^dV_d^2=\mathbf{0},\\&\big(U_d^2\big)^\top K_{11}^dQ=\mathbf{0},\qquad \big(U_d^2\big)^\top K_{11}^dV_d^2=\hat{\sigma}_d(V_d^2)^\top Q_\perp,\end{aligned}\tag{3.10b}$$

$$\Delta_1^d=\Delta_2^d,\quad \Delta_1^d+\big(\Delta_1^d\big)^\top=\mathbf{0},\tag{3.10c}$$

$$\mathrm{Sym}\big(\big(Q_d^1\big)^\top\big(K_{12}^d+K_{21}^d\big)Q\big)=\mathbf{0},\tag{3.10d}$$

where $\mathrm{Sym}(A)=(A+A^\top)/2$.

Proof. According to [25, Lemma 10], under Assumption 3.1, there exists $\hat{d}\in\{1,\ldots,p\}$ such that $k_{\hat{d}}\leq 2$. Hence, $\bar{k}=\min_{1\leq d\leq p}\{k_d\}\leq 2$.

Eq. (3.10a) is a direct consequence of Lemma 3.1(iii). The equations in (3.10b), (3.10c), and (3.10d) can be obtained from the similar way to [25, Lemma 7]. ☐

By the definition of Q, $Q_1^1,\ldots,Q_p^1$ in Lemma 3.3, for any $(x_{st},z_{st}^1,\ldots,z_{st}^p)\in\Omega_{st}$, there exists unit vector y, such that $\overrightarrow{x_{st}}=Qy$ and $\overrightarrow{z_{st}^d}=Q_d^1 y, d=1,\ldots,p$. Let $U_d=[Q_d^1\ U_d^2]$ be given as in Lemma 3.3 and $Q_\perp$ be the orthogonal complement of Q. There exist u_x,v_x,u_{z_d}, and $v_{z_d}, d=1,\ldots,p$ such that $\overrightarrow{x_{\mathcal{I}}}=Qu_x+Q_\perp v_x$ and $\overrightarrow{z_{\mathcal{I}}^d}=Q_d^1 u_{z_d}+U_d^2 v_{z_d}, d=1,\ldots,p$. Then for any feasible point $(\overrightarrow{x_{st}},\overrightarrow{z_{st}^1},\ldots,\overrightarrow{z_{st}^p})$ of problem (3.9), we have

$$
\begin{aligned}
&0=\overrightarrow{x_{st}}^\top\overrightarrow{x_{\mathcal{I}}}=y^\top Q^\top(Qu_x+Q_\perp v_x)=y^\top u_x,\\
&0=\overrightarrow{z_{st}^d}^\top\overrightarrow{z_{\mathcal{I}}^d}=y^\top\left(Q_d^1\right)^\top\left(Q_d^1 u_{z_d}+U_d^2 v_{z_d}\right)=y^\top u_{z_d},\quad d=1,\ldots,p.
\end{aligned}
$$

Let $Y_\perp$ be the orthogonal complement of y, then there exist w_x and $\{w_{z_d}\}_{d=1}^p$ such that $u_x=Y_\perp w_x$ and $u_{z_d}=Y_\perp w_{z_d}, d=1,\ldots,p$. Denote $w_z=(w_{z_1},\ldots,w_{z_p})$ and $v_z=(v_{z_1},\ldots,v_{z_p})$. Problem (3.9) reduces to

$$
\min_{y,Y_\perp,w_x,v_x,w_z,v_z} g(y,Y_\perp,w_x,v_x,w_z,v_z)\quad \text{s.t.}\quad [y\ Y_\perp]^\top[y\ Y_\perp]=I_{\bar{k}}, \tag{3.11}
$$

where

$$
\begin{aligned}
&g(y,Y_\perp,w_x,v_x,w_z,v_z)=\|g_{\mathcal{I}}(\check{x},\mathbf{z})\|_2^2\\
=&\sum_{d=1}^p\sum_{i=1}^n\left\|\mathbf{M}\big((a_d^{(i)})_{st}\big)\overrightarrow{z_{\mathcal{I}}^d}+\mathbf{M}\big((a_d^{(i)})_{\mathcal{I}}\big)\overrightarrow{z_{st}^d}-\mathbf{W}\big(b_{\mathcal{I}}^{(i)}\big)\overrightarrow{x_{st}}-\mathbf{W}\big(b_{st}^{(i)}\big)\overrightarrow{x_{\mathcal{I}}}\right\|_2^2\\
=&\sum_{d=1}^p\Big\{nw_{z_d}^\top w_{z_d}+nv_{z_d}^\top v_{z_d}+2w_{z_d}^\top Y_\perp^\top\left(Q_d^1\right)^\top S_{12}^d Q_d^1 y-2w_{z_d}^\top Y_\perp^\top\left(Q_d^1\right)^\top K_{12}^d Qy\\
&\quad-2nw_{z_d}^\top w_x+2v_{z_d}^\top\left(U_d^2\right)^\top S_{12}^d Q_d^1 y-2v_{z_d}^\top\left(U_d^2\right)^\top K_{12}^d Qy-2v_{z_d}^\top\hat{\sigma}_d\left(V_d^2\right)^\top Q_\perp v_x\\
&\quad-2y^\top\left(Q_d^1\right)^\top S_{22}^d Qy-2y^\top\left(Q_d^1\right)^\top K_{21}^d QY_\perp w_x-2y^\top\left(Q_d^1\right)^\top K_{21}^d Q_\perp v_x\\
&\quad+2y^\top Q^\top S_{21}QY_\perp w_x+2y^\top Q^\top S_{21}Q_\perp v_x\Big\}+npw_x^\top w_x+npv_x^\top v_x+c\\
=&\sum_{d=1}^p\Big\{nw_{z_d}^\top w_{z_d}+nv_{z_d}^\top v_{z_d}+2w_{z_d}^\top Y_\perp^\top\left(\Delta_1^d\right)^\top y-2nw_{z_d}^\top w_x+2v_{z_d}^\top\left(D_2^d\right)^\top y\\
&\quad-2v_{z_d}^\top\hat{\sigma}_d\left(V_d^2\right)^\top Q_\perp v_x-2y^\top\left(Q_d^1\right)^\top S_{22}^d Qy-2y^\top\Delta_1^d Y_\perp w_x-2y^\top D_1^d v_x\Big\}\\
&+npw_x^\top w_x+npv_x^\top v_x+c_1,
\end{aligned}
$$

$c_1=\sum_{d=1}^p\sum_{i=1}^n\|\overrightarrow{(a_d^{(i)})_{\mathcal{I}}}\|_2^2+p\sum_{i=1}^n\|\overrightarrow{b_{\mathcal{I}}^{(i)}}\|_2^2$ follows the property $M(q)^\top M(q)=W(q)^\top W(q)=\|\vec{q}\|_2^2 I_4$, the second equality use the definition of $Q_d^1,U_d^2,Q,V_d^2,K_{12}^d,K_{21}^d,S_{12}^d,S_{22}^d$, and S_{21}, and Eq. (3.10b), and the last equation follows from the definition of $\Delta_1^d, d=1,\ldots,p$, (3.10c), $D_1^d=((Q_d^1)^\top K_{21}^d-Q^\top S_{21})Q_\perp$, and $D_2^d=((Q_d^1)^\top(S_{12}^d)^\top-Q^\top(K_{12}^d)^\top)U_d^2, d=1,\ldots,p$.

Lemma 3.4. *Problem* (3.11) *is equivalent to*

$$
\max_{y,Y_\perp} y^\top Sy+\frac{1}{n}\sum_{d=1}^p\left\|Y_\perp^\top\Delta_1^d y\right\|_2^2\quad \text{s.t.}\quad [y\ Y_\perp]^\top[y\ Y_\perp]=I_{\bar{k}},
$$

where

$$
\begin{aligned}
S=&\frac{1}{n}\sum_{d=1}^{p}D_2^d\left(D_2^d\right)^\top+\sum_{d=1}^{p}\left[\left(Q_d^1\right)^\top S_{22}^dQ+Q^\top\left(S_{22}^d\right)^\top Q_d^1\right]\\
&+\Theta^\top\left(npI_{4-\bar{k}}-\frac{1}{n}\sum_{d=1}^{p}\Gamma_d\Gamma_d^\top\right)^{-1}\Theta,\\
\Theta=&\sum_{d=1}^{p}\left[D_1^d-\frac{1}{n}D_2^d\Gamma_d^\top\right]^\top,\\
\Gamma_d=&Q_\perp^\top V_d^2\hat{\sigma}_d,\quad d=1,\ldots,p.
\end{aligned}
$$

Proof. Notice that the constrains in (3.11) does not contain w_x,v_x,w_z, and v_z, by the first-order optimality condition and the definition of $\Gamma_d,d=1,\ldots,p$, the solution $(w_x^*,v_x^*,w_z^*,v_z^*)$ of problem (3.11) satisfies

$$
-Y_\perp^\top\left(\sum_{d=1}^{p}\Delta_1^d\right)^\top y-n\sum_{d=1}^{p}w_{z_d}^*+npw_x^*=0, \tag{3.12a}
$$

$$
-\left(\sum_{d=1}^{p}D_1^d\right)^\top y-\sum_{d=1}^{p}\Gamma_dv_{z_d}^*+npv_x^*=0, \tag{3.12b}
$$

$$
Y_\perp^\top\left(\Delta_1^d\right)^\top y+nw_{z_d^*}-nw_x^*=0,\quad d=1,\ldots,p, \tag{3.12c}
$$

$$
\left(D_2^d\right)^\top y+nv_{z_d}^*-\Gamma_d^\top v_x^*=0,\qquad d=1,\ldots,p. \tag{3.12d}
$$

From (3.12c) and (3.10c), we have

$$
w_{z_d}^*=w_x^*+\frac{1}{n}Y_\perp^\top\Delta_1^dy,\quad d=1,\ldots,p. \tag{3.13}
$$

From (3.12d), we have

$$
v_{z_d}^*=\frac{1}{n}\Gamma_d^\top v_x^*-\frac{1}{n}\left(D_2^d\right)^\top y,\quad d=1,\ldots,p. \tag{3.14}
$$

Using (3.13) and (3.14), it holds that

$$
\begin{aligned}
&g(y,Y_\perp,w_x^*,v_x^*,w_z^*,v_z^*)\\
=&\sum_{d=1}^{p}\left\{-\frac{1}{n}y^\top\left(\Delta_1^d\right)^\top Y_\perp Y_\perp^\top\Delta_1^dy-y^\top\left(\left(Q_d^1\right)^\top S_{22}^dQ+Q^\top\left(S_{22}^d\right)^\top Q_d^1\right)y-\frac{1}{n}y^\top D_2^d\left(D_2^d\right)^\top y\right\}\\
&+\sum_{d=1}^{p}\left\{-\frac{1}{n}(v_x^*)^\top\Gamma_d\Gamma_d^\top v_x^*+\frac{2}{n}y^\top D_2^d\Gamma_d^\top v_x^*-2y^\top D_1^dv_x^*\right\}+np(v_x^*)^\top v_x^*+c_1.
\end{aligned}
$$

Recall that $\nabla_{v_x} g(y,Y_\perp,w_x^*,v_x^*,w_z^*,v_z^*)=0$, which yields $(npI_{4-\bar{k}}-\sum_{d=1}^p \Gamma_d\Gamma_d^\top/n)v_x^*=\Theta y$, and hence

$$v_x^*=\left(npI_{4-\bar{k}}-\frac{1}{n}\sum_{d=1}^p \Gamma_d\Gamma_d^\top\right)^{-1}\Theta y. \tag{3.15}$$

$npI_{4-\bar{k}}-\sum_{d=1}^p \Gamma_d\Gamma_d^\top/n$ is invertible by noting that

$$\begin{aligned} npI_{4-\bar{k}}-\frac{1}{n}\sum_{d=1}^p \Gamma_d\Gamma_d^\top &= \sum_{d=1}^p \left[nI_{4-\bar{k}}-\frac{1}{n}Q_\perp^\top V_d^2\hat{\sigma}_d^2\left(V_d^2\right)^\top Q_\perp\right] \\ &= \sum_{d=1}^p Q_\perp^\top\left[nI_{4-\bar{k}}-\frac{1}{n}V_d^2\hat{\sigma}_d^2\left(V_d^2\right)^\top\right]Q_\perp. \end{aligned}$$

For each $d\in\{1,\ldots,p\}$, the eigenvalues of $nI_{4-\bar{k}}-V_d^2\hat{\sigma}_d^2(V_d^2)^\top/n$ are n and $n-(\sigma_d^j)^2/n$, where σ_d^j denotes the j-th element of $\hat{\sigma}_d$, $0\le\sigma_d^j\le n, j=1,\ldots,4-\bar{k}$. Recall the definition of $\bar{k}$ in Lemma 3.3, there must exist $\bar{d}\in\{1,\ldots,p\}$ such that $0\le\sigma_{\bar{d}}^j<n$, which implies that

$$npI_{4-\bar{k}}-\frac{1}{n}\sum_{d=1}^p \Gamma_d\Gamma_d^\top = \sum_{d=1}^p Q_\perp^\top\left[nI_{4-\bar{k}}-\frac{1}{n}V_d^2\hat{\sigma}_d^2\left(V_d^2\right)^\top\right]Q_\perp$$

is a symmetric and positive definite matrix.

Using (3.13)-(3.15) and the definition of S, we have

$$g(y,Y_\perp,w_x^*,v_x^*,w_z^*,v_z^*)=-y^\top Sy-\frac{1}{n}\sum_{d=1}^p \left\|Y_\perp^\top\Delta_1^d y\right\|_2^2+c_1.$$

The desired result holds. □

Theorem 3.1. *Suppose Assumptions* 2.1 *and* 3.1 *holds. Let $k_d, d=1,\ldots,p$, be the minimum number of singular values of $\mathbf{K}_{11}^d$ equals to n and $\bar{k}=\min_{1\le d\le p}\{k_d\}$.*

(a) In case $\bar{k}=1$,

$$\overrightarrow{x_{st\star}}=\pm Q, \tag{3.16a}$$

$$\overrightarrow{z_{st\star}^d}=\pm Q_d^1,\quad d=1,\ldots,p, \tag{3.16b}$$

$$\overrightarrow{x_{\mathcal{I}\star}}=\pm Q_\perp\left(npI_3-\frac{1}{n}\sum_{d=1}^p\Gamma_d\Gamma_d^\top\right)^{-1}\Theta, \tag{3.16c}$$

$$\overrightarrow{z_{\mathcal{I}\star}^d}=\pm\frac{1}{n}\left[U_d^2\Gamma_d^\top\left(npI_3-\frac{1}{n}\sum_{d=1}^p\Gamma_d\Gamma_d^\top\right)^{-1}\Theta-U_d^2\left(D_2^d\right)^\top\right],\quad d=1,\ldots,p. \tag{3.16d}$$

(b) In case $\bar{k}=2$,

$$\overrightarrow{x_{st\star}}=\pm Qy^*, \tag{3.17a}$$

$$\overrightarrow{z^d_{st\star}}=\pm Q^1_d y^*, \quad d=1,\dots,p, \tag{3.17b}$$

$$\overrightarrow{x_{\mathcal{I}\star}}=QY^*_\perp w^*_x+Q_\perp\left(npI_2-\frac{1}{n}\sum_{d=1}^p\Gamma_d\Gamma_d^\top\right)^{-1}\Theta y^*, \tag{3.17c}$$

$$\begin{aligned}\overrightarrow{z^d_{\mathcal{I}\star}}=&Q^1_dY^*_\perp w^*_{z_d}+\frac{1}{n}U^2_d\Gamma_d^\top\left(npI_2-\frac{1}{n}\sum_{d=1}^p\Gamma_d\Gamma_d^\top\right)^{-1}\Theta y^*\\&-\frac{1}{n}U^2_d\left(D^d_2\right)^\top y^*,d=1,\dots,p,\end{aligned} \tag{3.17d}$$

where y^* is a unit eigenvector corresponding to the largest eigenvalue of S, $Y^*_\perp$ is the orthogonal complement of y^*, and $w^*_{z_d}-w^*_x=-(\Delta^d_1)(1,2)/n$ or $w^*_{z_d}-w^*_x=(\Delta^d_1)(1,2)/n$, $d=1,\dots,p$.

Then

$$(\check{x}_\star,\mathrm{z}_\star):=(\check{x}_\star,\check{z}_{1_\star},\dots,\check{z}_{p_\star})=\left(x_{st\star}+x_{\mathcal{I}\star}\epsilon,z^1_{st\star}+z^1_{\mathcal{I}\star}\epsilon,\dots,z^p_{st\star}+z^p_{\mathcal{I}\star}\epsilon\right)$$

is the solution to problem (3.9) *and the corresponding transformation matrices $(X_\star,Z_{1\star},\dots,Z_{d\star})$ is a subspace constrained least squares solution to Eq.* (1.1).

Proof. (a) If $\bar{k}=1$, then we have $(y^*)^2=1$ and $Y^*_\perp=0$. Hence, we have (3.16a) and (3.16b) since $\overrightarrow{x_{st\star}}=Qy^*$ and $\overrightarrow{z^d_{st\star}}=Q^1_dy^*$, $d=1,\dots,p$.

(3.16c) follows from $\overrightarrow{x_{\mathcal{I}\star}}=QY^*_\perp w^*_x+Q_\perp v^*_x$ and (3.15).

(3.16d) follows from $\overrightarrow{z^d_{\mathcal{I}\star}}=Q^1_dY^*_\perp w^*_{z_d}+U^2_dv^*_{z_d}$, (3.14), and (3.15).

(b) If $\bar{k}=2$, recall that $\Delta^d_1+(\Delta^d_1)^\top=\mathbf{0}$, from the proof of [25, Theorem 3], for any $(y,Y_\perp)$ satisfies $[y\ Y_\perp]^\top[y\ Y_\perp]=I_2$, we have

$$Y_\perp^\top\Delta^d_1y=\left(\Delta^d_1\right)(1,2)\quad\text{or}\quad Y_\perp^\top\Delta^d_1y=-\left(\Delta^d_1\right)(1,2),\quad d=1,\dots,p,$$

where $(\Delta^d_1)(1,2)$ denotes the $(1,2)$ entry of Δ^d_1, which implies that $w^*_{z_d}-w^*_x=\pm(\Delta^d_1)(1,2)/n$, $d=1,\dots,p$, and $\|Y_\perp^\top\Delta_1y\|_2^2$ is a constant. Hence, y^* is a unit eigenvector corresponding to the largest eigenvalue of C. (3.17a) and (3.17b) hold obviously.

(3.17c) follows from $\overrightarrow{x_{\mathcal{I}\star}}=QY^*_\perp w^*_x+Q_\perp v^*_x$ and (3.15).

(3.17d) follows from $\overrightarrow{z^d_{\mathcal{I}\star}}=Q^1_dY^*_\perp w^*_{z_d}+U^2_dv^*_{z_d}$, (3.13), and (3.14). □

3.1.2 Case 2. $\sigma_{\max}(K^d_{11})\neq n$ for some $d\in\{1,\dots,p\}$

It follows from Lemma 3.1(vi) that the value of $\mathrm{Sc}(g^*_{st}(\check{x},\mathrm{z})g_{\mathcal{I}}(\check{x},\mathrm{z}))$ is independent of $x_{\mathcal{I}},z^1_{\mathcal{I}},\dots,z^p_{\mathcal{I}}$ when $(\check{x},\check{z}_1,\dots,\check{z}_p)$ satisfies $(x_{st},z^1_{st},\dots,z^p_{st})\in\Omega^j_{st}$, $j=1,\dots,l$. Hence, problem

(3.8) reduces to

$$\min_{\check{x},\,\mathbf{z}} \mathrm{Sc}\big(g_{st}^*(\check{x},\mathbf{z})g_{\mathcal{I}}(\check{x},\mathbf{z})\big) \quad \text{s.t.} \quad \big(x_{st},z_{st}^1,\ldots,z_{st}^p\big)\in\Omega_{st}^j. \tag{Sc-j}$$

The matrix representation of problem (Sc-j) is given by

$$\max_{\overrightarrow{x}_{st},\overrightarrow{z_{st}^1},\ldots,\overrightarrow{z_{st}^p}} \sum_{d=1}^p \left(\overrightarrow{z_{st}^d}\right)^\top \left(K_{12}^d+K_{21}^d\right)\overrightarrow{x_{st}} \quad \text{s.t.} \quad \left(\overrightarrow{x_{st}},\overrightarrow{z_{st}^1},\ldots,\overrightarrow{z_{st}^p}\right)\in\overrightarrow{\Omega_{st}^j}, \tag{MSc-j}$$

where $\overrightarrow{\Omega_{st}^j}=\{(\vec{u},\vec{v}_1,\ldots,\vec{v}_p)\,|\,(u,v_1,\ldots,v_p)\in\Omega_{st}^j\}$. Let $(Q^j,Q_1^{1j},\ldots,Q_p^{1j})$ be matrices whose columns form an orthogonal basis of the singular subspace pair $\overrightarrow{\Omega_{st}^j}$. Then $\overrightarrow{x_{st}}=Q^jy$ and $\overrightarrow{z_{st}^d}=Q_d^{1j}y, d=1,\ldots,p$, for some y satisfies $y^\top y=1$. Problem (MSc-j) can be further represented as

$$\max_y y^\top T_j y \quad \text{s.t.} \quad \|y\|_2=1.$$

where $T_j=\sum_{d=1}^p(Q_d^{1j})^\top(K_{12}^d+K_{21}^d)Q^j$. The optimal solution y_j of the above problem is the unit eigenvector of $\mathrm{Sym}(T_j)$ corresponding to $\lambda_{\max}(\mathrm{Sym}(T_j))$, where $\lambda_{\max}(A)$ denotes the maximal eigenvalue of A. The solution $(\overrightarrow{x_{st\diamondsuit}}^j,\overrightarrow{z_{st\diamondsuit}^1}^j,\ldots,\overrightarrow{z_{st\diamondsuit}^p}^j)$ of problem (MSc-j) is given by $(Q^jy_j,Q_1^{1j}y_j,\ldots,Q_p^{1j}y_j)$. Let

$$j_\star=\operatorname*{argmin}_j\left\{\left|\lambda_{\max}\left(\mathrm{Sym}\left(\sum_{d=1}^p\left(Q_d^{1j}\right)^\top\left(K_{12}^d+K_{21}^d\right)Q^j\right)\right)\right|\right\}.$$

Then $(\overrightarrow{x_{st\star}},\overrightarrow{z_{st\star}^1},\ldots,\overrightarrow{z_{st\star}^p})=(Q^{j_\star}y_{j_\star},Q_1^{1j_\star}y_{j_\star},\ldots,Q_p^{1j_\star}y_{j_\star})$.

After obtaining $(x_{st\star},z_{st\star}^1,\ldots,z_{st\star}^p)$, we can determine $(x_{\mathcal{I}\star},z_{\mathcal{I}\star}^1,\ldots,z_{\mathcal{I}\star}^p)$ by solving problem (3.7), that is,

$$\begin{aligned}&\min_{x_{\mathcal{I}},z_{\mathcal{I}}^1,\ldots,z_{\mathcal{I}}^p} \|g_{\mathcal{I}}(\check{x},\mathbf{z})\|_2^2\\ &\text{s.t.} \quad {x_{st\star}}^*x_{\mathcal{I}}+x_{\mathcal{I}}^*x_{st\star}=0, \quad \left(z_{st\star}^d\right)^*z_{\mathcal{I}}^d+\left(z_{\mathcal{I}}^d\right)^*z_{st\star}^d=0, \quad d=1,\ldots,p.\end{aligned}$$

Denote $X_\perp$ and $Z_{d\perp}$ as the orthogonal complement of $\overrightarrow{x_{st\star}}$ and $\overrightarrow{z_{st\star}^d}, d=1,\ldots,p$, respectively. The feasible point of the above problem satisfies $\overrightarrow{x_{\mathcal{I}}}=X_\perp u$ and $\overrightarrow{z_{\mathcal{I}}^d}=Z_{d\perp}v_d$ for some u and $v_d, d=1,\ldots,p$. Notice that $\mathbf{M}(a)^\top\mathbf{M}(a)=\mathbf{W}(a)^\top\mathbf{W}(a)=\|\vec{a}\|_2^2I_4$ for any $a\in\mathbb{Q}$ and $X_\perp^\top X_\perp=Z_{1\perp}^\top Z_{1\perp}=\ldots=Z_{p\perp}^\top Z_{p\perp}=I_3$. The above problem can be further formulated as

$$\min_{u,v} h(u,v_1,\ldots,v_p), \tag{3.18}$$

where

$$
\begin{aligned}
&h(u,v_1,\dots,v_p)=\|g_{\mathcal{I}}(\check{x},\mathbf{z})\|_2^2\\
=&\sum_{d=1}^{p}\Big(nv_d^\top v_d-2v_d^\top Z_{d\perp}^\top K_{11}^d X_\perp u+2\Big(\overrightarrow{z_{st\star}^d}^\top\big(S_{12}^d\big)^\top-\overrightarrow{x_{st\star}}^\top\big(K_{12}^d\big)^\top\Big)Z_{d\perp}v_d-2\overrightarrow{z_{st\star}^d}^\top K_{21}^d X_\perp u\Big)\\
&+2p\overrightarrow{x_{st\star}}^\top S_{21}X_\perp u+npu^\top u.
\end{aligned}
$$

Theorem 3.2. *(u_*,v_*) is a solution of problem* (3.18) *if and only if it satisfies*

$$
\begin{bmatrix} npI_3 & H\\ H^\top & nI_{3p}\end{bmatrix}\begin{bmatrix}u_*\\ v_*\end{bmatrix}=\begin{bmatrix} g_0\\ \tilde{g}\end{bmatrix}, \tag{3.19}
$$

where

$$
\begin{aligned}
&v_*=(v_{1*},\dots,v_{p*}),\\
&H=\big[-X_\perp^\top\big(K_{11}^1\big)^\top Z_{1\perp}\dots -X_\perp^\top\big(K_{11}^p\big)^\top Z_{p\perp}\big],\\
&g_0=X_\perp^\top\left(-pS_{21}^\top\overrightarrow{x_{st\star}}+\sum_{d=1}^{p}\big(K_{21}^d\big)^\top\overrightarrow{z_{st\star}^d}\right),\\
&\tilde{g}=\begin{bmatrix} Z_{1\perp}^\top\Big(K_{12}^1\overrightarrow{x_{st\star}}-S_{12}^1\overrightarrow{z_{st\star}^1}\Big)\\ \vdots\\ Z_{p\perp}^\top\Big(K_{12}^p\overrightarrow{x_{st\star}}-S_{12}^p\overrightarrow{z_{st\star}^p}\Big)\end{bmatrix}.
\end{aligned}
$$

Proof. Notice that h is quadratic in (u,v). It is sufficient to show that h is convex and $\nabla h(u_*,v_*)=0$. We first show that $\nabla^2 h(u,v)\succeq 0$ for any $u\in\mathbb{R}^3$ and $v\in\mathbb{R}^{3p}$.

By [6, Theorem 7.7.9], it is sufficient to show that there is a contraction $C\in\mathbb{R}^{3\times 3p}$ such that $H=n\sqrt{p}C$. Notice that

$$
K_{11}^d=\big(M_d^1\big)^\top W_1,\quad \big(M_d^1\big)^\top M_d^1=W_1^\top W_1=nI_4,\quad d=1,\dots,p.
$$

Denote $G_1\triangleq W_1X_\perp\in\mathbb{R}^{4n\times 3}$ and $G_2^d\triangleq M_d^1 Z_{d\perp}\in\mathbb{R}^{4n\times 3}$. Let $G_1=L_1\Sigma_1R_1^\top$ and $G_2^d=L_2^d\Sigma_2^d(R_2^d)^\top$ be SVDs of G_1 and $G_2^d, d=1,\dots,p$, respectively. Then we have

$$
\begin{aligned}
&-X_\perp^\top\big(K_{11}^d\big)^\top Z_{d\perp}=-G_1^\top G_2^d=-R_1\Sigma_1^\top L_1^\top L_2^d\Sigma_2^d\big(R_2^d\big)^\top,\quad d=1,\dots,p,\\
&\sqrt{n}I_3=\big(X_\perp^\top W_1^\top W_1X_\perp\big)^{1/2}=\big(G_1^\top G_1\big)^{1/2}\\
&\qquad=\big(R_1\Sigma_1^\top\Sigma_1R_1^\top\big)^{1/2}=R_1\widehat{\Sigma}_1R_1^\top,\\
&\sqrt{n}I_3=\big(Z_{d\perp}^\top\big(M_d^1\big)^\top M_d^1Z_{d\perp}\big)^{1/2}=\big(\big(G_2^d\big)^\top G_2^d v\big)^{1/2}\\
&\qquad=\big(R_2^d\big(\Sigma_2^d\big)^\top\Sigma_2^d\big(R_2^d\big)^\top\big)^{1/2}=R_2^d\widehat{\Sigma}_2^d\big(R_2^d\big)^\top,\qquad d=1,\dots,p,
\end{aligned}
$$

where $\widehat{\Sigma}_1=(\Sigma_1^\top\Sigma_1)^{1/2}=\sqrt{n}I_3$ and $\widehat{\Sigma}_2^d=((\Sigma_2^d)^\top\Sigma_2^d)^{1/2}=\sqrt{n}I_3$ since R_1 and $R_2^d, d=1,\dots,p$, are orthogonal. Denote

$$\widetilde{C}=-\frac{1}{\sqrt{p}}R_1\widehat{L}_1^\top\left[\widehat{L}_2^1(R_2^1)^\top \quad \cdots \quad \widehat{L}_2^p(R_2^p)^\top\right]\in\mathbb{R}^{3\times 3p},$$

where $\widehat{L}_1$ and $\widehat{L}_2^d$ as the first three columns of L_1 and $L_2^d, d=1,\dots,p$, respectively. $\widetilde{C}$ is contraction by noting that $\sigma_{\max}(\widetilde{C})\le 1$. Moreover,

$$\widehat{\Sigma}_1^\top\widehat{L}_1^\top\widehat{L}_2^d\widehat{\Sigma}_2^d=\Sigma_1^\top L_1^\top L_2^1\Sigma_2^d,\quad d=1,\dots,p \tag{3.20}$$

and

$$\begin{aligned}
&n\sqrt{p}\widetilde{C}=-(\sqrt{n}I_3)(\sqrt{p}\widetilde{C})(\sqrt{n}I_{3p})\\
=&-R_1\widehat{\Sigma}_1R_1^\top R_1\widehat{L}_1^\top\left[\widehat{L}_2^1(R_2^1)^\top \quad \dots \quad \widehat{L}_2^p(R_2^p)^\top\right]\begin{bmatrix}R_2^1\widehat{\Sigma}_2^1(R_2^1)^\top & & \\ & \ddots & \\ & & R_2^p\widehat{\Sigma}_2^p(R_2^p)^\top\end{bmatrix}\\
=&-R_1\widehat{\Sigma}_1R_1^\top R_1\widehat{L}_1^\top\left[\widehat{L}_2^1(R_2^1)^\top R_2^1\widehat{\Sigma}_2^1(R_2^1)^\top \quad \cdots \quad \widehat{L}_2^p(R_2^p)^\top R_2^p\widehat{\Sigma}_2^p(R_2^p)^\top\right]\\
=&-\left[R_1\widehat{\Sigma}_1\widehat{L}_1^\top\widehat{L}_2^1\widehat{\Sigma}_2^1(R_2^1)^\top \quad \cdots \quad R_1\widehat{\Sigma}_1\widehat{L}_1^\top\widehat{L}_2^p\widehat{\Sigma}_2^p(R_2^p)^\top\right]\\
=&-\left[R_1\Sigma_1^\top L_1^\top L_2^1\Sigma_2^1(R_2^1)^\top \quad \cdots \quad R_1\Sigma_1^\top L_1^\top L_2^p\Sigma_2^p(R_2^p)^\top\right]\\
=&-\left[X_\perp^\top(K_{11}^1)^\top Z_{1\perp} \quad \cdots \quad X_\perp^\top(K_{11}^p)^\top Z_{p\perp}\right]=H,
\end{aligned}$$

where the last second equality follows from (3.20). Hence, problem (3.18) is convex and (3.19) holds. ☐

Remark 3.1. Notice that $\begin{bmatrix} npI_3 & H\\ H^\top & nI_{3p}\end{bmatrix}$ is a block arrowhead matrix. We have

$$\begin{bmatrix} npI_3 & H\\ H^\top & nI_{3p}\end{bmatrix}=\begin{bmatrix} I_3 & 0\\ \frac{1}{np}H^\top & I_{3p}\end{bmatrix}\begin{bmatrix} npI_3 & 0\\ 0^\top & nI_{3p}-\frac{1}{np}H^\top H\end{bmatrix}\begin{bmatrix} I_3 & \frac{1}{np}H\\ 0 & I_{3p}\end{bmatrix},$$

which yields $u_*=(g_0-Hv_*)/(np)$ and v_* is the solution of the equation

$$\left(nI_{3p}-\frac{1}{np}H^\top H\right)v=\tilde{g}-\frac{1}{np}H^\top g_0.$$

If we further have $\sigma_{\max}(H)<n\sqrt{p}$ (or equally, $\sigma_{\min}(\widetilde{C})<1$ in the proof of Theorem 3.2), then by using Sherman-Morrison-Woodbury formula, we have

$$v_*=\frac{1}{n}\left(I_{3p}-H^\top\left(HH^\top-n^2pI_3\right)^{-1}H\right)\left(\tilde{g}-\frac{1}{np}H^\top g_0\right).$$

Corollary 3.1. *When there exists* $d \in \{1,\dots,p\}$ *such that* $\sigma_{\max}(K_{11}^d) \neq n$,

$$(\check{x}_\star, \mathbf{z}_\star) := (\check{x}_\star, \check{z}_{1_\star}, \dots, \check{z}_{p_\star}) = \left(x_{st\star} + x_{\mathcal{I}\star}\epsilon, z_{st\star}^1 + z_{\mathcal{I}\star}^1\epsilon, \dots, z_{st\star}^p + z_{\mathcal{I}\star}^p\epsilon\right)$$

is a subspace constrained least squares solution of Eq. (1.3), *where*

$$\overrightarrow{x_{st\star}} = Q^{j_\star} y_{j_\star}, \quad \overrightarrow{x_{\mathcal{I}\star}} = X_\perp u_*, \quad \overrightarrow{z_{st\star}^d} = Q_d^{1j_\star} y_{j_\star}, \quad \overrightarrow{z_{\mathcal{I}\star}^d} = Z_{d\perp} v_d^*, \quad d = 1,\dots,p,$$

$y_{j_\star}$ *is the unit eigenvector corresponding to the maximal eigenvalue of* $\mathrm{Sym}(T_{j_\star})$ *with*

$$j_\star = \arg\min_j \left\{ \left| \lambda_{\max}\left(\mathrm{Sym}(T_j)\right) \right| \right\},$$

$T_j = \sum_{d=1}^p (Q_d^{1j})^\top (K_{12}^d + K_{21}^d) Q^j, Q^j, Q_1^{1j}, \dots, Q_p^{1j}$ *be matrices whose columns form an orthogonal basis of the singular subspace pair* $\overrightarrow{\Omega_{st}^j}$, $X_\perp$ *and* $Z_\perp$ *are the orthogonal complement of* $\overrightarrow{x_{st\star}}$ *and* $\overrightarrow{z_{st\star}^d}$, *respectively, and* (u_*, v_*) *is the solution of Eq.* (3.19). *The corresponding transformation matrix* $(X_\star, Z_{1\star}, \dots, Z_{d\star})$ *is a subspace constrained least squares solution to Eq.* (1.1).

Remark 3.2. Note that $\min_j \{|\lambda_{\max}(\mathrm{Sym}(T_j))|\} = 0$ implies $\mathrm{Sym}(T_{j_\star}) = \mathbf{0}$. In this case, the subspace constrained least squares solution of Eq. (1.3) reduces to the solution of the problem

$$\min \|g_{\mathcal{I}}(\check{x}, \mathbf{z})\|_2^2 \quad \text{s.t.} \quad \left(x_{st}, z_{st}^1, \dots, z_{st}^p\right) \in \Omega_{st}^{j_\star}, \quad |\check{x}| = |\check{z}_1| = \cdots = |\check{z}_p| = 1,$$

which was given in Theorem 3.1 except replacing Q, Q_d^1, and U_d^2 by $Q^{j_\star}, Q_d^{1j_\star}$, and $U_d^{2j_\star}$, where

$$K_{11}^d = \left[Q_d^{1j_\star} \; U_d^{2j_\star} \right] \begin{bmatrix} \sigma_d I_{\bar{k}} & \\ & \tilde{\sigma}_d \end{bmatrix} \left[Q^{j_\star} \; V_d^{2j_\star} \right]^\top$$

be the SVD of $\mathbf{K}_{11}^d, d = 1,\dots,p$ and $(Q^{j_\star}, Q_1^{1j_\star}, \dots, Q_p^{1j_\star})$ be matrices whose columns form an orthogonal basis of the singular subspace pair $\overrightarrow{\Omega_{st}^{j_\star}}$.

3.2 Solution to multi-camera robot-world hand-eye calibration

As mentioned in Section 2.1, Assumption 2.1 can be verified by checking whether matrices $\{K_{11}^d\}_{d=1}^p$ share common right singular vectors. We consider the following preprocessing on $\{\{(A_d^{(i)}), B^{(i)})\}_{i=1}^n\}_{d=1}^p$ when it does not hold.

S1. The estimate of $X_\Diamond$. Notice that for any $d \in \{1,\dots,p\}$ and $1 \le i < j \le n$, from (1.1), we have

$$A_d^{(i)} X = Z_d B^{(i)}, \quad A_d^{(j)} X = Z_d B^{(j)},$$

which yields $Z_d = A_d^{(i)} X (B^{(i)})^{-1} = A_d^{(j)} X (B^{(j)})^{-1}$, or equally,

$$\left(A_d^{(j)}\right)^{-1} A_d^{(i)} X = X (B^{(j)})^{-1} B^{(i)}.$$

Denote $\widehat{A}_d^{ji} = (A_d^{(j)})^{-1} A_d^{(i)}$ and $\widehat{B}^{ji} = (B^{(j)})^{-1} B^{(i)}, 1 \le i < j \le n, d = 1,\dots,p$. The above equation becomes to

$$\widehat{A}_d^{ji} X = X \widehat{B}^{ji}, \quad 1 \le i < j \le n, \quad d = 1,\dots,p. \tag{3.21}$$

Eq. (3.21) takes the form of the hand-eye calibration problem. We can estimate $X_\Diamond$ by the subspace constrained least squares solution presented in [25].

S2. The estimate of $\{R_{Z_{d_\Diamond}}\}_{d=1}^p$. For each $d \in \{1,\dots,p\}$, define $R_{Z_d^{(i)}} \triangleq R_{A_d^{(i)}} R_X (R_{B^{(i)}})^{-1}$. Let $z_d^{(i)}, i = 1,\dots,n$ be the unit quaternion corresponding to $R_{Z_d^{(i)}}$. Then we set $R_{Z_{d_\Diamond}}$ as the rotation matrix with respect to unit quaternion $z_{d_\Diamond}$, where

$$\overrightarrow{z_{d_\Diamond}} = \arg\min_z \left\{ \frac{1}{n} \sum_{i=1}^n \left\| z - \overrightarrow{z_d^{(i)}} \right\|_2^2 : \text{s.t.} \|z\|_2 = 1 \right\} = \frac{\sum_{i=1}^n \overrightarrow{z_d^{(i)}}}{\left\| \sum_{i=1}^n \overrightarrow{z_d^{(i)}} \right\|_2}.$$

S3. Correct $\{\{R_{A_d^{(i)}}\}_{i=1}^n\}_{d=1}^p$ using $R_{X_\Diamond}$ and $\{R_{Z_{d_\Diamond}}\}_{d=1}^p$. Define

$$R_{A_d^{(i*)}} := R_{Z_{d_\Diamond}} R_{B^{(i)}} R_{X_\Diamond^{-1}}, \quad i = 1,\dots,n, \quad d = 1,\dots,p.$$

Notice that $\{\{A_d^{(i*)}\}_{i=1}^n\}_{d=1}^p$ and $B^{(i)}$ satisfies

$$R_{A_d^{(i*)}} R_{X_\Diamond} = R_{Z_{d_\Diamond}} R_{B^{(i)}}, \quad i = 1,\dots,n, \quad d = 1,\dots,p.$$

Assumption 2.1 holds on the corrected data set $\{\{(A_d^{(i*)}, B^{(i)})\}_{i=1}^n\}_{d=1}^p$ and we replace $A_d^{(i)}$ by $A_d^{(i*)}$.

Moreover, Lemma 3.2 indicates that when $\sigma_{\max}(K_{11}^d) \ne n$ for some $d \in \{1,\dots,p\}$, there exist $i \ne j$ such that $\sigma_d^i = 1$ and $\sigma_d^j = 0$. Motivated by the numerical results presented in [25, Tables 3 and 7] for (robot-world) hand-eye calibration, we correct the sign of Eq. (1.3) after solving $(x_{st\star}, z_{st\star}^1, \dots, z_{st\star}^p)$ as follows. We set $\{\{\sigma_d^i\}_{d=1}^p\}_{i=1}^n$ by

$$\sigma_d^i = \begin{cases} 0, & \text{if} \quad \left| \left(a_d^{(i)}\right)_{st} z_{st\star}^d - x_{st\star} b_{st}^{(i)} \right| \le \left| \left(a_d^{(i)}\right)_{st} z_{st\star}^d + x_{st\star} b_{st}^{(i)} \right|, \\ 1, & \text{otherwise}, \end{cases} \quad i = 1,\dots,n, \quad d = 1,\dots,p. \tag{3.22}$$

Define

$$\widetilde{M}_d^1 = \begin{bmatrix} (-1)^{\sigma_d^1} M\left(\left(a_d^{(1)}\right)_{st}\right) \\ \vdots \\ (-1)^{\sigma_d^n} M\left(\left(a_d^{(n)}\right)_{st}\right) \end{bmatrix}, \quad \widetilde{M}_d^2 = \begin{bmatrix} (-1)^{\sigma_d^1} M\left(\left(a_d^{(1)}\right)_{\mathcal{I}}\right) \\ \vdots \\ (-1)^{\sigma_d^n} M\left(\left(a_d^{(n)}\right)_{\mathcal{I}}\right) \end{bmatrix}, \quad d = 1,\dots,p.$$

Replace $\{K_{11}^d\}_{d=1}^p, \{K_{21}^d\}_{d=1}^p$, and $\{K_{12}^d\}_{d=1}^p$ by

$$\widetilde{K}_{11}^d = \big(\widetilde{M}_d^1\big)^\top W_1, \quad \widetilde{K}_{21}^d = \big(\widetilde{M}_d^2\big)^\top W_1, \quad \widetilde{K}_{12}^d = \big(\widetilde{M}_d^1\big)^\top W_2, \quad d=1,\dots,p. \tag{3.23}$$

Solve $\overrightarrow{x_{\mathcal{I}\star}}$ and $\{\overrightarrow{z_{\mathcal{I}\star}^d}\}_{d=1}^p$ as discussed in Section 3.1.2 by

$$\overrightarrow{x_{\mathcal{I}*}^c} = X_\perp u_*^c, \quad \text{and} \quad \overrightarrow{z_{\mathcal{I}*}^d} = Z_\perp v_{d*}^c, \quad d=1,\dots,p, \tag{3.24}$$

where u_*^c and $\{v_{d*}^c\}_{d=1}^p$ is the solution of the equation

$$\begin{aligned}
&\begin{bmatrix} npI_3 & -X_\perp^\top\big(\widetilde{K}_{11}^1\big)^\top Z_{1\perp} & \dots & -X_\perp^\top\big(\widetilde{K}_{11}^p\big)^\top Z_{p\perp} \\ -Z_{1\perp}^\top \widetilde{K}_{11}^1 X_\perp & nI_3 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ -Z_{p\perp}^\top \widetilde{K}_{11}^p X_\perp & \mathbf{0} & \dots & nI_3 \end{bmatrix} \begin{bmatrix} u^c \\ v_1^c \\ \vdots \\ v_p^c \end{bmatrix} \\
&= \begin{bmatrix} X_\perp^\top\Big(-pS_{21}^\top \overrightarrow{x_{st*}} + \sum_{d=1}^p \big(\widetilde{K}_{21}^d\big)^\top \overrightarrow{z_{st*}^d}\Big) \\ Z_{1\perp}^\top\big(\widetilde{K}_{12}^1 \overrightarrow{x_{st*}} - \widetilde{S}_{12}^1 \overrightarrow{z_{st*}^1}\big) \\ \vdots \\ Z_{p\perp}^\top\big(\widetilde{K}_{12}^p \overrightarrow{x_{st*}} - \widetilde{S}_{12}^p \overrightarrow{z_{st*}^p}\big) \end{bmatrix}.
\end{aligned} \tag{3.25}$$

We summarize the whole procedure in Algorithm 1.

Remark 3.3. Algorithm 1 involves computing the SVD of matrices $\{K_{11}^d\}_{d=1}^p \subseteq \mathbb{R}^{4\times 4}$, and may require inverse of $(npI_{4-\bar{k}} - \sum_{d=1}^p \Gamma_d \Gamma_d^\top / n)$ (where $\bar{k}=1$ or 2) or computation of the orthogonal complement of $\overrightarrow{x_{st\star}} \in \mathbb{R}^{4\times 1}$ (or $\overrightarrow{z_{st\star}^d}$). Owing to the small dimensionality of these objects, each of these operations can be implemented with a negligible number of floating-point operations (FLOPs). In contrast, the computation of $\{\{(\hat{A}_d^{(i)}, \hat{B}^{(i)})\}_{i=1}^n\}_{d=1}^p$, $\{\{(\check{a}_d^{(i)}, \check{b}^{(i)})\}_{i=1}^n\}_{d=1}^p$, $\{K_{11}^d\}_{d=1}^p$, $\{K_{12}^d\}_{d=1}^p$, $\{K_{21}^d\}_{d=1}^p$, S_{21}, $\{S_{12}^d\}_{d=1}^p$, and $\{\{\sigma_d^i\}_{i=1}^n\}_{d=1}^p$ dominate the computational cost, requiring at most $347np+194n-32p-16$ FLOPs. When Assumption 2.1 holds, the overall computation of Algorithm 1 is bounded by $781np+184n+623p+346+p\cdot f_{svd}+(p+1)\cdot f_\perp$, where f_{svd} and $f_\perp$ denote the FLOP counts for the SVD of a 4×4 matrix and the orthogonal complement of a 4×1 vector, respectively. If the pre-processing procedure is applied on $\{\{(A_d^{(i)}), B^{(i)})\}_{i=1}^n\}_{d=1}^p$, the additional cost is primarily attributable to computing matrices $\widehat{A}_d^{ji}$ and $\widehat{B}^{ji}$ for $1\le i<j\le n, d=1,\dots,p$, which requires at most $56n(p+1)(n-1)$ FLOPs. Consequently, the worst-case computational complexity of Algorithm 1 is $\mathcal{O}(n^2p)$. Under Assumption 2.1, the complexity reduces to $\mathcal{O}(np)$.

Algorithm 1 The Closed-Form Dual Quaternion Solution for the Multi-Camera Robot-World Hand-Eye Calibration Problem.

Input: $\{\{(A_d^{(i)},B^{(i)})\}_{i=1}^n\}_{d=1}^p$.

1: Compute $\{\{(\hat{A}_d^{(i)},\hat{B}^{(i)})\}_{i=1}^n\}_{d=1}^p$ and $\{K_{11}^d\}_{d=1}^p$.

2: Compute the SVD of $K_{11}^d = U_d S_d V_d, d=1,\ldots,p$.

3: **if** $\{K_{11}^d\}_{d=1}^p$ have common right singular vector **then**

4: **if** $\sigma_{\max}(K_{11}^d)=n$ for all $d=1,\ldots,p$ **then**

5: **if** $k=1$ **then**

6: Compute $(\overrightarrow{x_{st\star}},\overrightarrow{x_{\mathcal{I}\star}},\overrightarrow{z_{st\star}^d},\overrightarrow{z_{\mathcal{I}\star}^d})$ as in (3.16a)-(3.16d).

7: **else if** $k=2$ **then**

8: Compute $(\overrightarrow{x_{st\star}},\overrightarrow{x_{\mathcal{I}\star}},\overrightarrow{z_{st\star}^d},\overrightarrow{z_{\mathcal{I}\star}^d})$ as in (3.17a)-(3.17d).

9: **end if**

10: **else if** there exists $d\in\{1,\ldots,p\}$ such that $\sigma_{\max}(K_{11}^d)\neq n$ **then**

11: Compute $T_j=\sum_{d=1}^p (Q_d^{1j})^\top (K_{12}^d+K_{21}^d)Q^j$.

12: Compute $j_\star = \mathrm{argmin}_j\{|\lambda_{\max}(\mathrm{Sym}(T_j))|\}$.

13: **if** $\lambda_{\max}(\mathrm{Sym}(T_j))\neq 0$ **then**

14: Compute the unit eigenvector y_j corresponding to $\lambda_{\max}(\mathrm{Sym}(T_j))$.

15: Compute $\overrightarrow{x_{st\star}} = Q^{j_*}y_{j_\star}, \overrightarrow{z_{st\star}^d} = Q_d^{1j_\star}y_{j_\star}$, and orthogonal complements $X_\perp$ and $\{Z_{d\perp}\}_{d=1}^p$.

16: Determine $\{\{(-1)^{\sigma_d^i}\}_{d=1}^p\}_{i=1}^n$ as in (3.22).

17: Compute $\{\widetilde{K}_{11}^d\}_{d=1}^p, \{\widetilde{K}_{21}^d\}_{d=1}^p, \{\widetilde{K}_{12}^d\}_{d=1}^p$, and $\{\widetilde{S}_{12}^d\}_{d=1}^p$ as in (3.23).

18: Compute $(u_*,v_{1*}^c,\ldots,v_{p*}^c)$ as the solution of Eq. (3.25).

19: Compute $\overrightarrow{x_{\mathcal{I}\star}^c}$ and $\{\overrightarrow{z_{\mathcal{I}\star}^d}\}_{d=1}^p$ as in (3.24).

20: **else**

21: Compute $(\overrightarrow{x_{st\star}},\overrightarrow{x_{\mathcal{I}\star}},\overrightarrow{z_{st\star}^d},\overrightarrow{z_{\mathcal{I}\star}^d})$ as in Remark 3.2.

22: **end if**

23: **end if**

24: **else**

25: Estimate $X_\Diamond$ and $\{Z_{d\Diamond}\}_{d=1}^p$.
Correct $\{\{A_d^{(i*)}\}_{i=1}^n\}_{d=1}^p$.
Go to Step 1 with data set $\{\{(A_d^{(i)},B^{(i)})\}_{i=1}^n\}_{d=1}^p \leftarrow \{\{(A_d^{(i*)},B^{(i)})\}_{i=1}^n\}_{d=1}^p$.

26: **end if**

27: $\check{x}=x_{st}+x_{\mathcal{I}}\epsilon$, $\check{z}_d=z_{st}^d+z_{\mathcal{I}}^d\epsilon$, x_{st}, $x_{\mathcal{I}}$, z_{st}^d, and $z_{\mathcal{I}}^d$ are quaternions corresponding to $\overrightarrow{x_{st\star}}$, $\overrightarrow{x_{\mathcal{I}\star}}$, $\overrightarrow{z_{st\star}^d}$, and $\overrightarrow{z_{\mathcal{I}\star}^d}$, $d=1,\ldots,p$ respectively.

28: Compute R_X and R_{Z_d} by (2.2) with x_{st} and z_{st}^d, $t_X=2x_{\mathcal{I}}x_{st}^*$, and $t_{Z_d}=2z_{\mathcal{I}}^d(z_{st}^d)^*$, $d=1,\ldots,p$.

29: **return** (R_X,t_X), $\{(R_{Z_d},t_{Z_d})\}_{d=1}^p$.

4 Numerical experiment

In this section, we evaluate the effectiveness and efficiency of our proposed method to solve the multi-camera robot-world hand-eye calibration problem on both simulated and real data. All experiments were performed with MATLAB R2023a on an HP StarBook 14 laptop.

4.1 Simulated data

Algorithm 1 (denoted as "ours") is compared with the methods introduced in [20]. There are two types of cost functions (c1 and c2), which can be further categorized into simultaneous and separable forms, rotation parameters represented in three different forms: axis-angle, Euler angles, and quaternions, were considered in [20]. These methods are named according to the convention "cost function type_rotation parameter form_separability" (denoted as "c1_euler_simultaneous", "c1_axis_simultaneous", "c1_ quaternion_ simultaneous", "c1_euler_separable", "c1_axis_separable", "c1_quaternion_separable", "c2_euler_simultaneous", "c2_axis_simultaneous", "c2_quaternion_simultaneous", "c2_ euler_separable", "c2_axis_separable", "c2_quaternion_separable"). Additionally, our algorithm is also compared with the method proposed by Wang [22] (denoted as "Wang2022"). We employ the following error metrics to evaluate and compare the performance of different methods:

a) Rotation error on X: $e_{R_X} = \|\hat{R}_X - R_X\|_F$,
b) Translation error on X: $e_{t_X} = \|\hat{t}_X - t_X\|_2$,
c) Average rotation error on $Z_d : e_{R_Z} = \sum_{i=1}^{3} \|\hat{R}_{Z_i} - R_{Z_i}\|_F/3$,
d) Average translation error on Z_d: $e_{t_Z} = \sum_{i=1}^{3} \|\hat{t}_{Z_i} - t_{Z_i}\|_2/3$,

where $\hat{\cdot}$ denotes results returned by each method.

4.1.1 Noiseless simulated data

In this test, we generate $\{\{(A_d^{(i)}, B^{(i)}, Z_d, X)\}_{i=1}^{25}\}_{d=1}^{3}$ as follows. We set the rotational component of X as the identity matrix and generate the rotational components of $B^{(i)}$ and Z_d from the random unit quaternion following formula (2.2), where elements of unit quaternion follow the uniform distribution on the surface of a 4-sphere as described in [14]. The translational components of $X, B^{(i)}$, and Z_d follow the uniform distribution within the interval $[-0.25, 0.25]$. We construct $A_d^{(i)}$ by

$$A_d^{(i)} := Z_d B^{(i)} X^{-1}, \quad i = 1, \ldots, 25, \quad d = 1,2,3. \tag{4.1}$$

From (4.1), we have $A_d^{(i)} X = Z_d B^{(i)}$, which yields $R_{A_d^{(i)}} R_X = R_{Z_d} R_{B^{(i)}}, i = 1, \ldots, 25, d = 1,2,3$. Hence, Assumption 2.1 holds by noting that $R_{A_d^{(i)}}^\top = R_{A_d^{(i)}}^{-1}$ and $R_{B^{(i)}}^\top = R_{B^{(i)}}^{-1}$. Table 1 reports

Table 1: Average error results for noiseless simulated data over 100 trials.

Method	Time (s)	e_{R_X}	e_{R_Z}	e_{t_X}	e_{t_Z}
c1_euler_simultaneous	1.6149	1.24E-04	4.19E-04	1.02E-05	1.78E-05
c1_axis_simultaneous	0.2569	2.51E-06	2.25E-06	1.83E-06	5.68E-06
c1_quaternion_simultaneous	0.6082	2.07E-06	8.31E-07	2.74E-06	8.71E-06
c1_euler_separable	1.9431	8.50E-02	8.53E-02	2.29E-03	1.87E-03
c1_axis_separable	0.4998	2.46E-06	5.69E-06	5.03E-07	1.14E-06
c1_quaternion_separable	0.4309	2.52E-06	5.76E-06	4.97E-07	1.12E-06
c2_euler_simultaneous	1.3469	1.35E-04	4.20E-04	9.60E-06	1.49E-05
c2_axis_simultaneous	0.2690	2.34E-06	1.99E-06	2.32E-06	6.36E-06
c2_quaternion_simultaneous	0.5623	1.93E-06	8.27E-07	2.95E-06	8.87E-06
c2_euler_separable	1.4595	2.84E-02	2.87E-02	1.86E-03	4.01E-04
c2_axis_separable	0.4794	2.48E-06	5.68E-06	6.20E-07	8.42E-07
c2_quaternion_separable	0.4476	2.56E-06	5.77E-06	5.96E-07	7.65E-07
Wang2022	0.0029	1.91E-15	6.78E-16	1.19E-16	1.49E-16
ours	0.0086	1.89E-16	1.64E-15	8.50E-17	3.67E-16

average error results generated by the simulated data over 100 trials. It can be observed from Table 1 that our presented method and Wang2022 achieve comparable performance in terms of both runtime and error metrics, while outperforming all other comparison methods. It should be noted that Wang2022 employs the Kronecker product to formulate the transformation equation, which represents a conventional tool for handling such equations. The comparable results obtained by our proposed method and Wang2022 provide partial validation for the rationality of our subspace constrained least squares solution. We also employ the axis-angle representation of rotation matrices to generate the rotational component of X, where the rotation axis is rotated around the y-axis by $\pi/8, \pi/4, \pi/2$, and π, respectively. The obtained results are similar to those shown in Table 1 so we omit it.

4.1.2 Noisy simulated data

In this test, we generate $B^{(i)}$, Z_d, and the translational components of X as in Section 4.1, and generate the rotational component of X from the random unit quaternion as for Z_d in Section 4.1. We then introduce noise into the transformation matrices after generating $\{\{(A_d^{(i)}\}_{i=1}^{25}\}_{d=1}^{3}$ by (4.1). Let $q_{A_d^{(i)}}$ be the unit quaternion representation of $R_{A_d^{(i)}}$. Then we first compute the noisy unit quaternion using the formula

$$q_{A_{dnoisy}^{(i)}} = \frac{q_{A_d^{(i)}} + \sigma_{\text{noise}} \cdot n_4}{\left| q_{A_d^{(i)}} + \sigma_{\text{noise}} \cdot n_4 \right|},$$

where $\vec{n}_4 \in \mathbb{R}^4$ follows the standard normal distribution and $\sigma_{\text{noise}} \in (0,0.2)$. Finally, we generate the noisy rotation matrices $\{\{A_d^{(i)}\}_{i=1}^{25}\}_{d=1}^{3}$ from $q_{A_{dnoisy}^{(i)}}$ using formula (2.2). Assumption 2.1 no longer holds because the rotational component of the matrix $A_d^{(i)}$ is corrupted by additive noise. The pre-processing procedure (Step 25 in Algorithm 1) is invoked.

The numerical results presented in Section 4.1.1 demonstrate that the performance of methods c_1 and c_2 from [20] are highly similar for different choices of rotation parameterization and "axis" performs a little better. Hence, we report only the axis-based parameterization in this test. Fig. 2 displays the average errors over 100 trials under varying noise levels. The key observations are as follows:

i) For the rotation errors in X and $\{Z_d\}_{d=1}^3$, the proposed method "ours" performs comparably to Wang2022 and outperforms the other comparison methods.

ii) For the translation errors in X and $\{Z_d\}_{d=1}^3$, the proposed method "ours" outperforms all comparison methods.

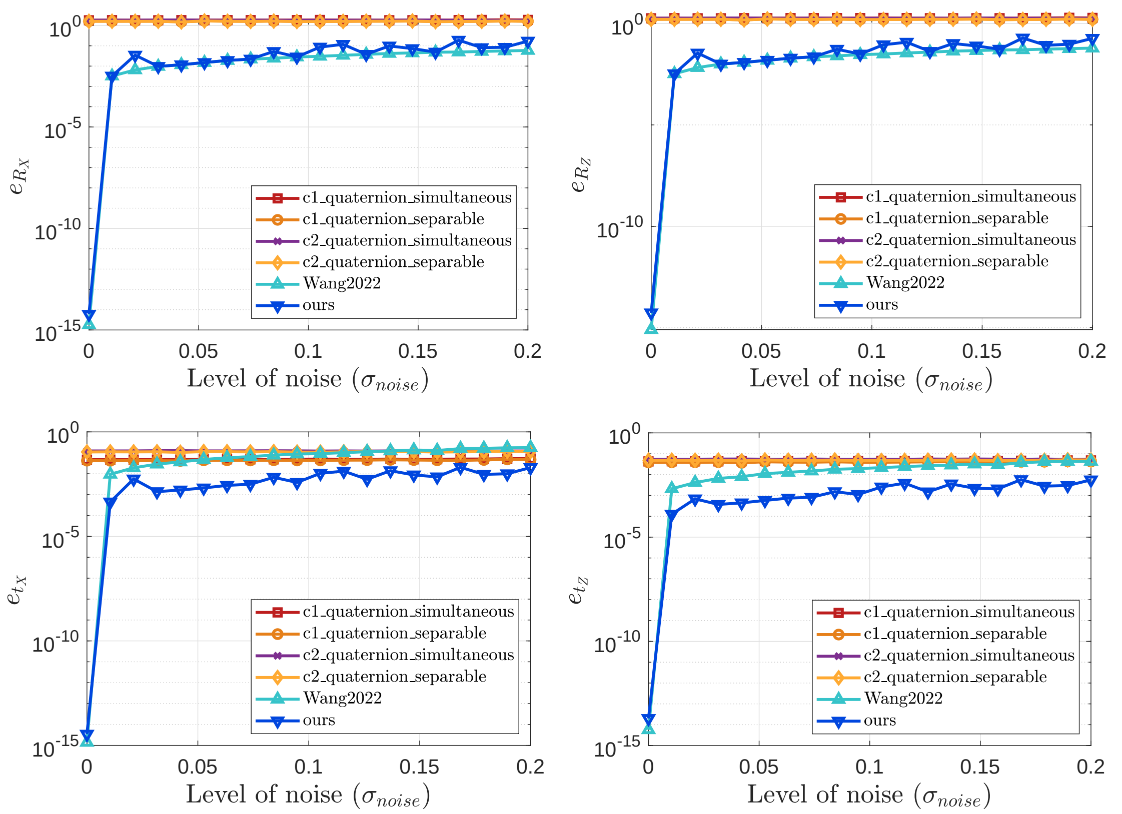

Figure 2: Average results when noise level σ_{noise} varying from 0 to 0.2 over 100 trials.

4.2 Real data

In this section, we employ real multi-camera datasets "Dataset 6", "Dataset 7", and "Dataset 8" from Tabb [20] for testing Algorithm 1. We compare "ours" with the "c1" and "c2" class methods, and the "rp1" and "rp2" class methods (different parameters are used to represent rotatio, denoted as "rp1_euler", "rp1_axis", "rp1_quaternion", "rp2_euler", "rp2_axis", and "rp2_quaternion") in Tabb [20], the two methods in Eurico [15] (denoted as "Eurico2021-1" and "Eurico2021-2", and the method proposed by Wang [22]. As in [20], we consider the following error metrics:

a) Average rotation error on X in terms of matrix representation

$$e_{R_1} = \frac{1}{pn}\sum_{d=1}^{p}\sum_{i=1}^{n}\left\| R_{A_d^{(i)}}\hat{R}_X - \hat{R}_{Z_d}R_{B^{(i)}}\right\|_F^2.$$

b) Average rotation error on X in terms of axis-angle representation

$$e_{R_2} = \frac{1}{pn}\sum_{d=1}^{p}\sum_{i=1}^{n}\text{angle}\big(R_d^{(i)}\big),$$

where $R_d^{(i)} = (R_{Z_d}R_{B^{(i)}})^\top (R_{A_d^{(i)}}R_X)$.

c) Average translation error

$$e_t = \frac{1}{pn}\sum_{d=1}^{p}\sum_{i=1}^{n}\left\| \Big(R_{A_d^{(i)}}t_X + t_{A_d^{(i)}}\Big) - \Big(R_{Z_d}t_{B^{(i)}} + t_{Z_d}\Big)\right\|^2.$$

d) Average transformation error

$$e_c = \frac{1}{pn}\sum_{d=1}^{p}\sum_{i=1}^{n}\|A_d^{(i)}X - Z_dB^{(i)}\|_F^2.$$

e) The reprojection root mean squared error

$$\text{rrmse}_d = \left(\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\left\|\vec{\mathbf{x}}_{ij} - f\Big(K, \big[Z_dB^{(i)}X^{-1}\big]_{3\times 4}\overrightarrow{\mathcal{X}}_j\Big)\right\|^2\right)^{1/2},$$

where m is the number of grid points in the checkerboard (calibration board), $\vec{x}_{ij}$ denotes the checkerboard points in the pixel coordinate system, $f(K,v) = Kv$ is the projection function that projects the checkerboard points in the world coordinate system to the pixel coordinate system, $K \in \mathbb{R}^{3\times 3}$ is the intrinsic parameter matrix, $[\cdot]_{3\times 4}$ denotes the top three row of matrix "$\cdot$", and $\overrightarrow{\mathcal{X}}_j \in \mathbb{R}^{4\times 1}$ represents the points on the board in the world coordinate system.

Tables 2-4 report results for Datasets 6 to 8. We observed that Assumption 2.1 is violated by Dataset 6, but is satisfied by Datasets 7 and 8. It can be seen that:

i) The performs of Wang2022 and "ours" are significantly faster compared to other methods, while the computation times of the rp1- and rp2- class methods and "Eurico2021-1", "Eurico2021-2" are considerably longer. A potential reason is that these methods employ iterative approach to solve nonlinear programming problems, where the performance is sensitive to the initial starting point.

ii) For each dataset, our proposed method "ours" reaches the lowest average rotation errors in terms of e_{R_1} and e_{R_2}.

iii) The average errors of translation vector e_t obtained by "ours" are lower than most comparison algorithms.

iv) The proposed method "ours" achieves lower reprojection root mean squared errors compared to quaternion-based methods in both c1- and c2- classes.

Table 2: Results for Dataset 6, where $(n,p)=(13,2)$.

Method	Time (s)	e_{R_1}	e_{R_2}	e_t	e_c	$rrmse_1$	$rrmse_2$
c1_euler_simultaneous	7.83	2.98E-05	2.06E-01	9.95E-05	1.29E-04	5.01E+00	5.01E+00
c1_axis_simultaneous	8.14	2.59E-05	1.87E-01	4.19E-05	6.78E-05	3.22E+00	3.20E+00
c1_quaternion_simultaneous	10.62	3.52E-05	2.27E-01	4.49E-04	4.84E-04	1.13E+01	1.13E+01
c2_euler_simultaneous	11.21	2.37E-05	1.86E-01	1.06E-04	1.30E-04	3.21E+00	3.24E+00
c2_axis_simultaneous	8.66	2.41E-05	1.88E-01	1.08E-04	1.32E-04	3.23E+00	3.26E+00
c2_quaternion_simultaneous	13.07	3.84E-05	2.40E-01	7.16E-04	7.54E-04	1.27E+01	1.27E+01
c1_euler_separable	22.07	1.54E-04	4.52E-01	6.38E-03	6.54E-03	2.21E+01	2.24E+01
c1_axis_separable	21.59	2.27E-05	1.76E-01	1.96E-02	1.96E-02	3.23E+01	3.33E+01
c1_quaternion_separable	32.03	2.29E-05	1.75E-01	2.55E-02	2.55E-02	3.73E+01	3.87E+01
c2_euler_separable	30.62	3.45E-05	2.17E-01	4.52E-02	4.53E-02	5.33E+01	5.56E+01
c2_axis_separable	25.77	2.27E-05	1.76E-01	1.97E-02	1.97E-02	3.23E+01	3.34E+01
c2_quaternion_separable	21.65	2.29E-05	1.75E-01	2.57E-02	2.57E-02	3.73E+01	3.87E+01
rp1_euler	1275.82	1.60E-04	4.82E-01	3.86E-04	5.45E-04	6.84E-01	6.96E-01
rp1_axis	1233.82	1.58E-04	4.78E-01	3.82E-04	5.39E-04	6.83E-01	6.95E-01
rp1_quaternion	1337.55	2.54E-03	2.03E+00	6.00E-03	8.54E-03	3.23E+00	3.18E+00
rp2_euler	946.34	2.76E-04	6.47E-01	6.76E-04	9.52E-04	1.40E+00	1.42E+00
rp2_axis	955.28	2.08E-04	5.55E-01	5.19E-04	7.27E-04	1.39E+00	1.41E+00
rp2_quaternion	3114.99	2.54E-03	2.03E+00	6.00E-03	8.54E-03	3.23E+00	3.18E+00
Eurico2021-1	490.66	2.34E-03	1.94E+00	5.68E-03	8.02E-03	5.45E+00	5.13E+00
Eurico2021-2	507.30	2.34E-03	1.94E+00	5.71E-03	8.05E-03	5.40E+00	5.33E+00
Wang2022	0.04	2.27E-05	1.77E-01	1.77E-02	1.77E-02	3.02E+01	2.93E+01
ours	0.27	2.27E-05	1.76E-01	7.51E-03	7.54E-03	1.80E+01	1.77E+01

Table 3: Results for Dataset 7, where $(n,p)=(33,3)$.

Method	Time (s)	e_{R_1}	e_{R_2}	e_t	e_c	$rrmse_1$	$rrmse_2$	$rrmse_3$
c1_euler_simultaneous	24.91	3.23E-05	2.14E-01	6.46E-05	9.70E-05	2.57E+00	2.56E+00	1.68E+00
c1_axis_simultaneous	21.84	2.61E-05	1.89E-01	5.06E-05	7.67E-05	2.37E+00	2.39E+00	1.55E+00
c1_quaternion_simultaneous	26.16	3.76E-05	2.24E-01	1.83E-04	2.20E-04	5.30E+00	5.22E+00	3.41E+00
c2_euler_simultaneous	23.73	2.85E-05	1.95E-01	7.87E-05	1.07E-04	1.35E+00	1.36E+00	9.32E-01
c2_axis_simultaneous	21.13	2.84E-05	1.94E-01	7.66E-05	1.05E-04	1.34E+00	1.34E+00	9.23E-01
c2_quaternion_simultaneous	27.57	2.90E-05	1.99E-01	2.23E-04	2.52E-04	4.95E+00	4.88E+00	3.17E+00
c1_euler_separable	44.81	3.07E-05	2.08E-01	4.41E-05	7.49E-05	2.71E+00	2.72E+00	1.79E+00
c1_axis_separable	51.28	2.41E-05	1.79E-01	4.43E-05	6.84E-05	2.52E+00	2.58E+00	1.69E+00
c1_quaternion_separable	52.95	2.44E-05	1.81E-01	4.05E-05	6.48E-05	2.57E+00	2.64E+00	1.71E+00
c2_euler_separable	44.81	2.41E-05	1.79E-01	4.41E-05	6.82E-05	2.50E+00	2.57E+00	1.67E+00
c2_axis_separable	44.09	2.41E-05	1.79E-01	4.49E-05	6.89E-05	2.50E+00	2.56E+00	1.67E+00
c2_quaternion_separable	61.92	2.44E-05	1.81E-01	4.10E-05	6.54E-05	2.55E+00	2.62E+00	1.69E+00
rp1_euler	3631.24	6.57E-05	3.12E-01	1.81E-04	2.47E-04	1.06E+00	1.09E+00	7.70E-01
rp1_axis	3593.28	6.88E-05	3.18E-01	1.89E-04	2.57E-04	1.06E+00	1.09E+00	7.67E-01
rp1_quaternion	4177.53	1.69E-04	5.03E-01	5.21E-04	6.90E-04	1.62E+00	1.62E+00	1.10E+00
rp2_euler	200.23	3.38E-05	2.24E-01	1.03E-04	1.37E-04	1.18E+00	1.19E+00	8.37E-01
rp2_axis	192.94	3.26E-05	2.19E-01	9.63E-05	1.29E-04	1.18E+00	1.18E+00	8.34E-01
rp2_quaternion	2908.34	5.23E-04	9.09E-01	1.49E-03	2.01E-03	2.03E+00	2.00E+00	1.36E+00
Eurico2021-1	2583.59	1.95E-04	5.38E-01	6.87E-04	8.83E-04	3.26E+00	3.55E+00	7.35E-01
Eurico2021-2	2756.49	1.93E-04	5.35E-01	6.79E-04	8.72E-04	3.23E+00	3.50E+00	7.35E-01
Wang2022	0.04	2.41E-05	1.79E-01	4.54E-05	6.94E-05	2.50E+00	2.56E+00	1.67E+00
ours	0.30	2.41E-05	1.79E-01	4.57E-05	6.97E-05	2.48E+00	2.54E+00	1.67E+00

5 Conclusion

In this paper, we study the definition and expression of the closed-form solutions to multi-unit dual quaternion vector equations by combining the application of multi-camera calibration with the correspondence between transformation matrices and dual quaternions. The definition of the subspace constrained least solution is motivated by a key property satisfied by the exact solution of the rotation component in the transformation matrix. This work extends the subspace constrained least squares solution for robot-world hand-eye calibration presented in [25]. To ensure the validity of the underlying basis assumption in real-world data, we further propose a correction strategy. Extensive numerical experiments – conducted on both synthetic and real-world datasets – validate the effectiveness of the proposed method.

Table 4: Results for Dataset 8, where $(n,p)=(31,3)$.

Method	Time (s)	e_{R_1}	e_{R_2}	e_t	e_c	$rrmse_1$	$rrmse_2$	$rrmse_3$
c1_euler_simultaneous	33.19	3.21E-05	2.08E-01	6.68E-05	9.89E-05	2.78E+00	2.90E+00	1.40E+00
c1_axis_simultaneous	22.13	2.59E-05	1.85E-01	5.52E-05	8.11E-05	2.61E+00	2.77E+00	1.32E+00
c1_quaternion_simultaneous	26.24	3.93E-05	2.27E-01	1.78E-04	2.17E-04	6.02E+00	5.98E+00	2.94E+00
c2_euler_simultaneous	24.15	2.71E-05	1.86E-01	7.90E-05	1.06E-04	1.64E+00	1.80E+00	8.56E-01
c2_axis_simultaneous	21.10	2.71E-05	1.86E-01	7.72E-05	1.04E-04	1.63E+00	1.80E+00	8.53E-01
c2_quaternion_simultaneous	29.11	2.85E-05	1.93E-01	2.25E-04	2.54E-04	6.20E+00	6.13E+00	3.02E+00
c1_euler_separable	47.40	3.05E-05	2.03E-01	4.81E-05	7.85E-05	2.95E+00	3.10E+00	1.48E+00
c1_axis_separable	44.35	2.37E-05	1.76E-01	4.82E-05	7.20E-05	2.85E+00	3.03E+00	1.44E+00
c1_quaternion_separable	55.12	2.41E-05	1.77E-01	4.59E-05	6.99E-05	2.93E+00	3.09E+00	1.46E+00
c2_euler_separable	43.66	2.37E-05	1.76E-01	4.97E-05	7.35E-05	2.74E+00	2.87E+00	1.38E+00
c2_axis_separable	44.36	2.37E-05	1.76E-01	5.00E-05	7.37E-05	2.73E+00	2.87E+00	1.38E+00
c2_quaternion_separable	56.87	2.41E-05	1.77E-01	4.78E-05	7.18E-05	2.80E+00	2.94E+00	1.41E+00
rp1_euler	3349.04	5.11E-05	2.74E-01	1.49E-04	2.00E-04	1.33E+00	1.51E+00	7.42E-01
rp1_axis	3345.76	5.11E-05	2.72E-01	1.48E-04	1.99E-04	1.34E+00	1.52E+00	7.44E-01
rp1_quaternion	4171.58	2.78E-04	6.59E-01	8.41E-04	1.12E-03	2.11E+00	2.24E+00	1.29E+00
rp2_euler	162.78	2.86E-05	1.98E-01	8.46E-05	1.13E-04	1.52E+00	1.70E+00	8.27E-01
rp2_axis	117.33	2.88E-05	1.97E-01	8.32E-05	1.12E-04	1.51E+00	1.70E+00	8.25E-01
rp2_quaternion	1361.96	5.48E-04	9.37E-01	1.75E-03	2.30E-03	3.51E+00	3.50E+00	2.32E+00
Eurico2021-1	2266.11	2.30E-04	5.72E-01	9.01E-04	1.13E-03	6.06E+00	4.37E+00	7.10E-01
Eurico2021-2	2898.65	2.43E-04	5.89E-01	9.27E-04	1.17E-03	5.75E+00	4.98E+00	7.01E-01
Wang2022	0.04	2.37E-05	1.76E-01	4.88E-05	7.26E-05	2.80E+00	2.95E+00	1.41E+00
ours	0.31	2.37E-05	1.76E-01	5.00E-05	7.37E-05	2.72E+00	2.87E+00	1.38E+00

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