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SPLIT-STEP FORWARD MILSTEIN METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider the problem of computing numerical solutions for stochastic differential equations (SDEs) of Itô form. A fully explicit method, the split-step forward Milstein (SSFM) method, is constructed for solving SDEs. It is proved that the SSFM method is convergent with strong order $\gamma = 1$ in the mean-square sense. The analysis of stability shows that the mean-square stability properties of the method proposed in this paper are an improvement on the mean-square stability properties of the Milstein method and three stage Milstein methods.

Key Words. Stochastic differential equation, Explicit method, Mean convergence, Mean square convergence, Stability, Numerical experiment.

1. Introduction

In this paper, we consider d-dimensional Itô stochastic differential equations (SDEs) of the following form

$$(1)\qquad \begin{cases} dY(t) = f\left(Y(t)\right)dt + g\left(Y(t)\right)dW(t), t \in [t_0, T], \\ Y(t_0) = Y_0, \end{cases}$$

where $Y(t)$ is a random variable with value in R^d, $f : R^d \to R^d$ is called the drift function, $g : R^d \to R^d$ is called the diffusion function, and $W(t)$ is a Wiener process whose increments $\Delta W(t) = W(t + \Delta t) - W(t)$ are Gaussian random variables $N(0, \Delta t)$.

Stochastic differential equations have come to play an important role in many branches of science and industry. The importance of numerical methods for SDEs can not be overemphasized as SDEs are used in modeling of many chemical, physical, biological and economical systems [2]. SDEs arising in many applications can not be solved analytically, hence one needs to develop effective numerical methods for such systems. In recent years, many efficient numerical methods have been constructed for solving different type of SDEs with different properties, for example, Wang et al.[8], Higham [1], Platen [5], Wang [7]. These numerical schemes are now abundant and classified according to their type (strong or weak) and order of convergence [2]. In this paper, we focus our attention on schemes that converge in the strong sense. The concepts of strong convergence concern the accuracy of a numerical method over a finite interval $[t_0, T]$ for small step sizes Δt.

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2. Motivation and background

Milstein et al.[3] studied the fully implicit methods for Itô SDEs. The fully implicit methods have been constructed for stiff SDEs where some components of a stiff multidimensional system have a vanishing drift term for which semi-implicit methods can not improve the stability of the numerical solution. In this paper, we propose to solve SDEs of type (1). For such equations semi-implicit methods are applicable, however the Newton iteration is necessary for semi-implicit methods, which makes such methods expensive. Hence to avoid this issue, we need explicit methods.

In order to improve the stability properties of the explicit methods for solving SDEs, some attempts have been made to propose modified explicit Euler and Milstein methods. For example, Wang et al.[8] studied the split-step forward methods for Itô SDEs. Wang [7] studied the three-stage stochastic Runge-Kutta methods for Stratonovich SDEs. In this paper, as a fully explicit method, we discuss the split-step forward Milstein (SSFM) method which has better stability properties than the Milstein and three-stage Milstein methods. The SSFM method has unbounded stability region whereas the Milstein method has bounded stability region. In Section 5, an example is presented in order to show that the accuracy and convergence property of SSFM method are better than that of the Milstein method and three stage Milstein methods.

This paper is organized as follows. In Section 3, we introduce some notation and hypotheses of Eq. (1). In the same section we discuss the convergence of the SSFM method. The stability properties of the SSFM method are reported in Section 4. In Section 5, examples are presented in order to illustrate the applicability of our results. Conclusions are given in Section 6.

3. Numerical analysis of the method

3.1. General framework. Let there be a common underlying complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with index $t \in [t_0, T]$ on which the vector stochastic process $Y(t)$ consists of component-wise collections of random variables. Along a given sample path w, $Y(t; w)$ denotes the value taken by the random variable Y_t. We consider the numerical integration of the initial value Itô SDEs with noise in the form of

$$dY(t) = f(Y(t))\,dt + g(Y(t))\,dW(t) \tag{2}$$

with

$$Y(t_0) = Y_0.$$

Let $|x|$ be the Euclidean norm of vector $x \in R^d$. Let $\mathbf{E}$ denote the expectation.

3.2. Assumptions. Let gg' denote a vector of length d with ith component equal to $(gg')_i = \sum_{k=1}^{d} g_k \frac{\partial g_i}{\partial y_k}$.
The following assumptions can be found in [4, 8] when considering the convergence properties of splitting schemes for Itô SDEs.
A1. The functions f, g and gg' satisfy the Lipschitz condition; that is, there exists a positive constant L_1 such that for any x_1, $x_2 \in R^d$,

$$|\, f(x_1) - f(x_2) \,| \quad \leq \quad L_1 \,|\, x_1 - x_2 \,|, \tag{3}$$

$$|\, g(x_1) - g(x_2) \,| \quad \leq \quad L_1 \,|\, x_1 - x_2 \,|, \tag{4}$$