INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 4, Pages 688–714

FINITE ELEMENT APPROXIMATION OF OPTIMAL CONTROL FOR SYSTEM GOVERNED BY IMMISCIBLE DISPLACEMENT IN POROUS MEDIA

YANZHEN CHANG, WEIDONG CAO, DANPING YANG, TONGJUN SUN, AND WENBIN LIU

(Communicated by Yanping Lin)

Abstract. In this work, we study the finite element approximation of a model optimal control problem governed by the system describing the two-phase incompressible flow in porous media, with the aim to maximize production of oil from petroleum reservoirs. We first give the proof for the existence of the solutions of the control problem. The optimality conditions are then obtained and the existence of the solution of the adjoint equations is shown. After that we consider its finite element approximation. We have obtained the a priori error estimates with the optimal orders and minimum regularity requirements. Finally, we carry out some numerical tests.

Key words. Finite element approximation, optimal control problem, immiscible displacement.

1. Motivation

The field of petroleum engineering is concerned with the search for ways to extract more oil and gas from the earths subsurface. In a world in which an increase in production of tenths of a percentage may result into a growth in profit of millions of dollars, no stone is left unturned.

A common technique in oil recovery, known as “water flooding”, makes use of two types of wells: injection and production wells. The production wells are used to transport liquid and gas from the reservoir to the subsurface. The injection wells inject water into the oil reservoir with the aim to push the oil towards the production wells and keep up the pressure difference. The oil-water front progresses toward the production wells until water breaks through into the production stream. An increasing amount of water is used, while the oil production rate diminishes, until at some time the recovery is no longer profitable and production is brought to an end. Using water flooding, up to about 35 percent of the oil can be recovered economically. Due to the strongly heterogeneous nature of oil reservoirs, the oil-water front does not travel uniformly towards the production wells, but is usually irregularly shaped. As a result, large amounts of oil may be still trapped within the reservoir as water breakthrough occurs and production is brought to an end.

Recent advances in petroleum engineering allow for advanced well downhole measurement and control devices, which expand the possibilities to manipulate and control fluid flow paths through the oil reservoir. The ability to manipulate the progression of the oil-water front provides the possibility to search for a control strategy that will result in maximization of oil recovery. A straightforward approach to find such a control strategy is to use the optimal control technique to increase recovery by delaying water breakthrough and increasing sweep, based on a predictive reservoir model. Obviously, this problem can be described as an optimal control problem of PDEs where the goal is to find a control q over a time interval

Received by the editors November 24, 2012 and, in revised form, November 30, 2013.
2000 *Mathematics Subject Classification.* 65N30, 49J20.

$[0, T]$ that maximizes a certain performance measure $\mathcal{J}(q)$. Meantime, one realistic way to control the flow paths through the reservoir is to manipulate the quantity of water injected by the control valve settings.

Reservoir simulators use conservation of mass and momentum equations to describe the flow of oil, water or gas through the reservoir rock. Although oil consists of a large number of chemical components with varying properties, in many reservoir modeling cases the Black Oil Model is adopted for simplicity reasons. This model distinguishes between three phases: water, oil and gas. For further simplification, in the oil reservoirs models used within this work no gas is assumed present, hence reducing the number of phases to two.

In this study we carry out some initial investigations on the finite element approximation of this kind of optimal control problem, which to our best knowledge is not much studied in the literature. In our first model we assume that the reservoir is isolated so that with the water being injected, the remaining oil at any time can be estimated by the integral of the concentration over the reservoir at that time. This is of course a much simplified model, but the essential mathematical difficulties to be dealt in this kind of control problems are clearly displayed in it. Thus hopefully this will pave the way to the study of more complex and realistic situations. Although our initial objective is to minimize the remaining oil by adjusting the amount of what injected, the water injected needs to be purified and is expensive. Therefore the cost of water injection needs to be considered as well. We then extend our objective functional into weighted sum of the total remaining oil and the total water injected. Of course it is natural to consider a linear functional to express the cost of oil remained and water injected. However in order to effectively compute such a control problem, still it needs to be reglazed by adding quadratic terms. Thus in this work we will directly consider a quadratic model. Assume the water injection period is between [0,T]. Then in the case of 2-d, our model is governed by a nonlinear coupled system of equations for the movement of two-phase incompressible and completely immiscible fluids in a reservoir $\Omega \subset R^2$ of unit thickness:

$$\min\ J(q) = \min_{q\in K} \frac{1}{2}\int_\Omega \tilde{\omega} c^2(T) + \frac{\alpha_0}{2}\int_0^T \int_\Omega \sum_{i=1}^{N_w} \delta_{w,i} q_i^2 \tag{1}$$

subject to

$$\begin{cases} \phi(x)\dfrac{\partial c}{\partial t} + b(c)u\cdot\nabla c - \nabla\cdot(d(c)\nabla c) = f(c)\displaystyle\sum_{i=1}^{N_w}\delta_{w,i}q_{w,i}, \\ \nabla\cdot u = \displaystyle\sum_{i=1}^{N_w}\delta_{w,i}q_{w,i} - \sum_{j=1}^{N_o}\delta_{o,j}q_{o,j}, \\ u = -a(c)\nabla p, \\ q = \displaystyle\sum_{i=1}^{N_w} q_{w,i} = \sum_{j=1}^{N_o} q_{o,j}, \end{cases} \tag{2}$$

where N_w, N_o are the total numbers of the injective wells and the production wells respectively, $\delta_{w,i}$, $\delta_{o,j}$ are the Dirac functions located at the $i-$th injection well and the $j-$th production well respectively, $K = \{q \in L^\infty[0,T] : \quad 0 \le q \le \hat{q}\}$ and $\hat{q}$ is a