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MIXED FINITE VOLUME METHOD FOR ELLIPTIC PROBLEMS ON NON-MATCHING MULTI-BLOCK TRIANGULAR GRIDS

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Abstract. This article presents a mixed finite volume method for solving second-order elliptic equations with Neumann boundary conditions. The computational domains can be decomposed into non-overlapping sub-domains or blocks and the diffusion tensors may be discontinuous across the sub-domain boundaries. We define a conforming triangular partition on each sub-domains independently, and employ the standard mixed finite volume method within each sub-domain. On the interfaces between different sun-domains, the grids are non-matching. The Robin type boundary conditions are imposed on the non-matching interfaces to enhance the continuity of the pressure and flux. Both the solvability and the first order rate of convergence for this numerical scheme are rigorously proved. Numerical experiments are provided to illustrate the error behavior of this scheme and confirm our theoretical results.

Key words. Mixed finite volume method, error estimate, multi-block domain, non-matching grids.

1. Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. Consider the following single phase flow model for the pressure p and the velocity $\boldsymbol{u}$:

$$\boldsymbol{u} = -K(\boldsymbol{x})(\nabla p - \boldsymbol{\beta}(\boldsymbol{x})p) \quad \text{in } \Omega, \tag{1}$$

$$c(\boldsymbol{x})p + \nabla \cdot \boldsymbol{u} = f \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Here $\boldsymbol{n}$ is the outward unit normal vector with respect to $\partial\Omega$, the coefficient $K(\boldsymbol{x})$ is a symmetric and uniformly positive-definite matrix representing the permeability divided by the viscosity, $\boldsymbol{\beta}(\boldsymbol{x})$ is a vector representing gravity effect, $c(\boldsymbol{x}) > 0$ represents the compressibility of the medium, and f is a source or sink term. In many applications, due to the complexity of the domain geometry or the solution itself, the computational domain Ω is required to be a multi-block domain with grids defined independently on each block. Just as introduced in [19, 20], there are two such examples. One is the modeling of flow in a porous medium with known faults [24], in which material properties would have incontinuity. Another one is the modeling of wells, whose solutions are more desired to be carried out on locally refined grids.

In the numerical simulation of (1)-(3) defined on a multi-block domain, each block is independently covered by a local grid and the standard mixed finite element(MFE) methods could be used within each block. However, since grids do not match on the interfaces between different blocks, the normal trace of the velocity space is no longer continuous across these interfaces. In order to overcome this obstacle, several efficient techniques have been developed to enhance the continuity of the pressure and flux. In [20], the MFE method with mortar elements is presented, in which a mortar finite element space is introduced to approximate the

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trace of the pressure on the non-matching interfaces, and a continuity condition of the flux is also imposed weakly. This method is optimally convergent if the mortar finite element space has one order higher approximability than the normal trace of the velocity space. In [19], the authors constructed a non-mortar MFE method by imposing Robin type conditions on the non-matching interfaces to unite the sub-domain problems. This method could achieve optimal convergent rate for both the pressure and the velocity, and is more convenient for locally refined grids. Both forementioned methods have to solve interface problems resulting from the additional flux-matching conditions. In [10], the authors studied an alternative approach based on enhancing the velocity space along the sub-domain interfaces. The characteristic of this construction is that it yields a flux-continuous velocity space, and thus no interface problems are required to be solved. These three methods have their respective advantages and have been extended to other different physical and numerical models. Readers are referred to [14, 1, 3, 18, 21, 5] and references therein for their recent developments.

Finite volume(FV) methods have been one class of the most commonly used numerical methods for solving partial differential equations in practice, because they can keep a certain conservation property and have flexibility in handling complicated domain geometries and boundary conditions. On the other hand, it could lead to a better numerical treatment on the velocity by discretizing the equations (1)-(3) directly than just computing it from the pressure. Motivated by these reasons, mixed finite volume(MFV) methods have been proposed and analyzed in [15, 16, 17, 6, 25, 7, 8, 9, 22, 2, 23]. However, their construction and corresponding theoretical analysis are all executed on matching grids. In this article we consider MFV approximation of equations (1)-(3) on non-matching triangular grids. Assume that Ω is a union of non-overlapping polygonal blocks, each covered by a conforming triangular grid. On each block, we employ the standard MFV method based on the lowest order Raviart-Thomas space to discretize equations (1)-(3). On the interfaces between different blocks we use the same technique as that investigated in [19] to keep the continuity of the pressure and flux. The Robin type conditions are imposed weakly on the non-matching interfaces by using double-valued Lagrange multipliers to approximate the trace of the pressure. Since the normal components of the velocity space are no longer continuous across the non-matching interfaces and the term related to Lagrange multipliers also needs to be estimated, it is difficult to extend the theoretical analysis used in the standard MFV method on matching grids to this numerical scheme. Under proper assumptions about the regularity of exact solutions, we give the solvability and convergence analysis of this MFV method on non-matching grids by the main ideas employed in [19]. But some details are quite different.

The rest of the paper is organized as follows. In the next section we introduce some necessary notations, assumptions and definitions. Section 3 is devoted to formulating the MFV method on non-matching multi-block triangular grids and presenting several lemmas which are indispensable in the theoretical analysis. Then, we prove error estimates in Section 4. In Section 5, several numerical examples are presented to test the computational efficiency of this numerical scheme and confirm our theoretical results. Finally, we draw a brief conclusion in Section 6.

2. Preliminaries

We assume that Ω can be divided into non-overlapping sub-domains $\Omega_i, i = 1, 2, \cdots, n$, i.e. $\Omega = \bigcup_{i=1}^n \Omega_i$. Let $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$, $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and $\Gamma = \bigcup_{i=1}^n \Gamma_i$