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COMBINED METHODS FOR SOLVING TIME-VARYING SEMILINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH THE USE OF SPECTRAL PROJECTORS AND APPLICATIONS

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Abstract. Two combined methods for computing solutions of time-varying semilinear differential-algebraic equations (descriptor systems) are obtained. When constructing the methods, time-varying spectral projectors which can be found numerically are used. This enables one to numerically solve the differential-algebraic equation (DAE) in the original form without additional analytical transformations. The convergence and correctness of the developed methods are proved. The methods are applicable to the semilinear DAEs with the continuous nonlinear part which may not be differentiable in time. The global Lipschitz condition and other conditions of this kind are not used in the presented theorems on the global solvability of DAEs and on the convergence of the methods. This extends the scope of the methods. The obtained theorems ensure both the existence of a unique global exact solution and the convergence of the methods, which enables one to compute an approximate solution on any given time interval. Numerical examples illustrating the capabilities of the methods and their effectiveness in various situations are provided. To demonstrate the practical application of the obtained methods and theorems, the numerical and theoretical analyses of mathematical models of the dynamics of electric circuits are carried out. It is shown that their results are consistent.

Key words. Descriptor system, differential-algebraic equation, numerical method, global solution, time-varying operator pencil, spectral projector.

1. Introduction

Consider an implicit differential equation of the form

$$\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t, x(t)), \tag{1}$$

where $t \in [t_+, \infty)$, $t_+ \geq 0$, $f \in C([t_+, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and A, $B \in C([t_+, \infty), \mathrm{L}(\mathbb{R}^n))$ ($\mathrm{L}(X, Y)$ denotes the space of continuous linear operators acting from the vector space X to the vector space Y, and $\mathrm{L}(X) := \mathrm{L}(X, X)$). The operators $A(t)$, $B(t)$ (depending on the parameter t) can be degenerate (i.e., noninvertible). Equations of the type (1) with a degenerate (for some t) operator $A(t)$ are called *degenerate differential equations* or *differential-algebraic equations* (*DAEs*). DAEs of the form (1) are commonly referred to as *semilinear*. Since the operators $A(t)$, $B(t)$ are time-varying, equation (1) is called a *time-varying* semilinear DAE or a time-varying degenerate differential equation (DE). In what follows, for the sake of generality, equation (1), where $A \in C([t_+, \infty), \mathrm{L}(\mathbb{R}^n))$ is an arbitrary operator function (i.e., the operator $A(t)$ is not necessarily degenerate), will be called a *time-varying semilinear differential-algebraic equation*.

The initial condition for the DAE is given by

$$x(t_0) = x_0, \tag{2}$$

where $t_0 \geq t_+$. The presence of a degenerate operator at the derivative in the DAE means the presence of algebraic constraints, namely, the graphs of the solutions

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must lie in the manifold generated by the "algebraic part" of the DAE and the initial points (t_0, x_0) must also belong to this manifold (see Remark 3.1).

A function $x \in C([t_0,t_1),\mathbb{R}^n)$ $([t_0,t_1) \subseteq [t_+,\infty))$ is said to be a *solution of equation* (1) *on* $[t_0,t_1)$ if the function $A(t)x(t)$ is continuously differentiable on $[t_0,t_1)$ and $x(t)$ satisfies (1) on $[t_0,t_1)$. If the solution $x(t)$ of (1) satisfies the initial condition (2), then it is called a *solution of the initial value problem* (*the IVP*) (1), (2).

Notice that if we consider a time-varying semilinear DAE of the form

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) = f(t,x(t)) \tag{3}$$

instead of (1), then we must require greater smoothness for its solution than for (1). Namely, a function $x \in C^1([t_0,t_1),\mathbb{R}^n)$ satisfying equation (3) on $[t_0,t_1)$ is called a solution of (3).

DAEs or degenerate DEs are also called descriptor equations, algebraic-differential systems and differential equations (or dynamical systems) on manifolds. These equations are used to describe mathematical models in control theory, radio-electronics, cybernetics, mechanics, economics, ecology, chemical kinetics and gas industry (see, e.g., [3–5, 17, 18, 20, 24]). It is known that the dynamics of electrical circuits is modeled using DAEs which, in general, cannot be reduced to explicit ordinary differential equations (ODEs). In Sections 4.1 and 4.2 we will consider two mathematical models having the form of the DAE (1), which describes transient processes in electrical circuits.

In the present paper, two combined numerical methods for the time-varying semilinear DAEs, which have the first and second orders of convergence, are developed. To obtain these methods, we use, in particular, the time-varying spectral projectors described in Sections 2.1 and 3.1 and the scheme with recalculation (the "predictor-corrector" scheme). The methods are called *combined* since each of them is essentially a combination of two methods, namely, a difference method for the "differential part" and an iteration method for the "algebraic part" of the DAE (this combination is used in method 1 presented in Section 3.2), and in addition recalculation is used in method 2 presented in Section 3.3. The theorems on the convergence and the orders of accuracy of the methods are proved in Sections 3.2, 3.3. These theorems contain conditions for the existence and uniqueness of exact solutions, which in conjunction with conditions for the convergence of the methods ensures the correctness of the methods as well.

Earlier, numerical methods for time-invariant semilinear DAEs were obtained (see [11]) using time-invariant spectral projectors. The presence of time-varying operators in the DAE (1) (as well as in (3)) significantly complicates the construction of the numerical methods and the proof of their convergence; however, the approach developed for the proof of the existence and uniqueness of global solutions in [8] enables one to solve this problem as well. Furthermore, in [11] an approximation by a centered difference was used in obtaining the second-order method, but this increases the quantitative characteristic of stability and, accordingly, a smaller step size may be required in calculations. Therefore, in the present paper, instead of a centered difference we use the recalculation technique to achieve the second order of convergence.

The *main aims of the present work* were: (1) to develop numerical methods, which have certain advantages described below, for time-varying semilinear DAEs, and (2) to demonstrate that the analytical methods, developed earlier to study the solvability and stability of time-varying semilinear DAEs, can be applied to the