INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 23, Number 3, Pages 353–373

doi: 10.4208/ijnam2026-1014

ENHANCED GRADIENT RECOVERY-BASED A POSTERIORI ERROR ESTIMATOR AND ADAPTIVE FINITE ELEMENT METHOD FOR ELLIPTIC EQUATIONS

YING LIU[†], JINGJING XIAO[‡,*], NIANYU YI[§], AND HUIHUI CAO[§]

Abstract. Recovery type a posteriori error estimators are popular, particularly in the engineering community, for their computationally inexpensive, easy to implement, and generally asymptotically exactness. Unlike the residual type error estimators, one can not establish upper and lower a posteriori error bounds for the classical recovery type error estimators without the saturation assumption. In this paper, we first present three examples to show the unsatisfactory performance in the practice of standard residual or recovery-type error estimators, then, an improved gradient recovery-based a posteriori error estimator is constructed. The proposed error estimator contains two parts, one is the difference between the direct and post-processed gradient approximations, and the other is the residual of the recovered gradient. The reliability and efficiency of the enhanced estimator are derived. Based on the improved recovery-based error estimator and the newest-vertex bisection refinement method with a tailored mark strategy, an adaptive finite element algorithm is designed. We then prove the convergence of the adaptive method by establishing the contraction of gradient error plus oscillation. Numerical experiments are provided to illustrate the asymptotic exactness of the new recovery-based a posteriori error estimator and the high efficiency of the corresponding adaptive algorithm.

Key words. Gradient recovery, a posteriori error estimator, adaptive finite element method, elliptic equation.

1. Introduction

Adaptive finite element methods [1, 2, 3] are widely used for numerically solving partial differential equations, especially for solutions with singularity or multiscale properties. Based on the principle of uniform distribution of errors, the adaptive algorithm adjusts the mesh such that the errors are "equally" distributed over the computational mesh. A posteriori error estimation [4, 5, 6], which provides information about the size and the distribution of the error, is an essential ingredient of adaptive finite element methods. Error estimators are computable quantities depending on finite element approximations and known data that locate accurate sources of global and local error. Globally, we use the a posteriori error estimator as the stop criterion to determine whether the finite element solution is an acceptable approximation. Locally, we use the a posteriori error estimator as an indicator which shows the error distribution and guides the local mesh adaptation.

There is a large numerical analysis literature on adaptive finite element methods [7, 8, 9, 10], and various kinds of a posteriori estimates have been proposed for different problems [5, 11, 12]. Basically, there are two types of a posteriori error estimations, one is residual-type error estimation, and the other is the recovery type error estimation. Residual type error estimations, originally introduced by Babuška and Rheinboldt [13], to estimate errors, consider local residuals of the numerical solution. Today, residual type a posteriori error estimates are well studied for a large class of elliptic model problems [14, 15, 16]. It is proved that the residual

Received by the editors on October 29, 2024 and accepted on August 1, 2025.
2000 *Mathematics Subject Classification.* 65N30, 65N50.
*Corresponding author.

estimator provides upper and lower bounds of the actual error in a suitable norm. However, in practical simulation, the residual error estimator is usually larger than the actual error. Hence, it may guide the adaptive algorithm not being terminated timely in practical applications (see Example 3.1). Recovery type a posteriori error estimations, which adopt a certain norm of the difference between the direct and post-processed approximations of the gradient (or other quantities) as an indicator, have gained wide popularity since the work of Zienkiewicz and Zhu [17]. Estimators of the recovery type possess many attractive features. In particular, their ease of implementation, generality, and ability to produce quite accurate estimators have led to their widespread adoption. Recovery type a posteriori error estimator is based on a suitable finite element post-processing technique including the gradient recovery [18, 19, 20, 21, 22], flux recovery [23, 24] or functional recovery [25]. Gradient recovery is a post-processing technique that reconstructs improved gradient approximations from finite element solutions, which is widely used in engineering practice for its superconvergence of recovered derivatives and its robustness as an a posteriori error estimator. Now, different kinds of post-processing techniques are developed based on weighted averaging [26], the local least-squares methods [17, 27, 28], the local or global projections, post-processing interpolation, and so on. Under the assumption that the recovery operator is superconvergent [4], the corresponding error estimator is asymptotically exact [11]. Unfortunately, without the saturation assumption that the recovery provides a better approximation than the numerical approximation does, one can not derive the reliability and efficiency of the recovery type estimator. In some cases, such as the diffusion problem with discontinuous coefficients of the load function is orthogonal to the finite element space [2], the recovery process may not provide improved gradient approximation, then the corresponding error estimator is not reliable and can lead to adaptive refinement completely failing to reduce the global error, which in turn produces a wrong finite element approximation. The purpose of this paper is to derive a gradient recovery-based error estimator with guaranteed upper and lower bounds of the actual error. In view of some interesting numerical findings, an improved error estimator, which contains an additional term that ensures reliability, is proposed and proved to be reliable and efficient. We formulate an adaptive algorithm that is driven by this estimator and a tailored mark strategy and establish its convergence theory.

The rest of this paper is organized as follows. In Section 2, we give a description of the linear elliptic model problem and its finite element method. The poor performance of the classical error estimators, i.e., residual estimators are not asymptotically accurate, the gradient recovery method may not provide a better approximation, and the gradient recovery-based error estimator may lead to over-refinement, are explained and demonstrated by three examples in Section 3. Then an improved gradient recovery-based error estimator is presented and analyzed. In Section 4, we present the adaptive finite element algorithm and prove its convergence result. Several numerical experiments are reported in Section 5 to demonstrate the performance of our improved estimator.

2. Model problem and its finite element scheme

Let $\Omega \subset R^d (d = 2, 3)$ be an open and bounded polyhedral domain. Following the standard notation, we use $L^2(\Omega)$ to denote the space of all square-integrable functions, and its norm is denoted by $\|\cdot\|$. Let $H^s(\Omega)$ be the standard Sobolev space with norm $\|\cdot\|_s$ and seminorm $|\cdot|_s$. Furthermore, $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ denote the norm