INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 23, Number 3, Pages 374–389

doi: 10.4208/ijnam2026-1015

AN EFFICIENT RVE-EMC APPROACH FOR MULTISCALE EQUATIONS OF RANDOM HETEROGENEOUS MATERIALS

YANFU CHEN, XIXIN WU, ZIHAO YANG, JIEQIONG ZHANG, AND JUNFENG ZHAO*

Abstract. Resolving multiscale equations in heterogeneous materials presents significant computational challenges arising from rapid spatial oscillations and random variations in solutions induced by intricate microstructural configurations. This study proposes and analyzes a two-stage stochastic homogenization framework designed to efficiently compute homogenized solutions for multiscale diffusion equations. The methodology unfolds through two distinct phases. In the first stage, each realization of the random microstructure undergoes spatial homogenization through a representative volume element (RVE)-based approach, effectively replacing the original multiscale diffusion equation with a random counterpart featuring piecewise constant coefficients. Through ensemble-based Monte Carlo (EMC) averaging of diffusion coefficients, we reformulate the random diffusion equation into a deterministic diffusion problem with a random source term in the second stage. This critical reformulation enables the implementation of an efficient fixed-point iteration scheme for solving the resultant constant-coefficient diffusion equation. In addition, the convergence of the homogenized solution to the solution of multiscale diffusion equation is proved. Numerical examples are provided to demonstrate the ability and accuracy of the proposed method.

Key words. Stochastic homogenization, representative volume element, ensemble-based Monte Carlo, convergence, random heterogeneous materials.

1. Introduction

As is well known, heterogeneous materials are widely used in the field of engineering due to their excellent properties. Heterogeneous materials, such as concrete, short-fiber-reinforced composites, polymer composites and damaged composites [1, 2, 3], often have uncertainties originating from variations in microstructures or material parameters. Accurately evaluating the physical and mechanical responses of these materials requires quantifying the uncertainty of microstructure and material parameters. Mathematically, these responses can be described using multiscale partial differential equations (PDEs) with random, rapidly oscillating coefficients. A notable example is the class of random multiscale diffusion equations, which have been widely applied in the study of elastic mechanics, heat conduction, and electromagnetics of heterogeneous materials (see [4, 5, 6, 7, 8, 9]). Given the inherent randomness and highly fluctuating nature of material parameters, directly obtaining an accurate numerical solution to the multiscale diffusion problem in random heterogeneous materials is computationally demanding. The requirement for an extremely fine mesh and large-scale sampling leads to excessive computational costs in terms of both memory and processing time. Therefore, it is imperative to develop novel, efficient numerical methods to address these challenges.

Traditional stochastic homogenization theory [10, 11, 12] has been extensively developed to derive the homogenized diffusion equation from the original multiscale equation by formulating a random elliptic cell problem over the entire spatial domain. However, obtaining its numerical solution remains a significant challenge

Received by the editors on February 5, 2025 and accepted on June 17, 2025.
2000 *Mathematics Subject Classification.* 35B27, 60H15.
*Corresponding author.

due to computational complexity. To address this, various techniques, such as periodization and cut-off methods, have been introduced to locally approximate the homogenized coefficients [13, 14]. Nevertheless, ensuring the accuracy of these approximations necessitates a sufficiently large bounded domain, making numerical computations highly demanding.

Another widely used approach in engineering computations is the representative volume element (RVE) method, which is employed to estimate the effective parameters of highly heterogeneous or random composite materials. In this method, the cell problem is defined on a reasonably large representative cell [15, 16, 17, 18, 19, 20, 21], where the size of the cell does not necessarily tend to infinity. However, this method is commonly used in engineering and lacks a mathematical theoretical foundation. Once the homogenized coefficients are approximated, the primary computational challenge shifts to solving the random diffusion equation incorporating these coefficients. Directly applying Monte Carlo methods [22] or the stochastic Galerkin method [23, 24, 25, 26, 27] to solve this equation is computationally prohibitive. The Monte Carlo approach requires repeated evaluations of the random problem for different sample coefficients, leading to excessive computational cost. Similarly, the stochastic Galerkin method often results in a high-dimensional deterministic system, which demands substantial computational resources and may be infeasible for large-scale problems. To overcome the computational challenge, several approaches have been proposed in literature including variants of the ensemble method [28, 29, 30, 31] and the multi-modes method [32, 33, 34, 35]. Resolving multiscale equations in heterogeneous materials still exist significant computational challenges arising from rapid spatial oscillations and random variations in solutions induced by intricate microstructural configurations.

This paper proposes an effective strategy to address the computational challenges arising from the spatial fast oscillations and the inherent randomness of the solution to the multiscale diffusion equation of random heterogeneous materials. Building on the RVE method, the ensemble-based Monte Carlo (EMC) method and our previous work [35], we introduce a practical two-stage stochastic homogenization method to efficiently compute homogenized solutions for multiscale diffusion equations. The key advantage of the proposed method is its ability to decouple the computational difficulties caused by spatial fast oscillations and those resulting from randomness, enabling each to be addressed separately using distinct strategies. In the first stage, the RVE-based spatially homogenization technique is proposed to deal with the computational difficulty caused by the spatial fast oscillation of the solution. And the EMC method is applied to deal with the computational difficulty caused by the randomness of the solution in the second stage. Besides, we also prove the convergence of the homogenized solution to the solution of multiscale diffusion equation. It should be pointed out that we give a mathematical interpretation and justification for the RVE method.

The remainder of the paper is structured as follows. Section 2 describes the setting of the multiscale diffusion problem arising from the random heterogeneous materials. In Section 3, we introduce the RVE-EMC two-stage approach and related convergence analysis for the multiscale diffusion problem. Section 4 presents the finite element discretization and a detailed implementation algorithm for the proposed method. In Section 5, numerical experiments are conducted to demonstrate the effectiveness of the proposed method. Finally, Section 6 provides concluding remarks.