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ANALYSIS OF HARMONIC AVERAGE METHOD FOR INTERFACE PROBLEMS WITH DISCONTINUOUS SOLUTIONS AND FLUXES

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Abstract. Harmonic average method has been widely utilized to deal with heterogeneous coefficients in solving differential equations. One remarkable advantage of the harmonic averaging method is that no derivative of the coefficient is needed. Furthermore, the coefficient matrix of the finite difference equations is an M-matrix which guarantees the stability of the algorithm. It has been numerically observed but not theoretically proved that the method produces second order pointwise accuracy when the solution and flux are continuous even if the coefficient has finite discontinuities for which the method is inconsistent ($O(1)$ in the local truncation errors). It has been believed that there are some fortunate error cancellations. The harmonic average method does not converge when the solution or the flux has finite discontinuities. In this paper, not only we rigorously prove the second order convergence of the harmonic averaging method for one-dimensional interface problem when the coefficient has a finite discontinuities and the solution and the flux are continuous, but also proposed an *improved harmonic average method* that is also second order accurate (in the L^∞ norm), which allows discontinuous solutions and fluxes along with the discontinuous coefficients. The key in the convergence proof is the construction of the Green's function. The proof shows how the error cancellations occur in a subtle way. Numerical experiments in both 1D and 2D confirmed the theoretical proof of the improved harmonic average method.

Key words. Harmonic average, improved harmonic average method, variable discontinuous coefficient, non-homogeneous jump conditions, Green function, discrete maximum principle, convergence analysis.

1. Introduction

Harmonic average, sometimes also called harmonic mean, has been applied to solve various differential equations when the material parameters have large variations, even with finite jump discontinuities, which are called interface problems. Applications include unsaturated flow in layered soils [11], non-linear heat conduction problems in [3]. Advantages of the harmonic average approach include the simplicity in implementation and preservation of some physical properties. There are limited references on study of the harmonic average method. A few discussions can be found in [1, 3, 11, 14]. It is observed that the harmonic average method works well for one-dimensional problems when the solution and the flux are continuous even if the coefficient have a finite number of jump discontinuities. However, to our best knowledge, there is no rigorous proof that can be found

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in the literature. In this paper, we have provided rigorous proof why and when the harmonic average method works; and more important, developed an improved harmonic average method that can work for interface problems with discontinuous solution and/or fluxes with second order accuracy in the L^∞ norm. The new improved harmonic average method does not need the derivative of the coefficient, an obvious advantage over some existing second order accurate methods.

We consider the following interface problem

$$(\beta(x)u_x)_x - \sigma(x)u = f(x) + v\delta(x-\alpha) + w\delta'(x-\alpha), \quad 0 < x < 1, \tag{1}$$

with specified boundary conditions of $u(x)$ at $x = 0$ and $x = 1$. We assume that $\beta(x) \ge \beta_0 > 0$, $\sigma(x) \ge 0$, and $\sigma(x), f(x) \in C[0,1]$, but allow $\beta(x) \in C(0,\alpha) \cup C(\alpha,1)$, which means that $\beta(x)$ can have a finite jump discontinuity at α. In [9], the authors showed that the problem is equivalent to the following interface problem,

$$\begin{aligned} &(\beta(x)u_x)_x - \sigma(x)u = f(x), \qquad \{0 < x < \alpha\} \cup \{\alpha < x < 1\}, \\ &[u]_\alpha = \frac{2w}{\beta^- + \beta^+}, \qquad [\beta u_x]_\alpha = v, \qquad 0 < \alpha < 1, \end{aligned} \tag{2}$$

where $\beta^- = \lim_{x\to\alpha-}\beta(x)$ and $\beta^+ = \lim_{x\to\alpha+}\beta(x)$. It is easier to use this formulation to prove the existence and uniqueness of the solution to the boundary value problem.

Harmonic average finite difference method. Given a uniform grid $0 = x_0 < x_1 < \cdots < x_N = 1$ with the step size h, the harmonic average finite difference equation at a grid point x_i is the following

$$\frac{1}{h^2}\Big(\beta_{i+\frac12}(U_{i+1} - U_i) - \beta_{i-\frac12}(U_i - U_{i-1})\Big)\, U_i - \sigma_i\, U_i = f_i, \tag{3}$$

where $\sigma_i = \sigma(x_i)$, $f_i = f(x_i)$, and $\beta_{i+\frac12}$ is the following harmonic average,

$$\beta_{i+\frac12} = \left(\frac{1}{h}\int_{x_i}^{x_{i+1}} \beta^{-1}(x)dx\right)^{-1}. \tag{4}$$

When $\beta(x) \in C[0,1]$, the above expression can be replaced with $\beta(x_{i+1/2})$. Numerically, second order accuracy in the pointwise L^∞ norm has been observed even if β has a finite jump discontinuity but the solution and the flux are continuous, that is, $w = 0$ and $v = 0$ assuming that the integration is accurate enough, see for example [5,13]. Note that the finite difference scheme is inconsistent at the two grid points adjacent to the discontinuity as we can see later. One misconception is that the local truncation errors would cancel out to $O(h)$, which would lead to second order convergence. The harmonic average finite difference method does not converge when the solution or flux has a finite jump, which is often referred as a non-homogeneous jump condition.

In this paper, we not only show the convergence of the harmonic average finite difference method for 1D interface problems with finite number of