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A HIGH-ORDER MIXED FINITE ELEMENT METHOD FOR SECOND ORDER ELLIPTIC EQUATIONS ON CURVED DOMAIN WITH BOUNDARY VALUE CORRECTIONS

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Abstract. This paper presents a boundary corrections mixed finite element method for second order elliptic equations with the non-homogeneous Neumann boundary condition on curved domains. A key feature of the boundary value corrections is the shift from the true boundary to a surrogate boundary, which avoids numerical integration formula on curved elements. We consider the high-order Raviart-Thomas element (RT_k) of degree $k \geq 1$ on triangular meshes, achieving an $O(h^{k+1/2})$ convergence in the L^2-norm estimate for the velocity field and an $O(h^k)$ convergence in the H^1-norm estimate for the pressure. Finally, numerical experiments validate our theoretical results.

Key words. Mixed finite element method, boundary value corrections, Neumann boundary condition, curved domain.

1. Introduction

Many practical problems arising in science and engineering often involve domains with curved boundaries. For the domain Ω with curved boundaries, the geometric error between the curved boundary Γ and the approximating boundary Γ_h leads to a loss of accuracy for high-order elements [33, 34]. There are two main strategies to address this issue. Both the isoparametric finite element method [21, 27] and the isogeometric analysis [16, 23] aim to reduce the geometric error without modifying the variational form. The second strategy is the boundary value correction method [7], which directly solves on a polygonal approximation domain Ω_h, and focuses on a modified variational formulation. We also mention that there are other approaches such as the discontinuous Galerkin method [14, 15], the shifted boundary method [2, 29], the cut finite element method (cutFEM) with boundary value correction [11], the cut-cell finite element method [30], etc.

In this paper, we consider the mixed finite method (MFEM) for second order elliptic problems with non-homogeneous Neumann boundary conditions on curved domains. Although analysis of discretizations in the primal formulation on curved domains has long been widely recognized, the research of the mixed formulation is relatively rare. In the context of the mixed finite element method, Neumann boundary conditions become essential. A subtlety emerges in that the Neumann boundary condition entails the outward normal vector on the boundary, which is different on Γ and Γ_h. To the best of our knowledge, such a problem, as well as subsequent MFEM error analysis on curved domains was first studied by Bertrand et al. [3, 4] for the Raviart-Thomas element in 2014. Later, they extended the analysis to parametric Raviart-Thomas elements in [5]. However, in [3, 4, 5], a homogeneous Neumann boundary was considered and enforced in the discrete space. In [17, 12], concerning the study on polygonal domain, authors used penalty method

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to weakly imposes the Neumann boundary conditions in the weak formulation. One of the purposes of our works is to research the possibility of weakening non-homogeneous Neumann curved boundary conditions. It seems that one only needs to apply the penalty method to curved boundary. However, one also has to be careful about the difference of the outward normal vector between Γ and Γ_h.

This paper adopts the boundary value correction technique on curved boundary, where Taylor expansion is used to transfer the boundary condition from curved boundary to polygonal approximation boundary. In view of the differences in outward normal vector between Γ and Γ_h, we introduce a map to connect Γ and Γ_h and then utilize the penalty method to weakly impose the Neumann boundary. To our knowledge, the best error results of MFEM for second order elliptic with non-homogeneous Neumann boundary is $O(h^k)$ for the velocity field in [31]; their work used the cutFEM to integrate directly on curved regions. We improve the error rates to $O(h^{k+1/2})$ and do not involve the curved elements. Furthermore, this work includes a rigorous analysis of the loss of approximation accuracy for high-order elements.

This paper is organized as follows. In section 2, we introduce some notations and preliminaries; In section 3, we describe the model problem and introduce the boundary value correction method; In section 4, we establish the discrete space and variational form and analyze its well-posedness; In section 5, the energy error estimate and the L^2 error estimate are proved; In section 6, we present several numerical experiments to verify the theoretical results; we conclude in 7 with our findings.

2. Notations and preliminaries

Throughout this paper, let Ω be a connected open set in $\mathbb{R}^2$ with Lipschitz continuous boundary Γ. We assume that Ω is approximated by a polygonal domain Ω_h and denote by Γ_h the boundary of Ω_h. Let $\mathcal{T}_h$ denote a family of triangular meshes for Ω_h. We require that all the vertices of $\mathcal{T}_h$ lying on Γ_h also lie on Γ, ensuring $\mathcal{T}_h$ is a body-fitted triangular partition of Ω. For each $K \in \mathcal{T}_h$, let $h_K = \text{diam(K)}$ and $h = \max_{K\in\mathcal{T}_h} h_K$. We assume the mesh is shape-regular; that is, there exists a constant $\sigma > 0$, independent of h, such that $\max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \leq \sigma$, where ρ_K is the diameter of the largest ball inscribed in K. Furthermore, we assume that the mesh is quasi-uniform; that is, there exists $\tau > 0$, independent of h, such that $\min_{K\in\mathcal{T}_h} h_K \geq \tau h$. Let $\mathcal{E}_h$ denote the set of all edges in $\mathcal{T}_h$. Define $\mathcal{E}_h^o$ as the set of all interior edges and $\mathcal{E}_h^b = \mathcal{E}_h \setminus \mathcal{E}_h^o$. Denote by $\mathcal{T}_h^b$ all mesh elements containing at least one edge in $\mathcal{E}_h^b$ and $\mathcal{T}_h^o = \mathcal{T}_h \setminus \mathcal{T}_h^b$. Let e be an interior edge shared by two elements K_1 and K_2, we define the jump $[\cdot]$ on e for scalar functions q as follows:

$$[q] = q|_{K_1} - q|_{K_2}.$$

We adopt standard definitions for the Sobolev spaces as presented in [10]. Let $H^m(S)$, for $m \in \mathbb{R}$ and $S \subset \mathbb{R}^2$ be the usual Sobolev space with associated norm $\|\cdot\|_{m,S}$ and seminorm $|\cdot|_{m,S}$. When $m = 0$, the space $H^0(S)$ coincides with the square integrable space $L^2(S)$. We define

$$H^m(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} H^m(K)$$

with seminorm

$$|\cdot|_{m,\mathcal{T}_h} := (\sum_{K\in\mathcal{T}_h} |\cdot|_{m,K}^2)^{1/2}.$$