

UNIFORM EQUIVALENCE OF L^2 - AND DISCRETE ℓ^2 -NORMS ON Q_1 -FINITE ELEMENT SPACES WITH MASS LUMPING IN ANY FINITE DIMENSION

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Abstract. The Q_1 -finite element spaces, in any finite d -dimension, are equipped with the discrete ℓ_h^2 -inner product generated by the simple row-sum mass lumping. The equivalence of the discrete ℓ_h^2 -norm and the L^2 -norm on the Q_1 -finite element spaces is uniform in mesh size h , in both cases of uniform and nonuniform partitions. Several representation formulae for these norms are derived. Using these, accurate bounds between these two norms are obtained, which is our major contribution. Examples show that these bounds are sharp. As an important application, the equivalence is established between discrete h_h^1 -norm and H^1 -norm. Numerical results are presented.

Key words. Finite element method, mass lumping, norm equivalence.

1. Introduction

In this section we will begin by the 1D simplest case for the motivation of our study. Then some notations and preliminaries will be given.

1.1. 1D motivation. Let us consider the simplest Sobolev space

$$H_0^1(0, 1) = \{f \in L^2(0, 1) \mid f' \in L^2(0, 1); \text{tr}_0(f) = 0, \text{tr}_1(f) = 0\},$$

where tr_ξ denotes the standard trace operator at ξ in the Sobolev space theory. A family of C^0 piecewise linear finite element spaces $(V_h)_{0 < h < 1}$ is defined in a standard fashion. For a positive integer l , let \mathcal{T}_h denote the standard mesh

$$0 = x^{(0)} < x^{(1)} < \dots < x^{(l-1)} < x^{(l)} = 1;$$

$$h^{(j)} = x^{(j)} - x^{(j-1)}, \quad j = 1, \dots, l,$$

with $h = \max_{j=1}^l h^{(j)}$. It is quite convenient, in analysis and in actual programming, to extend the meshes outside the domain. Thus, we assume that $x^{(-1)} = -h^{(1)}$ and $x^{(l+1)} = 1 + h^{(l)}$. Let V_h be the finite element subspace of $H_0^1(0, 1)$ associated with \mathcal{T}_h . For $j = 0, \dots, l$,

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denoting by ϕ_j the basis function

$$(1) \quad \phi_j(x) = \left[\frac{x - x^{(j-1)}}{h^{(j)}} \chi_{[x^{(j-1)}, x^{(j)}]}(x) + \frac{x^{(j+1)} - x}{h^{(j+1)}} \chi_{[x^{(j)}, x^{(j+1)}]}(x) \right] \chi_{[0,1]}(x),$$

where χ_S denotes the characteristic function of set S , one has $V_h = \text{span}\{\phi_1, \dots, \phi_{l-1}\}$. Let us look at the $L^2(0, 1)$ -norm on V_h in detail. For any v_h in V_h , i.e. $v_h = \sum_{j=1}^{l-1} v^j \phi_j$, $v^j \in \mathbb{R}$, (throughout we assume $v^0 = v^l = 0$)

$$\begin{aligned} \|v_h\|_{L^2(0,1)}^2 &= \int_0^1 |v_h|^2 dx = \sum_{j=1}^l \int_{x^{(j-1)}}^{x^{(j)}} |v^{j-1} \phi_{j-1}(x) + v^j \phi_j(x)|^2 dx \\ &= \sum_{j=1}^l \int_{x^{(j-1)}}^{x^{(j)}} \left[(v^{j-1})^2 \phi_{j-1}^2(x) \right. \\ &\quad \left. + 2v^{j-1}v^j \phi_{j-1}(x)\phi_j(x) + (v^j)^2 \phi_j^2(x) \right] dx \\ &= \sum_{j=1}^l \left[(v^{j-1})^2 \frac{h^{(j)}}{3} + 2v^{j-1}v^j \frac{h^{(j)}}{6} + (v^j)^2 \frac{h^{(j)}}{3} \right] \\ &= \frac{1}{3} \sum_{j=1}^l h^{(j)} [(v^{j-1})^2 + v^{j-1}v^j + (v^j)^2] \\ (2) \quad &= \frac{1}{6} \sum_{j=1}^l h^{(j)} (|v^{j-1}|^2 + |v^j|^2 + (v^{j-1} + v^j)^2). \end{aligned}$$

From this $L^2(0, 1)$ -norm representation, the $L^2(0, 1)$ -inner product on V_h is evidently deduced. Instead of the $L^2(0, 1)$ -inner product, the following (row-sum) mass-lumped ℓ_h^2 -inner product on V_h is frequently used:

$$(3) \quad (u_h, v_h)_{\ell_h^2} = \sum_{j=1}^{l-1} u^j v^j \bar{h}^{(j)} \quad \forall u_h = \sum_{j=1}^{l-1} u^j \phi_j(x), v_h = \sum_{j=1}^{l-1} v^j \phi_j(x) \in V_h,$$

where $\bar{h}^{(j)} = \frac{h^{(j)} + h^{(j+1)}}{2}$.

It is immediate to see that, for the V_h -interpolant $v_h \in V_h$ of any $v \in H_0^1(0, 1)$,

$$(4) \quad \|v_h\|_{\ell_h^2}^2 = \sum_{j=1}^{l-1} (v^j)^2 \bar{h}^{(j)} = \frac{1}{2} \sum_{j=1}^l [(v^{j-1})^2 + (v^j)^2] h^{(j)},$$