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Higher-Order Rogue Waves of the Nonlocal Nonlinear Schrödinger Equation in the Defocusing Regime

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Abstract. The previous studies have shown that the defocusing nonlinear Schrödinger equation (NLSE) has no the modulational instability, and was not found to admit the rogue wave phenomenon so far. In this paper, we address the question of the higher-order rogue wave solutions of the nonlocal $\mathscr{PT}$-symmetric NLSE in the defocusing regime. Based on Darboux transformation and iterations, we derive an explicit solution for the higher-order rogue waves by adopting a variable separation and Taylor expansion technique. The higher-order rogue wave solutions are expressed in separation-of-variables form. Furthermore, in order to understand these solutions better, patterns of the rogue waves for lowest three order are explored clearly and conveniently. The reported results may be useful for the design of experiments for observation of rogue waves in the defocusing nonlinear physical systems.

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Key words: Defocusing nonlocal nonlinear Schrödinger equation, variable separation technique, parity-time symmetric, rogue waves.

1. Introduction

It is well known that integrable nonlinear systems play a pivotal role in mathematical physics. Most of these integrable nonlinear systems are local equations. In other words, the solutions evolution relies only on local solution values. In recent years, integrable nonlocal nonlinear systems have attracted a lot of attention and are extensively studied. This type of equation is Parity-time ($\mathscr{PT}$) symmetric — i.e. it is invariant under complex conjugation and joint transformations. The first such system introduced by Ablowitz and Musslimani — c.f., e.g. Refs. [3–5, 20], has the form

$$\mathrm{i}u_t(x,t)+\frac{1}{2}u_{xx}(x,t)\pm u^2(x,t)\bar{u}(-x,t)=0, \tag{1.1}$$

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where the bar denotes the complex conjugation and the sign ± determines whether the Eq. (1.1) is focused or defocused. It is worth mentioning that $\mathscr{P}\mathscr{T}$ symmetric equations play a vital role in optics and other physical fields [26]. Inspired by the above nonlocal $\mathscr{P}\mathscr{T}$ symmetric nonlinear equation, new nonlocal nonlinear integrable equations have been proposed and studied over past few years [1, 2, 5, 8, 21, 27, 29, 30, 32, 33, 41, 59].

Rogue waves originally attracted a lot of attention due to mysterious and severely destructive oceanic surface waves [25, 38]. This types of waves are spontaneous large waves that "appear out of nowhere, disappear without a trace" [7]. The first analytical expression of rogue waves was derived for the cubic nonlinear Schrödinger equation by Peregrine [39]. Thereafter, higher-order rogue waves in the local NLSE were found, and their interesting dynamical patterns were also discussed [6, 9, 10, 12, 15–17, 22–24, 35–37, 40, 45, 50]. For instance, Kedziora *et al.* [24] studied circular rogue wave clusters of the local NLSE by adopting Darboux transformation (DT). Soon after Guo *et al.* [22] derived N-th order rogue wave solutions of the local NLSE using a generalized DT. In addition, Ohta and Yang [37] investigated N-th order rogue wave solutions of the local NLSE by using the Hirota bilinear method. A variable separation technique presented by Mu and Qin [12] is used to construct N-th order explicit rogue wave solutions of the local NLSE. In particular, Mu *et al.* [35] extend above own work to study general higher-order rogue waves of a vector NLSE using a DT with an asymptotic expansion method. Nowadays, rogue waves have been rapidly overspread to many research fields encompassing oceanography [18], nonlinear optics [43], Bose-Einstein condensation [14], superfluid helium [19], plasmas [34] and even finance [52, 53], quantum droplet [28], etc. As an unexplored and interesting subject, rogue waves in nonlocal integrable systems have received much attention recently [41, 42, 46–48, 55, 56].

Motivated by the works of Baronio *et al.* [12] and Mu *et al.* [35], we next attempt to study rogue wave solutions in several reverse-time integrable nonlocal nonlinear equations using the variable separation technique. As a typical example, we consider the scalar reverse-time nonlocal NLSE

$$\mathrm{i}q_t(x,t)+\frac{1}{2}q_{xx}(x,t)+\sigma q^2(x,t)\widetilde{q}(x,t)=0, \quad \sigma=\pm 1, \tag{1.2}$$

cf. [5, 44, 51, 54, 57, 58]. Here we define $\widetilde{q}(x,t)=q(x,-t)$. If $\sigma=1$, the Eq. (1.2) describes the focusing regime while the case $\sigma=-1$ is related to the defocusing regime. This nonlocal equation which also called $\mathscr{P}\mathscr{T}$-symmetric NLSE is a special reduction from the famous AKNS system. Due to this $\mathscr{P}\mathscr{T}$-symmetry, it is related to a cutting research area of contemporary physics [13, 26]. Therefore, this system have attracted a lot of attention in optics and other physical fields in recent years. Bounded multi-soliton solutions and their asymptotic analysis for the Eq. (1.2) had been explored [44]. Yang [57] had derived general multi-soliton solutions in the Eq. (1.2). In addition, there are also many other researchers who have made their own contributions to the study of reverse-time nonlocal NLSE. For instance, Ye and Zhang [58] had constructed the general soliton solutions with zero and non-zero background to a reverse-time nonlocal NLSE via binary DT. The Eq. (1.2) was generalized by Ma [31] to a multi-component one and construct its multi-soliton solutions. In this work, we mainly consider system (1.2) with the defocusing case.

The previous studies show that the defocusing NLSE has no modulational instability and does not possess the rogue wave phenomenon. As unexplored and interesting subject in the defocusing nonlocal $\mathscr{PT}$-symmetric NLSE, some natural problems arise:

1. Can we construct the rogue wave solution in the above nonlocal $\mathscr{PT}$-symmetric NLSE in the defocusing regime?
2. What about the patterns of rogue waves in the defocusing regime?

The main objective of the present work is to solve the problems listed above by considering the defocusing nonlocal $\mathscr{PT}$-symmetric NLSE. Firstly, we derive higher-order rogue wave solutions of a defocusing nonlocal $\mathscr{PT}$-symmetric NLSE by utilizing the DT via a variable separation technique. Moreover, the dynamic patterns of these solutions are graphically discussed by varying the available parameters. It is worth noting that rogue wave patterns in the defocusing regime also exhibit very rich structures.

The structure of this paper is as follows. In Section 2, the variable separation technique to treat the Lax pair of the nonlocal defocusing $\mathscr{PT}$-symmetric NLSE will be presented. Then the rogue waves of arbitrary order for these defocusing nonlocal $\mathscr{PT}$-symmetric NLSE are established via a Taylor expansion mechanism. In Section 3, a range of dynamic behaviors of these obtained rogue wave solutions are displayed graphically. Finally, our conclusions and a brief discussion of the obtained results are provided.

2. Variable Separation Technique

It is known that the Eq. (1.2) with $\sigma=-1$ is a compatibility condition, for the system of linear matrix differential equations (alias the Lax pair)

$$\Psi_x=\mathbf{U}\Psi, \quad \Psi_t=\mathbf{V}\Psi \tag{2.1}$$

with

$$\begin{aligned}
&\mathbf{U}=\mathrm{i}(\lambda\sigma_3+Q),\\
&\mathbf{V}=\mathrm{i}\lambda(\lambda\sigma_3+Q)+\frac{1}{2}\sigma_3(Q_x-\mathrm{i}Q^2),
\end{aligned}$$

and

$$Q=\begin{bmatrix} 0 & -\widetilde{q}(x,t) \\ q(x,t) & 0 \end{bmatrix},$$

where $\sigma_3=\mathrm{diag}(1,-1)$.

For convenience, we choose the following plane wave solution as an initial solution of the nonlocal defocusing NLSE:

$$q^{[0]}=\mathrm{i}\rho\exp(\mathrm{i}\rho^2 t), \tag{2.2}$$

where ρ is a real constant. Next we consider a family of the solutions of the Lax equation (2.1) corresponding to λ, it infers

$$\Psi = \begin{bmatrix} \psi \\ \phi \end{bmatrix} = \Lambda \mathscr{R} \mathscr{E} \mathscr{Z}, \quad \mathscr{R} = \exp(\mathrm{i}\Theta x), \quad \mathscr{E} = \exp(\mathrm{i}\Omega t), \quad \mathscr{Z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{2.3}$$

where $\mathscr{Z}$ is an arbitrary complex two dimensional vector and

$$\Lambda = \begin{bmatrix} \exp\left(-\mathrm{i}\rho^2 t\right) & 0 \\ 0 & \exp\left(\mathrm{i}\rho^2 t\right) \end{bmatrix}^{1/2}.$$

Here the matrices Θ and Ω are subject to

$$[\Theta, \Omega] = \Theta\Omega - \Omega\Theta = 0. \tag{2.4}$$

It follows from (2.1) and (2.4) that

$$\Theta = \begin{bmatrix} \lambda & -\mathrm{i}\rho \\ \mathrm{i}\rho & -\lambda \end{bmatrix}, \quad \Omega = \lambda\Theta.$$

Then the exponential matrices $\mathscr{R}$ and $\mathscr{E}$ in (2.3) can be written as

$$\mathscr{R} = \frac{1}{\tau} \begin{bmatrix} \tau\cos(\tau x) + \mathrm{i}\lambda\sin(\tau x) & \rho\sin(\tau x) \\ -\rho\sin(\tau x) & \tau\cos(\tau x) - \mathrm{i}\lambda\sin(\tau x) \end{bmatrix}, \tag{2.5}$$

$$\mathscr{E} = \frac{1}{\xi} \begin{bmatrix} \xi\cos(\xi t) + \mathrm{i}\lambda^2\sin(\xi t) & \rho\lambda\sin(\xi t) \\ -\rho\lambda\sin(\xi t) & \xi\cos(\xi t) - \mathrm{i}\lambda^2\sin(\xi t) \end{bmatrix}, \tag{2.6}$$

where $\tau = \sqrt{\lambda^2 + \rho^2}$ and $\xi = \lambda\tau$.

Next we focus on constructing higher-order rogue wave solutions of the nonlocal defocusing NLSE. Taking $\lambda = \mathrm{i}\rho(1+\epsilon)$ in (2.5) and (2.6), and utilizing the Taylor expansion for sines and cosines, we obtain and expansion at $\epsilon = 0$ for the matrix $\mathscr{R}$ in (2.5), viz.

$$\mathscr{R}\big|_{\lambda=\mathrm{i}\rho(1+\epsilon)} = \sum_{n=0}^{\infty} \mathscr{R}_n \epsilon^n,$$

where

$$\mathscr{R}_n = \begin{bmatrix} \alpha_n - \beta_n - \beta_{n-1} & \beta_n \\ -\beta_n & \alpha_n + \beta_n + \beta_{n-1} \end{bmatrix},$$

and

$$\begin{aligned} \alpha_n &= \sum_{l=0}^{[n/2]} \mathbf{C}_{n-l}^{l} 2^{n-2l} \mathbf{A}_{2(n-l)}, \\ \beta_n &= \sum_{l=0}^{[n/2]} \mathbf{C}_{n-l}^{l} 2^{n-2l} \mathbf{A}_{2(n-l)+1}, \\ \mathbf{C}_n^m &= \frac{n!}{m!(n-m)!}, \quad \mathbf{A}_m = \frac{(\rho x)^m}{m!}, \quad n \geq m, \quad m, n \in \mathbb{N}^+. \end{aligned}$$

Following the same way, the matrix $\mathscr{E}$ in (2.6) can be expanded around $\epsilon=0$ as

$$\mathscr{E}\Big|_{\lambda=\mathrm{i}\rho(1+\epsilon)}=\sum_{n=0}^{\infty}\mathscr{E}_n\epsilon^n,$$

where

$$\mathscr{E}_n=\begin{bmatrix}\gamma_n-\theta_n-\theta_{n-1} & \theta_n\\ -\theta_n & \gamma_n+\theta_n+\theta_{n-1}\end{bmatrix}$$

with

$$\begin{aligned}\gamma_n&=\sum_{l=0}^{[3n/4]}\sum_{m=0}^{l}(-1)^{n-l}\mathbf{C}_{n-l}^{m}\mathbf{C}_{2(n-l)}^{l-m}2^{n-l-m}\mathbf{B}_{2(n-l)},\\ \theta_n&=\mathrm{i}\sum_{l=0}^{[(3n+1)/4]}\sum_{m=0}^{l}(-1)^{n-l}\mathbf{C}_{n-l}^{m}\mathbf{C}_{2(n-l)+1}^{l-m}2^{n-l-m}\mathbf{B}_{2(n-l)+1},\\ \mathbf{B}_m&=\frac{(\rho)^{2m}t^m}{m!},\end{aligned}$$

and l is a nonnegative integer. Let us now assume that $\mathscr{Z}_k$ is a polynomial function of ϵ, i.e.

$$\mathscr{Z}(\epsilon)=\sum_{k=0}^{\infty}\mathscr{Z}_k\epsilon^k,\quad \mathscr{Z}_k=\begin{bmatrix}z_{1,k}\\ z_{2,k}\end{bmatrix},$$

so that the solution (2.3) has the form

$$\begin{aligned}\Psi\Big|_{\lambda=\mathrm{i}\rho(1+\epsilon)}&=\Lambda\sum_{n=0}^{\infty}\Psi_n\epsilon^n,\\ \Psi_n&=\begin{bmatrix}\psi_n\\ \phi_n\end{bmatrix}=\sum_{k=0}^{n}\sum_{j=0}^{n}\mathscr{R}_k\mathscr{E}_j\mathscr{Z}_{n-k-j}.\end{aligned}$$

It follows from the binomial theorem that

$$\begin{aligned}\mathrm{i}^j(1+\epsilon)^j\psi(\epsilon)&=\psi[j,0]+\psi[j,1]\epsilon+\cdots+\psi[j,N]\epsilon^N+\cdots,\\ \mathrm{i}^j(1+\epsilon)^j\phi(\epsilon)&=\phi[j,0]+\phi[j,1]\epsilon+\cdots+\phi[j,N]\epsilon^N+\cdots,\end{aligned}$$

and

$$\psi[j,N]=\sum_{k=0}^{n}\mathrm{i}^j\mathbf{C}_j^{n-k}\psi_k,\quad \phi[j,N]=\sum_{k=0}^{n}\mathrm{i}^j\mathbf{C}_j^{n-k}\phi_k.$$

Utilizing a generalized DT presented by Guo *et al.* [22], we immediately obtain the following N-th order rogue wave solutions for the nonlocal defocusing NLSE:

$$q^{[N]}=\mathrm{i}\left[1+2\mathrm{i}\frac{\nabla_{N2}}{\nabla_{N1}}\right]\exp(\mathrm{i}t),$$

where

$$\nabla_{N1}=\begin{vmatrix}\psi[N-1,0] & \cdots & \psi[N-1,N-1] & (-1)^N\widetilde{\phi}[N-1,0] & \cdots & (-1)^N\widetilde{\phi}[N-1,N-1]\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \psi[0,0] & \cdots & \psi[0,N-1] & (-1)^1\widetilde{\phi}[0,0] & \cdots & (-1)^1\widetilde{\phi}[0,N-1]\\ \phi[N-1,0] & \cdots & \phi[N-1,N-1] & (-1)^{N-1}\widetilde{\psi}[N-1,0] & \cdots & (-1)^{N-1}\widetilde{\psi}[N-1,N-1]\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \phi[0,0] & \cdots & \phi[N,N-1] & (-1)^0\widetilde{\psi}[0,0] & \cdots & (-1)^0\widetilde{\psi}[0,N-1]\end{vmatrix},$$

$$\nabla_{N2}=\begin{vmatrix}\phi[N,0] & \cdots & \phi[N,N-1] & (-1)^N\widetilde{\psi}[N,0] & \cdots & (-1)^N\widetilde{\psi}[N,N-1]\\ \psi[N-2,0] & \cdots & \psi[N-2,N-1] & (-1)^{N-1}\widetilde{\phi}[N-2,0] & \cdots & (-1)^{N-1}\widetilde{\phi}[N-2,N-1]\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \psi[0,0] & \cdots & \psi[0,N-1] & (-1)^1\widetilde{\phi}[0,0] & \cdots & (-1)^1\widetilde{\phi}[0,N-1]\\ \phi[N-1,0] & \cdots & \phi[N-1,N-1] & (-1)^{N-1}\widetilde{\psi}[N-1,0] & \cdots & (-1)^{N-1}\widetilde{\psi}[N-1,N-1]\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \phi[0,0] & \cdots & \phi[0,N-1] & (-1)^0\widetilde{\psi}[0,0] & \cdots & (-1)^0\widetilde{\psi}[0,N-1]\end{vmatrix}.$$

3. Lowest Three Order Rogue Wave Patterns

To verify the effectiveness of our method, we graphically display the first, second and third rogue wave solutions. In what follows, we let $\rho=1$ for the analysis convenience.

Case 1. Taking $N=1$ in the previous section gives

$$\mathscr{R}_0=\begin{bmatrix}1-x & x\\ -x & 1+x\end{bmatrix},\quad \mathscr{E}_0=\begin{bmatrix}1-\mathrm{i}t & \mathrm{i}t\\ -\mathrm{i}t & 1+\mathrm{i}t\end{bmatrix},$$

and

$$\begin{aligned}&\Psi_0=\begin{bmatrix}\psi_0\\ \phi_0\end{bmatrix}=\mathscr{R}_0\mathscr{E}_0\mathscr{Z}_0,\\ &\phi[0,0]=\phi_0,\quad \psi[0,0]=\psi_0,\\ &\phi[1,0]=\mathrm{i}\phi_0,\quad \psi[1,0]=\mathrm{i}\psi_0.\end{aligned}$$

Thus, the first-order rogue wave solution for the nonlocal defocusing NLSE is

$$q^{[1]}=\mathrm{i}\left[1+2\mathrm{i}\frac{\nabla_{12}}{\nabla_{11}}\right]\exp(\mathrm{i}t), \tag{3.1}$$

where

$$\nabla_{11}=\begin{vmatrix}\psi_0 & -\widetilde{\phi}_0\\ \phi_0 & \widetilde{\psi}_0\end{vmatrix},\quad \nabla_{12}=\begin{vmatrix}\mathrm{i}\phi_0 & -\mathrm{i}\widetilde{\psi}_0\\ \phi_0 & \widetilde{\psi}_0\end{vmatrix}.$$

The first-order rogue wave solution can be rewritten as

$$q^{[1]}=\mathrm{i}\left[-1+2\frac{(1+2\mathrm{i}t)(z_{1,0}-z_{2,0})^2}{\nabla}\right]\exp(\mathrm{i}t),$$

where

$$\nabla = 2\left(z_{1,0}-z_{2,0}\right)^2\left(x^2+t^2\right)-2\left(z_{1,0}^2-z_{2,0}^2\right)x+z_{1,0}^2+z_{2,0}^2.$$

When $z_{1,0}=z_{2,0}$, it can be reduced to the plane wave solution of the defocusing NLSE.

When $z_{1,0}=-z_{2,0}$, it can be reduced to

$$q^{[1]} = \mathrm{i}\left[-1+4\frac{(1+2\mathrm{i}t)}{4x^2+4t^2+1}\right]\exp(\mathrm{i}t),$$

this rogue wave is nonsingular, as shown in Fig. 1(a).

When $z_{1,0}=\mathrm{i}z_{2,0}$, it can be reduced to

$$q^{[1]} = \mathrm{i}\left[-1+\frac{4(1+2\mathrm{i}t)}{4(x+\mathrm{i}/2)^2+4t^2+1}\right]\exp(\mathrm{i}t),$$

this rogue wave would collapse at $x=0$. Based on the above analysis, we also know the properties of the first-order rogue wave solution.

Fig. 1 shows that this rogue wave is nonsingular. As shown in Fig. 1(a), the solution exhibits the classical x-symmetric Peregrine soliton by choosing the suitable free parameters. When the free parameters changes, this solution would be x-asymmetric, which is shown in Figs. 1(b) and 1(c). Thus, we declare that the free parameters can determine the central symmetry pattern of the rogue wave solution.

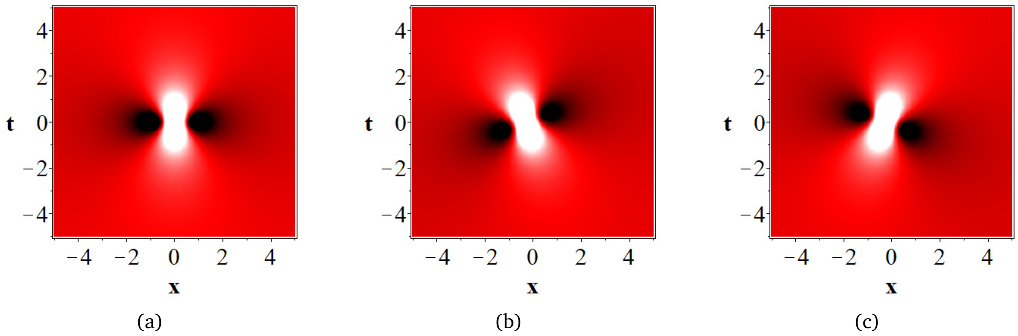


Figure 1: First-order rogue wave solution (3.1) to the nonlocal defocusing NLSE with parameters: (a) $(z_{1,0},z_{2,0})=(1,-1)$. (b) $(z_{1,0},z_{2,0})=(1,-2\mathrm{i})$. (c) $(z_{1,0},z_{2,0})=(1,2\mathrm{i})$.

Case 2. To obtain the second-order rogue wave solutions for the nonlocal defocusing NLSE, here we let $N=2$ in the previous section. Then

$$\mathscr{R}_1=\begin{bmatrix}\mathscr{I}_1 & x^3/3\\ -x^3/3 & \mathscr{I}_2\end{bmatrix},\quad \mathscr{E}_1=\begin{bmatrix}\mathscr{K}_1 & \mathrm{i}t\left(t^2-3\right)/3\\ -\mathrm{i}t\left(t^2-3\right)/3 & \mathscr{K}_2\end{bmatrix},$$

and

$$\begin{aligned}&\mathscr{I}_1=-x+x^2-\frac{1}{3}x^3, && \mathscr{I}_2=x+x^2+\frac{1}{3}x^3,\\ &\mathscr{K}_1=-2\mathrm{i}t-t^2+\frac{1}{3}\mathrm{i}t^3, && \mathscr{K}_2=2\mathrm{i}t-t^2-\frac{1}{3}\mathrm{i}t^3\end{aligned} \tag{3.2}$$

with

$$\Psi_1 = \begin{bmatrix} \psi_1 \\ \phi_1 \end{bmatrix} = (\mathscr{R}_0\mathscr{E}_1 + \mathscr{R}_1\mathscr{E}_0)\mathscr{Z}_0 + \mathscr{R}_0\mathscr{E}_0\mathscr{Z}_1,$$

$$\begin{aligned} &\phi[0,1] = \phi_1, && \psi[0,1] = \psi_1, \\ &\phi[2,0] = -\phi_0, && \psi[2,0] = -\psi_0, \\ &\phi[1,1] = \mathrm{i}(\phi_0 + \phi_1), && \psi[1,1] = \mathrm{i}(\psi_0 + \psi_1), \\ &\phi[2,1] = -(2\phi_0 + \phi_1), && \psi[2,1] = -(2\psi_0 + \psi_1). \end{aligned}$$

Hence, the second-order rogue wave solutions for the nonlocal defocusing NLSE are

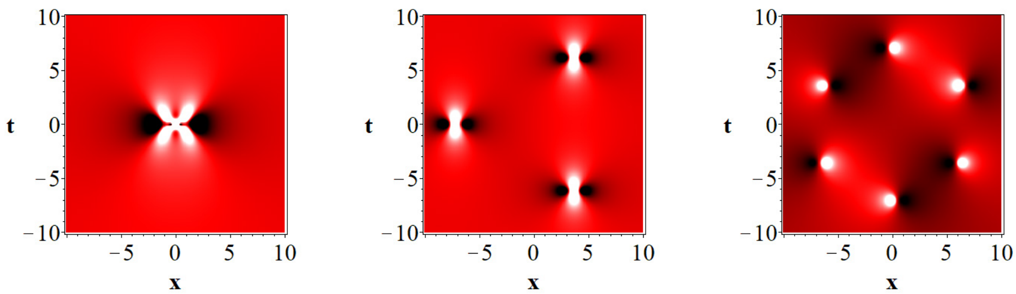
$$q^{[2]} = \mathrm{i}\left[1 + 2\mathrm{i}\frac{\nabla_{22}}{\nabla_{21}}\right]\exp(\mathrm{i}t), \tag{3.3}$$

where

$$\nabla_{21} = \begin{vmatrix} \psi[1,0] & \psi[1,1] & \widetilde{\phi}[1,0] & \widetilde{\phi}[1,1] \\ \psi[0,0] & \psi[0,1] & -\widetilde{\phi}[0,0] & -\widetilde{\phi}[0,1] \\ \phi[1,0] & \phi[1,1] & -\widetilde{\psi}[1,0] & -\widetilde{\psi}[1,1] \\ \phi[0,0] & \phi[0,1] & \widetilde{\psi}[0,0] & \widetilde{\psi}[0,1] \end{vmatrix},$$

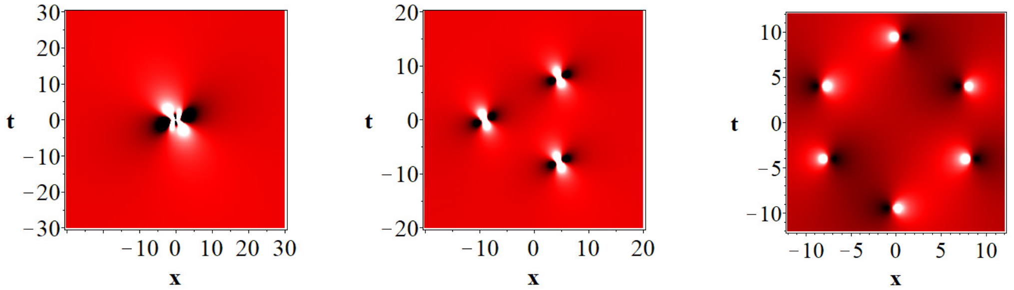
$$\nabla_{22} = \begin{vmatrix} \phi[2,0] & \phi[2,1] & \widetilde{\psi}[2,0] & \widetilde{\psi}[2,1] \\ \psi[0,0] & \psi[0,1] & -\widetilde{\phi}[0,0] & -\phi[0,1] \\ \phi[1,0] & \phi[1,1] & -\widetilde{\psi}[1,0] & -\widetilde{\psi}[1,1] \\ \phi[0,0] & \phi[0,1] & \widetilde{\psi}[0,0] & \widetilde{\psi}[0,1] \end{vmatrix}.$$

Here we present the main feature of second-order rogue wave solution, which are shown in Figs. 2 and 3. By tuning the free parameters $z_{1,0}, z_{2,0}, z_{1,1}, z_{2,1}$, we can get both nonsingular and singular (collapsing) solutions. A nonsingular solution can be observed in Fig. 2(a). In particular, the second-order x-symmetric rogue waves in Fig. 2(a) separate into three first-order x-symmetric rogue waves in Fig. 2(b) as $z_{2,1}$ are chosen a big enough value. Fig. 2(c)



(a) (b) (c)

Figure 2: Second-order rogue wave solution (3.3) to the nonlocal defocusing NLSE with parameters (a) $(z_{1,0}, z_{2,0}) = (1,-1), (z_{1,1}, z_{2,1}) = (0,0)$. (b) $(z_{1,0}, z_{2,0}) = (1,-1), (z_{1,1}, z_{2,1}) = (0,1000)$. (c) $(z_{1,0}, z_{2,0}) = (1,-1), (z_{1,1}, z_{2,1}) = (0,1000\mathrm{i})$.



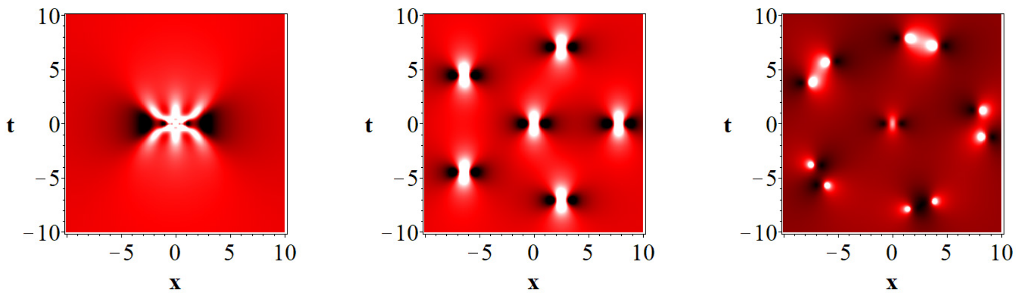
(a) (b) (c)

Figure 3: Second-order rogue wave solution (3.3) to the nonlocal defocusing NLSE with parameters (a) $(z_{1,0}, z_{2,0}) = (1, -i), (z_{1,1}, z_{2,1}) = (0, 0)$. (b) $(z_{1,0}, z_{2,0}) = (1, -i), (z_{1,1}, z_{2,1}) = (0, 1000i)$. (c) $(z_{1,0}, z_{2,0}) = (1, -i), (z_{1,1}, z_{2,1}) = (0, 1000)$.

show that the three nonsingular rogue waves separate into six singular rogue waves, since $z_{2,1}$ is chosen as 1000i in Fig. 2(c) instead of 1000 in Fig. 2(b). In addition, we can observe the second-order x-asymmetric rogue waves in Fig. 3(a), when the parameters $z_{2,0}$ is chosen as $-i$ in Fig. 3(a) instead of -1 in Fig. 2(a). Similarly, by turning the free parameters $z_{1,0}, z_{2,0}, z_{1,1}, z_{2,1}$, the three first-order x-asymmetric rogue waves and six singular rogue waves can be observed in Figs. 3(b) and 3(c), respectively.

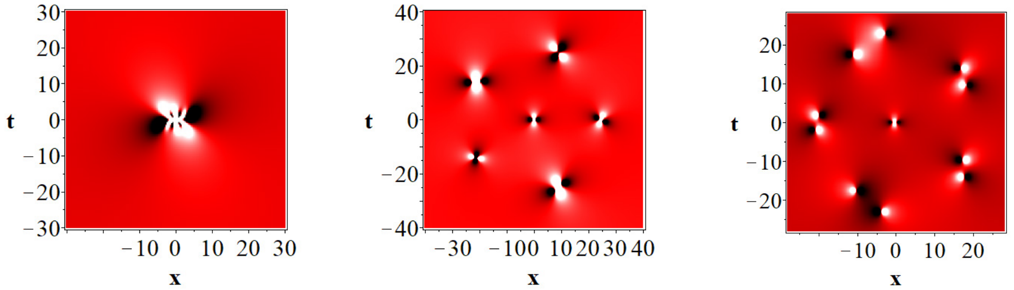
Case 3. In order to exhibit the effectiveness of these solution obtained in this work, we next discuss the third-order rogue wave solutions graphically.

From Fig. 4 to Fig. 6, we find that third-order rogue waves would display an even wider variety of solution patterns. From Fig. 4(a), we observe the third-order x-symmetric rogue waves. The third-order x-symmetric rogue waves separate into six first-order x-symmetric rogue waves in Fig. 4(b) as $z_{2,2}$ are chosen a big enough value. Fig. 4(b) displays six Peregrine-like peaks arranged in pentagon patterns. When $(z_{1,2}, z_{2,2})$ is chosen as 1000(i, 1) in Fig. 4(c) instead of (0,10000) in Fig. 4(b), the solution shows collapsing solutions in



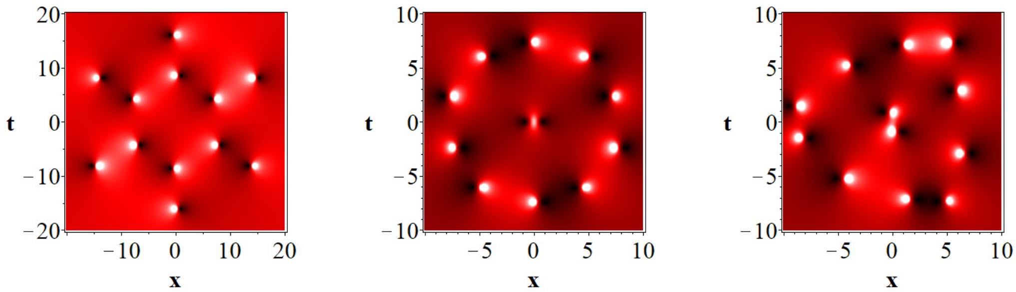
(a) (b) (c)

Figure 4: Third-order rogue wave solution to the nonlocal defocusing NLSE with parameters (a) $(z_{1,0}, z_{2,0}) = (1, -1), (z_{1,1}, z_{2,1}) = (0, 0), (z_{1,2}, z_{2,2}) = (0, 0)$. (b) $(z_{1,0}, z_{2,0}) = (1, -1), (z_{1,1}, z_{2,1}) = (0, 0), (z_{1,2}, z_{2,2}) = (0, 10000)$. (c) $(z_{1,0}, z_{2,0}) = (1, -1), (z_{1,1}, z_{2,1}) = (0, 0), (z_{1,2}, z_{2,2}) = 10000(i, 1)$.



(a) (b) (c)

Figure 5: Third-order rogue wave solution to the nonlocal defocusing NLSE with parameters (a) $(z_{1,0},z_{2,0}) = (1,-\mathrm{i}),(z_{1,1},z_{2,1}) = (0,0),(z_{1,2},z_{2,2}) = (0,0)$. (b) $(z_{1,0},z_{2,0}) = (1,-\mathrm{i}),(z_{1,1},z_{2,1}) = (1000\mathrm{i},1000\mathrm{i})$, $(z_{1,2},z_{2,2}) = (0,0)$. (c) $(z_{1,0},z_{2,0}) = (1,-\mathrm{i}),(z_{1,1},z_{2,1}) = (-100,1000),(z_{1,2},z_{2,2}) = (0,0)$.



(a) (b) (c)

Figure 6: Third-order rogue wave solution to the nonlocal defocusing NLSE with parameters (a) $(z_{1,0},z_{2,0}) = (1,-1),(z_{1,1},z_{2,1}) = (1000\mathrm{i},1000\mathrm{i}),(z_{1,2},z_{2,2}) = (100,100\mathrm{i})$. (b) $(z_{1,0},z_{2,0}) = (1,-1),(z_{1,1},z_{2,1}) = (0,0),(z_{1,2},z_{2,2}) = (0,10000\mathrm{i})$. (c) $(z_{1,0},z_{2,0}) = (1,-1),(z_{1,1},z_{2,1}) = (100,100)$, $(z_{1,2},z_{2,2}) = (0,10000\mathrm{i})$.

Fig. 4(c), which have ten singular peaks surrounding one Peregrine-like nonsingular peak. The similar phenomenon are also observed in Figs. 5(c) and 6(b). From Fig. 5(a), we observe the third-order x-asymmetric rogue waves. The third-order x-asymmetric rogue waves separate into six first-order rogue waves in Fig. 5(b) as $z_{1,1}, z_{2,1}$ are chosen the suitable parameters. Fig. 6(a) displays twelve singular peaks arranged in two circular patterns. Interestingly, Fig. 6(c) exhibits ten singular peaks surrounding two singular peaks, which have not been observed before.

Remark 3.1. In this work, we have verified that the defocusing nonlocal $\mathscr{P}\mathscr{T}$-symmetric NLSE could support the analytical rogue waves. For the focusing nonlocal $\mathscr{P}\mathscr{T}$-symmetric NLSE, its rogue wave solutions and rational solutions have been discussed in [44, 54]. Following the same way provided in this work, the fully explicit expressions of rogue wave solution for the focusing nonlocal NLSE (i.e., the Eq. (1.2) with $\sigma = 1$) can also be derived. Here we only give the explicit formula of first and second-order rogue wave solutions (the explicit formula of the solutions will be given in (3.4) and (3.5)) of the focusing nonlocal NLSE, which is different from the results obtained by [44, 54].

1. The first-order rogue wave solution for the nonlocal focusing NLSE reads

$$q^{[1]}=\left[1-2\frac{\nabla_{12}}{\nabla_{11}}\right]\exp(it), \tag{3.4}$$

where

$$\nabla_{11}=\begin{vmatrix} \psi_0 & -\widetilde{\phi}_0 \\ \phi_0 & \widetilde{\psi}_0 \end{vmatrix}, \quad \nabla_{12}=\begin{vmatrix} i\phi_0 & -i\widetilde{\psi}_0 \\ \phi_0 & \widetilde{\psi}_0 \end{vmatrix},$$

and

$$\begin{aligned}
&\Psi_0=\begin{bmatrix} \psi_0 \\ \phi_0 \end{bmatrix}=\mathscr{R}_0\mathscr{E}_0\mathscr{Z}_0,\\
&\phi[0,0]=\phi_0, \quad \psi[0,0]=\psi_0,\\
&\phi[1,0]=i\phi_0, \quad \psi[1,0]=i\psi_0
\end{aligned}$$

with

$$\mathscr{R}_0=\begin{bmatrix} 1-x & ix \\ ix & 1+x \end{bmatrix}, \quad \mathscr{E}_0=\begin{bmatrix} 1-it & -t \\ -t & 1+it \end{bmatrix}.$$

2. The second-order rogue wave solutions for the nonlocal focusing NLSE are

$$q^{[2]}=\left[1-2\frac{\nabla_{22}}{\nabla_{21}}\right]\exp(it), \tag{3.5}$$

where

$$\nabla_{21}=\begin{vmatrix} \psi[1,0] & \psi[1,1] & -\widetilde{\phi}[1,0] & -\widetilde{\phi}[1,1] \\ \psi[0,0] & \psi[0,1] & \widetilde{\phi}[0,0] & \widetilde{\phi}[0,1] \\ \phi[1,0] & \phi[1,1] & -\widetilde{\psi}[1,0] & -\widetilde{\psi}[1,1] \\ \phi[0,0] & \phi[0,1] & \widetilde{\psi}[0,0] & \widetilde{\psi}[0,1] \end{vmatrix},$$

$$\nabla_{22}=\begin{vmatrix} \phi[2,0] & \phi[2,1] & \widetilde{\psi}[2,0] & \widetilde{\psi}[2,1] \\ \psi[0,0] & \psi[0,1] & \widetilde{\phi}[0,0] & \phi[0,1] \\ \phi[1,0] & \phi[1,1] & -\widetilde{\psi}[1,0] & -\widetilde{\psi}[1,1] \\ \phi[0,0] & \phi[0,1] & \widetilde{\psi}[0,0] & \widetilde{\psi}[0,1] \end{vmatrix}$$

and

$$\begin{aligned}
&\mathscr{R}_1=\begin{bmatrix} \mathscr{I}_1 & ix^3/3 \\ ix^3/3 & \mathscr{I}_2 \end{bmatrix},\\
&\mathscr{E}_1=\begin{bmatrix} \mathscr{K}_1 & t\left(t^2-3\right)/3 \\ t\left(t^2-3\right)/3 & \mathscr{K}_2 \end{bmatrix}
\end{aligned}$$

with

$$\Psi_1=\begin{bmatrix} \psi_1 \\ \phi_1 \end{bmatrix}=(\mathscr{R}_0\mathscr{E}_1+\mathscr{R}_1\mathscr{E}_0)\mathscr{Z}_0+\mathscr{R}_0\mathscr{E}_0\mathscr{Z}_1,$$

$$\begin{aligned} &\phi[0,1]=\phi_1, && \psi[0,1]=\psi_1,\\ &\phi[2,0]=-\phi_0, && \psi[2,0]=-\psi_0,\\ &\phi[1,1]=\mathrm{i}(\phi_0+\phi_1), && \psi[1,1]=\mathrm{i}(\psi_0+\psi_1),\\ &\phi[2,1]=-(2\phi_0+\phi_1), && \psi[2,1]=-(2\psi_0+\psi_1). \end{aligned}$$

Here $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{K}_1$, and $\mathscr{K}_2$ agree with (3.2).

Remark 3.2. Similar to [11, 49, 60], we can also check the modulational instability of the plane wave solution (2.2) and find the criterion for the existence of rogue waves in the nonlocal defocusing NLSE. Here we assume that the perturbed plane-wave solution is given by $q = \mathrm{i}\rho\exp(\mathrm{i}\rho^2 t)(1+u)$, and u can be conveniently expressed as linear combinations of pure Fourier modes

$$u = f_+(x,t)\exp\big(\mathrm{i}\kappa(x+\omega t)\big) + f_-(x,-t)\exp\big(\mathrm{i}\kappa(x-\omega t)\big),$$

where κ and ω are the real wave number of the disturbance and the complex phase velocity, respectively. Then, imposing the modulation instability analysis for this constant background solution, we find that when $\kappa^2 > \rho^2$, this constant background is modulationally unstable with the growth rate $\mathrm{Im} = 2\sqrt{\rho^2-\kappa^2}$, which provides a physical explanation for the inception of rogue waves.

4. Conclusions and Discussions

In this work, we have derived N-th order rogue wave solutions of the nonlocal $\mathscr{P}\mathscr{T}$-symmetric NLSE in defocusing regime through a DT by a separation of variable approach. Moreover, the interesting and complicated dynamic patterns of these rogue waves have been discussed by varying the available parameters. More interesting are the collapsing solutions, which show more complex patterns and have not been observed in the corresponding local focusing and defocusing NLSE. Although our explicit solutions exhibited here are lowest three order rogue waves, a parallel way can be used to work out the N-th order rogue waves. Finally, it is worth to emphasize that the technique presented in this work may be available to construct rogue waves of matrix versions of the reverse-time nonlocal $\mathscr{P}\mathscr{T}$-symmetric NLSE, even its hierarchy. Additionally, the above considerations indicate that there may exist more abundant and novel rogue waves in nonlocal nonlinear equations than in the local ones.

The above results imply that the $\mathscr{P}\mathscr{T}$-symmetry plays an significant role in the study of the defocusing nonlocal NLSE. The method used in this work can also be extended to other nonlocal nonlinear equations such that these models with $\mathscr{P}\mathscr{T}$-symmetric may generate novel rogue wave phenomena. We will further study these questions in the near future.

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