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Exact Solutions for a Higher-Order Boussinesq Equation and a New Higher-Order Boussinesq-Like Equation

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Abstract. A kind of higher-order Boussinesq equation is studied in this work. Based on the Bell polynomials theories, bilinear representations of the equation are derived, and diverse interaction solutions are constructed through Hirota's bilinear method. These include interaction solutions characterized by hyperbolic cosine and cosine functions, as well as interactions between lump and soliton solutions. In addition, generalized bilinear operators are used in order to construct a new higher-order Boussinesq-like equation, while lump and breather solutions are also developed utilizing Hirota's bilinear technique. For the various explicit solutions obtained in this work, several of them are considered to selected particular values for the relevant parameters in order to plot different kind of three-dimensional surfaces with associated two-dimensional density profiles to give a comprehensive understanding of the evolution for various solutions.

AMS subject classifications: 65M10, 78A48

Key words: Higher-order Boussinesq equation, higher-order Boussinesq-like equation, binary Bell polynomial, Hirota's bilinear method, generalized bilinear method.

1. Introduction

In 1834, Russel [21] discovered solitary waves. Since then, soliton theory [5] has attracted increasing attention and has become a significant branch of nonlinear science. The emphasis of the soliton problems is on nonlinear evolution equations (NLEEs) [1,35], many of which are used to describe waves propagation in liquid or gas mixtures [19]. Therefore, exploring the physical meaning of the solutions determined by NLEEs is a crucial problem.

At present, there are various methods to obtain explicit solutions of NLEEs, including the Hirota bilinear representation [8], tanh-coth method [29], Lie symmetry analysis method [12], Darboux transformation [7, 18], Bäcklund transformation [26]. Note that

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the Hirota's bilinear method has proven to be the most straightforward and efficient approach for constructing exact solutions for NLEEs. Utilizing different kind of test functions, lumps [27], breathers [32], interaction phenomenon [34] and other explicit solutions could be obtained directly. Moreover, Ma [16,17] introduced a generalized bilinear approach, extending the conventional Hirota bilinear operators. This development led to a new class of bilinear differential equations, employing linear subspaces to construct their solutions.

Among integrable NLEEs, the Boussinesq equation serves as a nonlinear model for shallow water waves. The classical (1+1)-dimensional Boussinesq equation [3] is often expressed as

$$u_{tt}+u_{xx}-3(u^2)_{xx}-u_{xxxx}=0, \tag{1.1}$$

the Eq. (1.1), originally proposed by Boussinesq and later derived by Ursell [25] and Whitham [31], has been widely used to model the propagation of long waves in liquid mixtures [10]. Since (1.1) is a completely integrable system, it possesses an infinite number of conservation laws and exhibits Lie group symmetries, as previously established. If the ratio of the u_{tt} term to the u_{xxxx} term is negative, the Eq. (1.1) reduces to the "bad" Boussinesq equation. Conversely, reversing the sign of the u_{xxxx} term yields the "good" Boussinesq equation. The former supports smooth breather and rogue wave solutions, while the latter exhibits singular breather and rogue wave solutions [15].

In recent years, numerous enhanced Boussinesq equations have been developed and investigated. Kuma and Rani [11] obtained the exact solution of the extend (2+1)-dimensional Boussinesq equation employing Lie symmetry method, multiple-soliton solutions of the fourth-order Boussinesq equation are constructed by Wazwaz [30] utilizing Hirota's bilinear method and tanh-coth method. W. Sun and Y. Sun [24] employed the generalized Darboux transformation to derive a degenerate respiration solution of classical Boussinesq equation. In this paper, we consider a higher-order Boussinesq equation (HOBe), viz.

$$\begin{aligned} &u_{tt}+\gamma u_{xx}-\alpha\left(u_{xxxx}+6u_x^2+6uu_{xx}\right)\\ &\quad-\beta\left(15uu_{xxxx}+30u_xu_{xxx}+15u_{xx}^2+45u^2u_{xx}+90uu_x^2+u_{xxxxxx}\right)=0, \end{aligned} \tag{1.2}$$

where $u=u(x,t),\alpha,\beta,\gamma$ are arbitrary constants. Lump solutions and N-soliton solutions of the Eq. (1.2) were derived by Ren [20]. However, to the best our knowledge, the interaction solutions of Eq. (1.2) have not been studied.

It is also worth noting that for some NLEEs, the long-time asymptotics behavior for explicit solutions has an intriguing phenomenon. The investigation is based on Riemann-Hilbert technique and is a popular topic in resent years. Zhiqiang *et al.* [14] constructed a Riemann-Hilbert problem and employed $\overline{\partial}$-steepest descent method to investigate the long-time asymptotics behavior of the explicit solution of the WKI equation with weighted Sobolev initial data. The long-time asymptotics behavior for "good" Boussinesq equation was studied by Charlier *et al.* [4] and others authors, while a defocusing complex modified KdV equation was considered by Riemann-Hilbert approach [28]. Furthermore, utilizing N-fold Darboux transformation, the generalized perturbation $(n,N-n)$-fold Darboux transformation is used to solve a two-component nonlinear wave system [33].