East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2024-195.010325

Vol. **16**, No. 1, pp. 105-128
February 2026

High-Order Operator-Compensation Schemes with Mass and Energy Conservation for the Multi-Dimensional Zakharov System

Yanjing Li[1], Xianggui Li[1], Dongying Hua[1] and Yongyong Cai[2,*]

[1] *School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China.*

[2] *Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China.*

Received 21 November 2024; Accepted (in revised version) 1 March 2025.

Abstract. This paper presents high-order operator-compensation (OC) methods and the Crank-Nicolson (CN) technology to solve the Zakharov system (ZS). Firstly, the OC method is used for spatial discretization, and semi-discrete high-order OC schemes are obtained, which can maintain mass and energy conservation. Subsequently, the CN technology is adopted for temporal discretization, and full-discrete high-order OC schemes are obtained, which can inherit mass and energy conservation and achieve arbitrary even-order accuracy in space and second-order accuracy in time. In the process of solving ZS, we employ the linearization method to handle the nonlinear ZS, which significantly improves the computational efficiency. Considering the large stencil of numerical schemes, we propose a fast numerical iterative method to deal with the problem, which saves the computational cost. Numerical experiments are given to test accuracy order, verify conservation property, and simulate dynamic evolution.

AMS subject classifications: 65M06, 65M20, 35L70, 35Q55

Key words: Zakharov system, high-order operator-compensation method, conservation, Crank-Nicolson technology.

1. Introduction

Zakharov system is a nonlinear coupled wave equation describing the interaction of high-frequency Langmuir waves and low-frequency ionic acoustic waves in plasma. It is regarded as a general model for controlling the interaction between dispersive and non-dispersive waves [27]. The standard dimensionless form of ZS is

$$i\Psi_t + \Delta\Psi = W\Psi, \qquad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \tag{1.1}$$

$$W_{tt} - \Delta W = \Delta(|\Psi|^2), \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \tag{1.2}$$

*Corresponding author. *Email address:* `yongyong.cai@bnu.edu.cn` (Y. Cai)

where $i = \sqrt{-1}$ is the imaginary unit, $d = 1,2,3$ the spatial dimension, $\mathbf{x}$ the spatial coordinate, t the time variable, Δ the d-dimensional Laplace operator, $\Psi := \Psi(\mathbf{x},t)$ unknown complex function representing the envelope of a high-frequency electric field, and $W := W(\mathbf{x},t)$ unknown real function representing the deviation of the ion density from its equilibrium value. The initial conditions are

$$\Psi(\mathbf{x},0) = \Psi_0(\mathbf{x}), \quad W(\mathbf{x},0) = W_0(\mathbf{x}), \quad W_t(\mathbf{x},0) = \omega(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.3}$$

where $\Psi_0(\mathbf{x})$ represents a given smooth complex function, $W_0(\mathbf{x})$ and $\omega(\mathbf{x})$ represent two given smooth real functions.

The Zakharov system possesses the two essential conservation quantities — i.e. the mass

$$M(t) := \int_{\mathbb{R}^d} |\Psi(\mathbf{x},t)|^2 d\mathbf{x} \equiv M(0), \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0$$

and the energy

$$\begin{aligned} E(t) := & \int_{\mathbb{R}^d} |\nabla\Psi(\mathbf{x},t)|^2 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(\mathbf{x},t)|^2 d\mathbf{x} \\ & + \frac{1}{2}\int_{\mathbb{R}^d} |W(\mathbf{x},t)|^2 d\mathbf{x} + \int_{\mathbb{R}^d} |\Psi(\mathbf{x},t)|^2 W(\mathbf{x},t) d\mathbf{x} \\ \equiv & E(0), \quad t > 0, \end{aligned}$$

where the intermediate variable $u(\mathbf{x},t)$ satisfies

$$\Delta u = W_t, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0.$$

This system is applied in various fields such as plasma physics, fluid mechanics, and new energy. However, since the ZS is nonlinear, obtaining its analytical solutions for general initial conditions is challenging. It is necessary to study and solve the system through numerical methods.

Mathematically speaking, ZS can be seen as the coupling of the Schrödinger equation (1.1) and the wave equation (1.2). In theoretical analysis, there are numerous studies of the ZS — cf. [1, 16, 19, 20]. The global existence of weak solutions and the existence and uniqueness of smooth solutions with given smooth initial values for one-dimensional ZS were proven in [24]. Later, researchers improved the well posedness of ZS [5] and extended it to the case of generalized nonlinearity [10].

Extensive numerical research has been conducted on the ZS. Payne *et al.* [22] proposed the Fourier spectral method for the one-dimensional ZS, which suppresses aliasing errors by using only two-thirds of the Fourier components for a specific grid during fast Fourier transform. Glassey [11, 12] proposed an implicit finite difference scheme that could preserve energy conservation and proved its convergence in order $\mathcal{O}(h+\tau)$. A conservative difference scheme for ZS, which was implicit or semi-explicit depending on parameter choice,