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Asymptotic Behavior of Solutions for a Free Boundary Problem with a Monostable Nonlinearity

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Abstract. A free boundary problem with Dirichlet boundary conditions is studied. Such problems can be used for describing the spread of chemical substances or biological species, which live in a moving region $[0,h(t)]$. In this case, the free boundary $h(t)$ represents the spreading front. If the density of the substance or the population at the boundary exceeds a threshold value, they will be going to spread outwards. On the other hand, the outside environment may be not very beneficial for spreading and this generates a decay rate. We mainly analyze how the decay rate and threshold value affect the solutions. There is a trichotomy result: the solution is either shrinking, or the transition case, or spreading. Besides, if the decay rate or threshold value is large, only the shrinking happens.

AMS subject classifications: 35R35, 35B40, 35K57
Key words: Free boundary problem, asymptotic behavior, monostable, reaction-diffusion equation.

1. Introduction

Consider the following free boundary problem:

$$u_t = u_{xx} + f(u), \qquad t>0, \qquad 0<x<h(t), \tag{1.1a}$$
$$u(t,0)=0, \quad u(t,h(t))=\sigma, \qquad t>0, \tag{1.1b}$$
$$h'(t) = -u_x(t,h(t)) - \alpha, \qquad t>0, \tag{1.1c}$$
$$h(0)=h_0, \quad u(0,x)=u_0(x), \qquad 0\le x\le h_0, \tag{1.1d}$$

where $x=h(t)$ is a moving boundary, $f:[0,+\infty)\to\mathbb{R}$ a monostable nonlinearity, and $\alpha>0$, $\sigma\in(0,1)$ are given constants. In this paper, we always assume that the function $f\in C^1$ and satisfies the following conditions:

$$\begin{aligned} &f(u)>0, \quad u\in(0,1),\\ &f(u)<0, \quad u\in(1,+\infty),\\ &f(0)=f(1)=0, \quad f'(0)>0, \quad f'(1)<0. \end{aligned} \tag{1.2}$$

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The initial function u_0 belongs to $\mathscr{X}(h_0)$ for an $h_0 > 0$, where

$$\mathscr{X}(h_0) := \Big\{\phi \in C^2([0,h_0]) : \phi(0) = 0,\ \phi(h_0) = \sigma,\ \phi(x) \geq (\not\equiv) 0 \ \text{ in } \ (0,h_0)\Big\}.$$

Free boundary problems can describe the spreading of species or chemical substances invading into a new area. The solution $u(t,x)$ represents their density. The moving interval $[0,h(t)]$ is the area the species occupied at time $t > 0$, the free boundary $h(t)$ is moving depending on t, its speed is determined by the gradient of the species at the boundary. There is a resistance force with strength $\alpha > 0$ at the boundary caused by bad environment — cf. [4, 14, 24]. If species or chemical substances exceed a threshold value $\sigma > 0$, they will move outside. We assume $\sigma \in (0,1)$. As noted in [8], species are not too crowded for available resources in the given environment. However, it seems from $u(t,h(t)) = \sigma$ that the species may expand or shrink — i.e. spreading and shrinking of free boundary $h(t)$ is more complicated if $\sigma = 0$. We will analyze such cases and study the long time behavior of solutions.

Note that there are many free boundary problems with $\alpha = \sigma = 0$. If $u(t,0) = 0$ is replaced by $u_x(t,0)$, Du and Lin [9] studied the long time behaviour of solutions for a free boundary problem for $f(u)$ of logistic type. Besides spreading, they proved that vanishing — i.e. when $h(t)$ tends to a positive finite number and u tends to 0, may also happen. Apparently, in describing the spreading of species, free boundary problems are more reasonable than Cauchy problems, which have only hair-trigger result (any nonnegative solutions will spread) for monostable nonlinearity. For monostable nonlinearity, there are many works devoted to the long time behavior of solutions for free boundaries — cf. Refs. [2–4, 10, 12, 16, 18, 21]. Gu and Lou [17] added advection term βu_x and obtained different results, while Gu *et al.* [16] estimated the spreading speeds of the free boundaries. When nonlinearity $f(u)$ is bistable, Du and Lou [10] studied the Eq. (1.1a) with free boundaries

$$u\big(t,g(t)\big) = u\big(t,h(t)\big) = 0, \quad g'(t) = -\mu u\big(t,g(t)\big), \quad h'(t) = -\mu u\big(t,h(t)\big).$$

They obtained a trichotomy result — i.e. besides spreading and vanishing, there is also a transition case when $h(t) \to +\infty$ and u converges to a stationary solution defined on $\mathbb{R}$ as $t \to +\infty$. In [11], the authors answered several interesting questions left in [10], and completed the general theory for one-dimensional nonlinear free boundary problems. There are also some other free boundary problems with bistable nonlinearity, such as [17, 19], they obtained transition cases. Sharp estimates for the spreading speeds when spreading happens are obtained in [10, 12].

For $\sigma = 0$ and $\alpha > 0$, a free boundary problem for Fisher-KPP equation was studied in [2–4, 7]. Using the parameter α, the authors determined decay rate at the boundary, where species have growth resistance caused by bad environment outside of the existence interval (or say, the boundary). From their results, it is more difficult to spread for the solution when $\alpha > 0$ than $\alpha = 0$, and $h'(t) > 0$ if and only if $-u_x(t,h(t)) > \alpha$. They obtained different asymptotic behavior of solutions, especially the transition and vanishing cases.

If $\sigma > 0$, $\alpha = 0$, Du [8] studied the problem (1.1) with two free boundaries so that

$$u\big(t,g(t)\big)=u\big(t,h(t)\big)=\sigma,\quad h'(t)=-\frac{1}{\sigma}u_x\big(t,h(t)\big),$$

where $\sigma \in (0,1)$ is the preferred population density. Such a free boundary condition can describe the spreading of animals without a home for raising young ones — e.g. the wildebeests whose young one's can run with the mother almost immediately after birth. They showed that spreading happens — i.e. $-g_\infty = h_\infty + \infty$ and $u \to 1$.

The constant σ in the problem (1.1) has many explanations in tumor growth models. Byrne and Chaplain [1] proposed a tumor growth model with the influence of certain inhibitors. Assuming that the spherical-shaped tumor has no necrotic nucleus, they considered the following condition:

$$\begin{aligned}&\frac{\partial\sigma}{\partial r}(0,t)=0,\quad \sigma\big(R(t),t\big)=\sigma_R,\quad t>0,\\&\frac{\partial\beta}{\partial r}(0,t)=0,\quad \beta\big(R(t),t\big)=\beta_R,\quad t>0,\end{aligned}\tag{1.3}$$

where r is the polar radius and $R(t)$ the radius of tumor. In addition, $\sigma = \sigma(r,t)$ and $\beta = \beta(r,t)$ stand for the concentration of nutrients and inhibitors, respectively. The positive constants σ_R and β_R are the amount of nutrients and inhibitors the tumor gained from their surfaces, respectively. There are other models using the condition (1.3) [6, 13, 15]. Cui and Friedman [6] assumed that the concentration of external nutrient and external inhibitor is constants at the boundary — i.e. that the condition (1.3) is satisfied. Friedman and Reitich [15] used the condition (1.3) to describe the case when the birth rate of cells exceeds their death rate at the boundary of tumor.

Additionally, the condition (1.3) is also often used in the growth of the protocell [5, 14, 22, 24]. The building material of the protocell, denoted by C, will diffuse in the protocell. The molecules of building material can be polymerized to the constituent material of the protocell. However, since all sites inside the protocell are occupied, the polymerization only happens at the boundary, where the concentration of C must be exactly equal to C_1. Thus, on the boundary, we have

$$C\big(t,h(t)\big)=C_1.$$

If the concentration of C exceeds the threshold value C_1, the polymerization will take place, and the protocell will grow sequentially. In addition to polymerization, building materials may have a disintegration, denoted by β, which is caused by such factors as aging. This results in the tendency of the shrinking of the protocell — cf. Fig. 1. At the same time, the flux of building materials at the boundary causes cells to grow. So it satisfies the following boundary condition:

$$V_n=-\frac{\partial C}{\partial n}-\beta,$$

where n is the exterior normal, and V_n is the velocity along the direction n. In one dimensional space, this boundary condition is the one used in the problem (1.1).

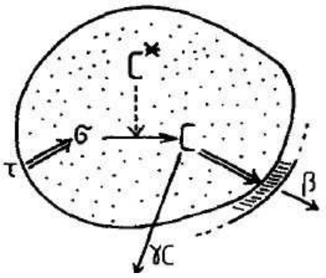

Figure 1: Illustration of protocell model.

In this paper, we mainly study the asymptotic behavior of solutions of the problem (1.1), which can model the spreading of species or chemical substances. When the density of species at the boundary exceeds a threshold number, spreading prepares to happen. However, the condition $u(t,h(t))=\sigma$ will make the free boundary $h(t)$ spread and shrink. Additionally, during the spreading process, the environment at the boundary $x=h(t)$ may be not very beneficial caused by some factors such as aging or predators, there is often a decay rate $\alpha>0$ at the boundary. On the other hand, the species try to spread since the density at the boundary exceeds the threshold number, they will look for living resources or food etc.. Of course, if the density reaches the maximum at the boundary $x=h(t)$, the species will shrink to the inside (move backwards). We will analyze how the threshold number and the decay rate affect the asymptotic behavior of solutions.

The main purpose of this paper is to study the asymptotic behavior of solutions for the problem (1.1). This depends on the parameters $\sigma>0$ and $\alpha>0$. For any $\sigma\in(0,1)$, there is a unique crucial point $\alpha^*(\sigma)$ determining the long time behavior of solutions — cf. Section 2. From Remark 3.2, [8] and [23], the problem (1.1) has a unique solution (u,h) with $u(t,x)\in C^{(1+\nu)/2,1+\nu}([0,T)\times[0,h(t)])$, $h(t)\in C^{1+\nu/2}([0,T))$ for $\nu\in(0,1)$ and $T\in(0,+\infty]$.

Theorem 1.1. *If $0<\alpha<\alpha^*(\sigma)$, then there are the following possibilities for the solution (u,h) of the problem* (1.1)*:*

(i) Shrinking. $0<T<+\infty$, $\lim_{t\to T}h(t)=0$ *and*

$$0\le\lim_{t\to T}\max_{x\in[0,h(t))}u(t,x)\le\sigma.$$

(ii) Transition. $T=+\infty$, $\lim_{t\to+\infty}h(t)=l_\alpha$ *and*

$$\lim_{t\to+\infty}u(t,x)=v_\alpha(x)\ \textit{ locally uniformly for }\ x\in(0,l_\alpha),$$

where $(q,l)=(v_\alpha,l_\alpha)$ is the unique solution of

$$\begin{aligned}&q''+f(q)=0, \quad z\in(0,l),\\&q(z)>0, \quad z\in(0,l),\\&q(0)=0, \quad q(l)=\sigma, \quad q'(l)=-\alpha.\end{aligned}$$

(iii) *Spreading.* $T = +\infty$, $\lim_{t\to+\infty} h(t) = +\infty$ *and* $\lim_{t\to+\infty} u(t,x) = U(x)$ *locally uniformly for* $x \in [0,+\infty)$*, where* $U(x)$ *is the unique solution of*

$$\begin{aligned} &v'' + f(v) = 0, \quad && x \in [0,+\infty), \\ &v(0) = 0, && v(+\infty) = 1. \end{aligned}$$

Moreover, if the initial data $u_0 = \tau\phi$ *for* $\phi \in \mathscr{X}(h_0)$*, then there is a constant* $\tau_* > 0$ *such that, shrinking happens when* $\tau < \tau_*$*, transition happens when* $\tau = \tau_*$ *and spreading happens when* $\tau > \tau_*$.

Theorem 1.2. *If* $\alpha \geq \alpha^*(\sigma)$*, then only the shrinking happens for any solution of the problem* (1.1).

This paper is organized as follows. In Section 2, we analyze the stationary solutions. In Section 3, we show the comparison principles, the estimates of $h'(t)$ and the existence of solutions. In Section 4, we give the asymptotic behavior of solutions and sufficient conditions for spreading, as well as shrinking and the transition case. In Section 5, we complete the proofs of main theorems.

2. Stationary Solutions

In this section, we analyze stationary solutions of the problem (1.1) with $\sigma \in (0,1)$ and $\alpha > 0$, which will be used in the main theorems.

Consider

$$v'' + f(v) = 0, \quad x > 0. \tag{2.1}$$

Here we focus on the nonnegative bounded stationary solutions. The defined domain J of $v(\cdot)$ is either $\mathbb{R}$, $[0,+\infty)$, $(-\infty,0]$ or $[0,a]$ for an $a > 0$, especially when $J = (0,a]$. Consider

$$v(a) = \sigma, \quad v'(a) = -\alpha < 0. \tag{2.2}$$

Let $p = v'$, so the Eq. (2.1) is changed into

$$\frac{\mathrm{d}p}{\mathrm{d}v} = -\frac{f(v)}{p}.$$

By the phase plane analysis [10, 17, 20], we have Fig. 2.

By (2.1), (2.2) and the phase plane analysis, the solutions of the problem (2.1) can be classified as follows:

(i) Constant solutions: $v(x) = 0$, 1 for $x \in J = \mathbb{R}$.

(ii) Strictly increasing solution on the half-line U: $v(x) = U(x)$ is the unique solution of (2.1) and it satisfies

$$v(0) = 0, \quad v(+\infty) = 1, \quad v'(\cdot) > 0 \quad \text{in} \quad [0,+\infty).$$

Denote $\alpha_0 := U^{'}(0)$. For any $0 < \sigma < 1$, we define $\alpha^*(\sigma) := U'|_{U_1=\sigma} > 0$ (cf. Fig. 2). Apparently, $\alpha^*(\sigma) \to \alpha_0$ as $\sigma \to 0$, and $\alpha^*(\sigma) \to 0$ as $\sigma \to 1$.

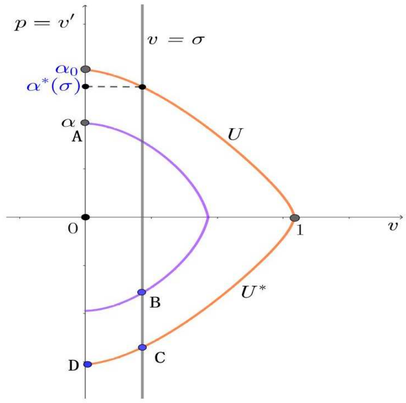

Figure 2: For any $\alpha \in (0, \alpha^*(\sigma))$, there is a unique compactly supported solution (v_α, l_α) satisfying boundary conditions $v_\alpha(l_\alpha) = \sigma$ and $-v_\alpha(l_\alpha) = \alpha$, see the trajectory from point A to point B. The trajectory from point $(0, \alpha_0)$ to $(1, 0)$ is the strictly increasing solution $U(x)$. The trajectory from point $(1, 0)$ to $(0, -\alpha_0)$ is the strictly decreasing solution $U^*(x)$.

(iii) Compactly supported solutions: From the phase plane analysis (cf. [10, 20]), when $0 < \alpha < \alpha_0$, there exists $l_0 > 0$, for any $l > l_0$, the problem

$$\begin{aligned} &q'' + f(q) = 0, \quad && z \in (0, l), \\ &q(z) > 0, && z \in (0, l), \\ &q(0) = 0, \quad q(l) = 0, \quad && q'(l) = -\alpha \end{aligned}$$

has a unique solution denoted by $(\widetilde{v}_\alpha, \widetilde{l}_\alpha)$.

For $0 < \alpha < \alpha^*(\sigma)$, the phase plane shows that the problem

$$\begin{aligned} &q'' + f(q) = 0, \quad && z \in (0, l), \\ &q(z) > 0, && z \in (0, l), \\ &q(0) = 0, \quad q(l) = \sigma, \quad && q'(l) = -\alpha \end{aligned} \tag{2.3}$$

has a unique solution, denoted by (v_α, l_α), such as the trajectory from point A to point B in Fig. 2. Note that v_α is the solution of the problem (1.1) satisfying the free boundary conditions, we will use them to construct upper and lower solutions.

(iv) Strictly decreasing solution on the half-line U^*: $v(x) = U^*(x)$ is the unique solution of (2.1) and it satisfies

$$v(0) = 0, \quad v(-\infty) = 1, \quad v'(\cdot) < 0 \quad \text{in } (-\infty, 0].$$

By the symmetry of the equation, we have $-(U^*)'(0) = U'(0)$, then $\alpha_0 = -(U^*)'(0)$. From phase plane analysis (cf. Fig. 2), there is a unique L^* such that

$$U^*(-L^*) = \sigma \quad \text{and} \quad -(U^*)'(-L^*) = \alpha^*(\sigma). \tag{2.4}$$

See point C in Fig. 2. The trajectory from point C to point D is a part of U^*, it satisfies (2.4) and $U^*(-\infty) = 1$.

(v) Compactly supported traveling waves $q(x-ct)$: $u(t,x)=q(x-ct)$ satisfies the following problem:

$$\begin{aligned} &q''+cq'+f(q)=0, \quad 0<x<l,\\ &q(0)=0, \quad q(l)=\sigma, \\ &q'(l)=-\alpha+c. \end{aligned} \tag{2.5}$$

When $0<\alpha<\alpha^*(\sigma)$, for any $c\in(-\alpha,c^*)$, the problem (2.5) has a unique solution (cf. [3]), denoted by (v_c^*,l_c^*), where c^* is the speed of traveling wave of the problem (1.1a). Actually, v_c^* continuously depends on c. In this paper, we only use the compactly supported traveling waves for small $c>0$.

3. Basic Results and Priori Estimates for h'

The forthcoming comparison theorems follow directly from [8, Lemmas 2.2-2.3].

Lemma 3.1. *Assume that* $T\in(0,\infty)$, $\overline{h}\in C^1([0,T])$, $\overline{u}\in C^{1,2}(D_T)\cap C(\overline{D}_T)$ *with* $D_T=\{(t,x)\in\mathbb{R}^2: 0<t\le T,\ 0<x<\overline{h}(t)\}$, *and*

$$\begin{aligned} &\overline{u}_t\ge\overline{u}_{xx}+f(\overline{u}), && t\in(0,T], \quad x\in\big(0,\overline{h}(t)\big),\\ &\overline{u}(t,0)=0, \quad \overline{u}\big(t,\overline{h}(t)\big)=\sigma, && t\in(0,T],\\ &\overline{h}'(t)\ge-\overline{u}_x\big(t,\overline{h}(t)\big)-\alpha, && t\in(0,T] \end{aligned}$$

with $u_0(x)\le\overline{u}(0,x)$ *for* $x\in[0,h_0]\subseteq[0,\overline{h}(0)]$. *Let* (u,h) *be the solution of the problem* (1.1) *with initial data* $u_0(x)$, *then*

$$u(t,x)\le\overline{u}(t,x), \quad x\in[0,h(t)]\subset[0,\overline{h}(t)], \quad t\in(0,T].$$

Proof. Note that u and $\overline{u}$ cannot obtain local minimal value smaller than σ, and $\overline{u}(t,x)>\sigma$ for $x\in[h(t),\overline{h}(t))$. We claim that $h(t)<\overline{h}(t)$ for all $t\in(0,T]$. Obviously, it holds for small $t>0$ since $h(0)=h_0<\overline{h}(0)$. Otherwise, we can find a first $t^*\le T$ such that $h(t)<\overline{h}(t)$ for $t\in(0,t^*)$ and $h(t^*)=\overline{h}(t^*)$, which follows that

$$h'(t^*)\ge\overline{h}'(t^*). \tag{3.1}$$

We now compare u and $\overline{u}$ over the region

$$\Omega_t:=\big\{(t,x)\in\mathbb{R}^2: 0<t\le t^*, 0<x<h(t)\big\}.$$

By the strong maximum principle, we have $u(t,x)<\overline{u}(t,x)$ in Ω_t. Therefore, $w(t,x):=\overline{u}(t,x)-u(t,x)>0$ in Ω_t, with $w(t^*,h(t^*))=0$. It follows from Hopf boundary lemma that $w_x(t^*,h(t^*))<0$. Then we deduce

$$\overline{h}'(t^*)-h'(t^*)\ge-\Big[\overline{u}_x(t^*,\overline{h}(t^*))-u_x(t^*,h(t^*))\Big]=-w_x\big(t^*,h(t^*)\big)>0.$$

However, this contradicts (3.1), which proves our claim that $h(t)<\overline{h}(t)$ for all $t\in(0,T]$. By the usual comparison principle over Ω_T, we conclude that $u<\overline{u}$ in Ω_T. $\square$

Lemma 3.2. *Assume* $T \in (0,\infty)$, $\overline{u} \in C^{1,2}(D_T) \cap C(\overline{D}_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ x > 0\}$, *and*

$$\begin{aligned} &\overline{u}_t \geq \overline{u}_{xx} + f(\overline{u}), \quad 0 < t \leq T, \quad x > 0, \\ &\overline{u}\big(t,h(t)\big) \geq \sigma, \quad \overline{u}(t,0) = 0, \quad t > 0. \end{aligned}$$

Let (u,h) *be the solution of the problem* (1.1) *with initial data* $u_0(x) \leq \overline{u}(0,x)$ *in* $[0,h_0]$. *Then* $u(t,x) \leq \overline{u}(t,x)$ *for* $x \in [0,h(t)]$ *and* $t \in (0,T]$.

Remark 3.1. Generally, we call $(\overline{u},\overline{h})$ the upper solution of the problem (1.1), and the lower solution is defined by revising all the inequalities in the above lemma.

Lemma 3.3. *Assume* (1.2). *Let* (u,h) *be the solution of the problem* (1.1) *defined for* $t \in [0,T)$ *with* $0 < T \leq \infty$. *Then there exists a constant* $M_1 \geq \sigma$ *such that*

$$0 < u(t,x) \leq M_1, \quad t \in [0,T), \quad x \in [0,h(t)].$$

There also exists M_2 *depending on* M_1 *but independent of* T *such that*

$$h'(t) \leq M_2, \quad t \in (0,T).$$

Proof. If $\|u_0\|_{L^\infty} \geq 1$, it can be derived from the property of f and the comparison principle that $u(t,x) < \|u_0\|_{L^\infty}$ for all $t \in [0,T)$ and $x \in [0,h(t)]$. Otherwise, $u(t,x) < 1$. Hence,

$$u(t,x) \leq M_1 := \max\{1, \|u_0\|_{L^\infty}\}, \quad t \in [0,T), \quad x \in [0,h(t)].$$

We next give the upper boundedness of $h'(t)$. In the following, we assume $u_x(t,h(t)) < 0$. Otherwise there is nothing left to prove since

$$h'(t) = -u_x\big(t,h(t)\big) - \alpha < -\alpha,$$

when $u_x(t,h(t)) > 0$. Choose a large enough constant

$$D = \max\left\{ \frac{M_1\alpha + \sqrt{M_1^2\alpha^2 + 2M_1N_1(M_1+\sigma)}}{2M_1}, \frac{\|u_0\|_{C^1([0,h_0])}}{(3/4)M_1+\sigma}, \frac{\|u_0'\|_{L^\infty([0,h_0])}}{2M_1} \right\},$$

where $N_1 := \max_{0<u<M_1} |f'(u)|$.

Define

$$F(t,x) := M_1 D\big(h(t)-x\big)\big[2 - D\big(h(t)-x\big)\big] + \sigma$$

for $0 < t < T$ and $x \in [\max\{h(t) - D^{-1}, 0\}, h(t)]$. It is easily seen that

$$F\big(t,h(t)\big) = u\big(t,h(t)\big) = \sigma, \quad t \in (0,T),$$

and when $h(t) - D^{-1} > 0$,

$$F\big(t,h(t)-D^{-1}\big) = M_1 + \sigma > M_1 \geq u(t,x), \quad t \in (0,T), \quad x \in \big[h(t)-D^{-1}, h(t)\big],$$

when $h(t)-D^{-1}<0$ and $h(t)>0$,

$$F(t,0)>\sigma>0=u(t,0).$$

According to the definitions of F, D, N_1, we get

$$F_t-F_{xx}-f(F)\geq 2D^2M_1-2M_1\alpha D-N_1(M_1+\sigma)\geq 0$$

for $0<t<T$ and $\max\{h(t)-D^{-1},0\}<x<h(t)$, and the last inequality is obtained by the definition of D. Additionally,

$$F(0,x)\geq u_0(x),\quad x\in\left[h_0-D^{-1},h_0\right].$$

The classical comparison principle gives

$$u(t,x)\leq F(t,x),\quad 0<t<T,\quad \max\left\{h(t)-D^{-1},0\right\}<x<h(t).$$

Therefore,

$$h'(t)=-u_x\big(t,h(t)\big)-\alpha\leq -F_x\big(t,h(t)\big)-\alpha\leq M_2:=2M_1D-\alpha.$$

The proof is complete. □

Remark 3.2. Based on [8,23], by the standard L^p estimates, Sobolev embedding theorem and the Hölder estimates for parabolic equations, the problem (1.1) has a unique solution (u,h) for $t\in[0,T)$ with $T\in(0,+\infty]$. Moreover, $u(t,x)\in C^{1+\nu/2,1+\nu}([0,T)\times[0,h(t)])$ and $h(t)\in C^{1+\nu/2}([0,T))$ for $\nu\in(0,1)$. Additionally, if $\inf_{0<t\leq T}h(t)>0$, then the solution can be extended to a bigger interval $(0,T_*)$ with $T_*>T$. From [9, Theorem 2.3] and [23], if $\inf_{0<t\leq T}h(t)>0$, then for any $\delta\in(0,T)$, there is $M_3>0$ depending on δ, M_1 and M_2 such that $\|u(t,\cdot)\|_{C^2([0,h(t)])}\leq M_3$ for $t\in[\delta,T)$. The limit $\lim_{t\to+\infty}h(t)$ always exists by [4, Lemma 2.8]. When $T=+\infty$, denote $h_\infty:=\lim_{t\to+\infty}h(t)$. There are three cases: $0<h_\infty<+\infty$, $h_\infty=+\infty$ and $h(t)\to 0$ as $t\to T$.

4. Asymptotic Behavior of Solutions and Sufficient Conditions

In this section, we first analyze the asymptotic behavior of solutions of the problem (1.1) with different $\alpha>0$. After that we give sufficient conditions for shrinking, spreading and the transition cases.

4.1. Asymptotic behavior of solutions

Lemma 4.1. *Let $0<\alpha<\alpha^*(\sigma)$ and (u,h) be the solution of the problem* (1.1)*. If $0<h_\infty<+\infty$, then*

$$h_\infty=l_\alpha,\quad \lim_{t\to+\infty}u(t,x)=v_\alpha(x),\quad x\in[0,\min\{h(t),l_\alpha\}],$$

where (v_α,l_α) is the compactly supported solution given in Section 2.

Proof. The regularity of solutions — cf. Remark 3.2, yields that for any sequence $\{t_n\}$, there is a subsequence $\widetilde{t}_n \to +\infty$, $u(\widetilde{t}_n,\cdot)$ which converges to $v(\cdot)$ in $C^{1+\nu}_{loc}([0,h_\infty])$, where v satisfies

$$\begin{aligned} &v'' + f(v) = 0, \quad 0 < x < h_\infty, \\ &v(0) = 0, \quad v(h_\infty) = \sigma. \end{aligned} \tag{4.1}$$

Furthermore, straighten the interval $[0,h(t)]$ to the fixed finite interval $[0,h_0]$ [8, 9, 23], and applying L^p estimates and Sobolev embedding theorems to the reduced equation with Dirichlet boundary conditions, by passing to a subsequence, we obtain

$$\lim_{n\to+\infty} \left\| u(\widetilde{t}_n,\cdot) - v(\cdot) \right\|_{C^{1+\nu/2}([0,h(\widetilde{t}_n)])} \to 0.$$

Therefore,

$$h'(\widetilde{t}_n) = -u_x\big(\widetilde{t}_n, h(\widetilde{t}_n)\big) - \alpha \to -v'(h_\infty) - \alpha$$

as $n \to +\infty$. On the other hand, since $h(t)$ is Hölder continuous when $h(t) > 0$. Combining this and $h_\infty \in (0,+\infty)$, we have $h'(t) \to 0$. Hence, $-v'(h_\infty) = \alpha$. From Section 2, when $0 < \alpha < \alpha^*(\sigma)$, the solution of (4.1) with $-v'(h_\infty) = \alpha$ is unique. Therefore, the solution of (4.1) is nothing but v_α. Thus,

$$h_\infty = l_\alpha, \quad V_{h_\infty}(x) \equiv v_\alpha, \quad x \in (0, h_\infty).$$

The proof is complete. □

Lemma 4.2. *Let* (u,h) *be the solution of the problem* (1.1). *Assume* $h_\infty = +\infty$, *then*

$$\lim_{t\to+\infty} u(t,x) = U(x)$$

locally uniformly for $x \in [0,+\infty)$.

Proof. We now consider the intersection points of $u(t,x)$ and $v_\alpha(x - nh_0)$ for any $n \in \mathbb{N}$. By the properties that $v_\alpha(x - nh_0) = 0$ for $x = nh_0$, $u(t,h(t)) = \sigma > 0$ and $h_\infty = +\infty$, there is a first time (denoted by t_n) such that $u(t_n,x)$ and $v_\alpha(x - nh_0)$ have only one intersection point. From $h_\infty = +\infty$, there also exists $T_n > t_n$ such that $h(T_n) = nh_0 + l_\alpha$ with $h(t) > nh_0 + l_\alpha$ for $t > T_n$. For all $n \geq 1$, define

$$W_n(t,x) := u(t,x) - v_\alpha(x - nh_0), \quad x \in I_n(t), \quad t \in [t_n, T_n],$$

where $I_n(t) := [nh_0, h(t)]$. Let $\mathscr{Z}_{I_n(t)}[W_n(t,\cdot)]$ be the zero number of $W_n(t,\cdot)$ on $I_n(t)$ for $t \in [t_n, T_n]$.

When $n = 1$ and $t = t_1$, $W_1(t_1,x)$ has only one zero point at $x = h(t_1)$. According to the zero number argument (cf. [3, Lemma 3.11]), $\mathscr{Z}_{I_1(t)}[W_1(t,\cdot)] = 1$ for $t \in [t_1, T_1]$. Especially, as $t \to T_1$, the unique zero point of W_1, namely the unique intersection point of u and v_α, moves to $x = h(T_1) = h_0 + l_\alpha$. Therefore,

$$u(T_1,x) \geq (\not\equiv) v_\alpha(x - h_0), \quad x \in [h_0, h_0 + l_\alpha].$$

Similarly, for all $n > 1$, we obtain

$$u(T_n, x) \geq (\not\equiv) v_\alpha(x - nh_0), \quad x \in [nh_0, nh_0 + l_\alpha].$$

From the comparison principle, for all $t > 0$, $n \geq 1$,

$$u(t + T_n, x) > v_\alpha(x - nh_0), \quad x \in [nh_0, nh_0 + l_\alpha],$$

and $h(t + T_n) > nh_0 + l_\alpha$ for $t > 0$. From our assumption $h_\infty = +\infty$, for any large $\ell > 0$, there exists T_ℓ such that $h(T_\ell) = \ell$ and $h(t) > \ell$ for $t > T_\ell$. Consider the problem

$$\begin{aligned} &w_t = w_{xx} + f(w), \quad 0 < x < \ell, \quad t > 0, \\ &w(t, 0) = 0, \quad w(t, \ell) = 0, \\ &w(0, x) \leq u(T_\ell, x), \quad 0 \leq x \leq \ell. \end{aligned}$$

Denote its solution by $w_\ell(t, x)$. The comparison principle derives that

$$u(t + T_\ell, x) \geq w_\ell(t, x), \quad x \in [0, \ell] \subset [0, h(t + T_\ell)].$$

Then

$$\liminf_{t \to +\infty} u(t + T_\ell, x) \geq v_\ell(x), \quad x \in [0, \ell],$$

where $v_\ell(x)$ is the solution of

$$\begin{aligned} &v'' + f(v) = 0, \quad x \in (0, \ell), \\ &v(0) = 0, \quad v(\ell) = 0. \end{aligned}$$

Letting $\ell \to +\infty$, by the convergence result, we have $v_\ell(x) \to U(x)$, where $U(x)$ is the solution of

$$\begin{aligned} &v'' + f(v) = 0, \quad x > 0, \\ &v(0) = 0, \quad v(+\infty) = 1. \end{aligned}$$

Thus,

$$\liminf_{t \to +\infty} u(t, x) \geq U(x), \quad x \in [0, +\infty). \tag{4.2}$$

On the other hand, choose a function $w_0(x)$ defined for $x \in [0, +\infty)$ and

$$\begin{aligned} &w_0(x) > u_0(x), \quad x \in [0, h_0], \\ &w_0(x) > \sigma, \qquad x > \delta_0, \end{aligned}$$

where $\delta_0 < h_0$. Now, consider the problem

$$\begin{aligned} &w_t = w_{xx} + f(w), \quad x > 0, \\ &w(t, 0) = 0, \qquad t > 0, \\ &w(0, x) = w_0(x), \quad x > 0. \end{aligned}$$

Its solution is denoted by $w(t,x)$. Since $\sigma \in (0,1)$ and $f(\sigma) > 0$, the classical comparison principle gives

$$w(t,x) > \sigma, \quad x > \delta(t),$$

where $\delta(t)$ is the rightmost zero point of $w(t,x) - \sigma$ and $\delta(0) = \delta_0$. Note that $u(t,h(t)) = \sigma < w(t,h(t))$. Then the classical comparison principle — Lemma 3.2, shows that

$$\delta(t) < h(t), \quad w(t,x) > u(t,x) \quad \text{for all} \quad t > 0, \quad x \in [0,h(t)].$$

It is well known that

$$w(t,\cdot) \to U(\cdot) \quad \text{in } [0,+\infty) \quad \text{as} \quad t \to +\infty.$$

Hence,

$$\limsup_{t\to+\infty} u(t,x) \le U(x) \quad \text{for} \quad x \ge 0.$$

Combining this and (4.2), we can derive the conclusion. □

Lemma 4.3. *Let* (u,h) *be the solution of the problem* (1.1) *defined on the maximal existence interval* $[0,\widehat{T})$ *with* $\widehat{T} \in (0,+\infty]$. *If* $\lim_{t\to\widehat{T}} h(t) = 0$, *then* $\widehat{T} < +\infty$ *and*

$$\lim_{t\to\widehat{T}} \max_{x\in[0,h(t)]} u(t,x) \le \sigma.$$

Proof. Lemma 3.3 shows that $F(t,x)$ is an upper solution near $x = h(t)$. Note that $F(t,x) \to \sigma$ as $x \to h(t)$ and the assumption $\lim_{t\to\widehat{T}} h(t) = 0$, we have

$$\lim_{t\to\widehat{T}} \max_{x\in[0,h(t))} u(t,x) \le \sigma.$$

So, for any small $\epsilon > 0$, there is $T_1 < \widehat{T}$ such that, when $t > T_1$, there hold $f(\sigma+\epsilon)h(t) < \sigma\alpha/2$ and $u(t,x) < \sigma + \epsilon$ for $x \in [0,h(t)]$. Define

$$U(t) := \int_0^{h(t)} u(t,x)dx$$

and

$$\Sigma_1 := \big\{t > T_1 : u(t,x) > \sigma,\ x \in [0,h(t))\big\},$$
$$\Sigma_2 := \big\{t > T_1 : u(t,x) < \sigma,\ x \in [0,h(t))\big\}.$$

By $\sigma \in (0,1)$ and the property of f, we see that u cannot reach the minimum in $(0,h(t))$. So in Σ_1, we have $u_x(t,h(t)) < 0$, this means that

$$h'(t) > -\alpha.$$

Therefore, for $t \in \Sigma_1$, we have

$$\begin{aligned} U'(t) &= h'(t)u\big(t,h(t)\big) + \int_0^{h(t)} u_t dx \\ &= \sigma h'(t) + \int_0^{h(t)} [u_{xx} + f(u)]dx \end{aligned}$$

$$
\begin{aligned}
&= \sigma h'(t) + u_x\big(t,h(t)\big) - u_x(t,0) + \int_0^{h(t)} f(u)dx \\
&= (\sigma-1)h'(t) - \alpha - u_x(t,0) + \int_0^{h(t)} f(u)dx \\
&\le (1-\sigma)\alpha - \alpha - u_x(t,0) + \frac{\sigma\alpha}{2} \le -\frac{\sigma\alpha}{2},
\end{aligned}
$$

i.e. $U'(t) \le U(0) - \sigma\alpha/2t$.

For $t \in \Sigma_2$, since $u(t,x) < \sigma + \epsilon$ for all $x \in [0,h(t)]$, we have $u_x(t,h(t)) > 0$, so $h'(t) = -u_x(t,h(t)) - \alpha < -\alpha$ and $h(t) < h_0 - \alpha t$. Hence,

$$
U(t) \le \int_0^{h_0-\alpha t} u(t,x)dx \le (\sigma+\varepsilon)(h_0-\alpha t).
$$

Therefore,

$$
U(t) \le \begin{cases} U(0) - \dfrac{\sigma\alpha}{2}t & \text{on } \Sigma_1, \\ (\sigma+\varepsilon)(h_0-\alpha t) & \text{on } \Sigma_2. \end{cases}
$$

So, $U(t) \to 0$ within a finite time. This and $u(t,x) = \sigma$ mean that $h(t)$ shrinks to 0 for a finite time $\widehat{T} < +\infty$. □

Remark 4.1. If $\widehat{T} < +\infty$, we also have $h(t) \to 0$ as $t \to \widehat{T}$. Otherwise the solution can be extended to an interval larger than $[0,\widehat{T}]$.

Lemma 4.4. *Let (u,h) be the solution of the problem* (1.1). *If $\alpha \ge \alpha^*(\sigma)$, then only shrinking happens.*

Proof. By Lemma 4.2, we have, for large t, $u(t,x) < U(x)$ for $x \in [0,h(t)]$. From Section 2, there is $L^* > 0$ such that $U^*(-L^*) = \sigma$ and $-(U^*)'(-L^*) = \alpha^*(\sigma)$. So, for any $T > 0$, there exists a constant $H > 0$ such that

$$
u(T,x) \le U^*(x-H-L^*)
$$

for $x \in [0,h(T)] \subset [0,H]$. By the property $-(U^*)'(-L^*) = \alpha^*(\sigma)$ and the comparison principle we have

$$
u(t,x) \le U^*(x-H-L^*), \quad x \in [0,h(t)] \subset [0,H], \quad t > T.
$$

Therefore, $0 < h_\infty \le H + L^*$ or $\lim_{t\to\widehat{T}} h(t) = 0$ for some $\widehat{T} > 0$.

If the former holds, then from the proof of Lemma 4.1, for some sequence $t_n \to +\infty$, $u(t_n,x) \to w(x)$ as $n \to +\infty$, where $w(x)$ satisfies

$$
\begin{aligned}
&w'' + f(w) = 0, \quad 0 < x < h_\infty, \\
&w(0) = 0, \quad w(h_\infty) = \sigma.
\end{aligned} \tag{4.3}
$$

Moreover, $h_\infty < +\infty$ implies that $h'(t) \to 0$ as $t \to +\infty$. So $-w'(h_\infty) = \alpha$. However, there is no such solution satisfying (4.3) with $-w'(h_\infty) = \alpha$ (note that $\alpha \ge \alpha^*(\sigma)$). Hence, $\lim_{t\to\widehat{T}} h(t) = 0$ for some $\widehat{T} > 0$, this means that shrinking happens. □

4.2. Sufficient conditions

The sufficient conditions for spreading, shrinking, and transition are provided by Lemmas 4.5, 4.6, and Remark 4.2.

Lemma 4.5. *Assume* $0 < \alpha < \alpha^*(\sigma)$. *Let* (u,h) *be the solution of the problem* (1.1) *with initial value* $u_0(x)$. *If* $u_0(x) > v_\alpha(x)$ *for* $x \in [0,l_\alpha] \subset [0,h_0]$, *then spreading happens.*

Proof. From the comparison principle, we have

$$h(t) > l_\alpha, \quad u(t,x) > v_\alpha(x) \quad \text{for} \quad x \in [0,l_\alpha] \subset [0,h(t)], \quad t > 0,$$

so that

$$\liminf_{t\to+\infty} u(t,x) \geq v_\alpha(x), \quad x \in [0,l_\alpha].$$

Since $v_c^*(x-ct)$ continuously depends on c, cf. Section 2, for sufficiently small $c > 0$, we have $u(1,x) \geq v_c^*(x)$ for $x \in [0,l_c^*] \subset [0,h(1)]$. This and the definition of $v_c^*(x)$ imply that $v_c^*(x-ct)$ is a lower solution. By the comparison principle, we have

$$h(t+1) > ct + l_c^*, \quad u(t+1,x) > v_c^*(x-ct)$$

for $x \in [ct, ct+l_c^*] \subset [0,h(t+1)]$, $t > 0$. Hence, $h_\infty = +\infty$. From Lemma 4.2 we have spreading. □

Lemma 4.6. *Assume* $0 < \alpha < \alpha^*(\sigma)$. *Choose* $\gamma < \alpha$. *Let* (v_γ, l_γ) *be the solution of the problem* (2.3) *with* α *replaced by* γ. *Suppose that* (u,h) *is the solution of the problem* (1.1) *with initial value* $u_0(x) < v_\gamma(x+l_0)$ *for* $x \in [0,h_0] \subset [0,l_\gamma - l_0]$, *where* $v_\gamma(l_0) = \sigma$. *Then shrinking happens.*

Proof. Note that $v_\gamma(l_0) = \sigma > 0$, so $u(t,0) = 0 < v_\gamma(x+l_0)|_{x=0}$. By this, $u_0(x) < v_\gamma(x+l_0)$, $\gamma < \alpha$ and the comparison principle, we have $h(t) < l_\gamma - l_0$ and $u(t,x) < v_\gamma(x+l_0)$ for all $t > 0$. Therefore,

$$0 < h_\infty \leq l_\gamma - l_0 \quad \text{or} \quad \lim_{t\to\widehat{T}} h(t) \to 0 \quad \text{for some} \quad 0 < \widehat{T} < +\infty. \tag{4.4}$$

If the former holds, Lemma 4.1 implies that transition happens, so $h_\infty = l_\alpha$. This is impossible from (4.4) and $l_\gamma < l_\alpha$, cf. the property of compactly supported solutions in Section 2. Combining this, (4.4) and Lemma 4.3, only shrinking happens. □

Remark 4.2. Let (u,h) be the solution of the problem (1.1) with initial data $u_0(x)$. When $0 < \alpha < \alpha^*(\sigma)$, we assume $h_0 = l_\alpha$ and $u_0(x) \equiv v_\alpha(x)$ for $x \in [0,h_0]$, then $u(t,x) \equiv v_\alpha(x)$, this indicates that transition happens.

5. Proofs of Main Theorems

Proof of Theorem 1.1. For any fixed $\phi \in \mathscr{X}(h_0)$ and $h_0 > 0$, we consider the initial data $u_0(x) = \tau\phi$ with constant $\tau > 0$. To emphasize the influence of such initial value on the asymptotic behavior of solutions, we denote the solution of the problem (1.1) by $(u(t,x;\tau\phi), h(t,x;\tau\phi))$, and define

$$\tau_* = \tau_*(h_0,\phi) := \sup\big\{\tau \geq 0 : \text{ shrinking happens for } (u,h)\big\}.$$

Lemma 4.6 indicates that $\tau_* > 0$.

(i) Shrinking happens when $\tau \in (0,\tau_*)$. It can be proved by the comparison principle and the definition of τ_*.

(ii) Transition happens when $\tau = \tau_*$. We prove this conclusion by showing that the following cases are impossible.

- Shrinking is impossible. Otherwise, there exist $T_1 > 0$ and $\gamma < \alpha$ such that $h(T_1) < l_\gamma - l_0$ and

$$u(T_1,x;\tau_*\phi) < v_\gamma(x+l_0), \quad x \in [0,h(T_1)],$$

where $(v_\gamma(x+l_0), l_\gamma)$ is the compactly supported solution given in Lemma 4.6. Because the solution depends continuously on the initial data, for small $\rho > 0$, let (u_ρ, h_ρ) be the solution of the problem (1.1) with initial data $\widetilde{u}_0(x) := (\tau_* + \rho)\phi$. Then we obtain that

$$u_\rho\big(T_1,x;(\tau_*+\rho)\phi\big) < v_\gamma(x+l_0)$$

for $x \in [0, h_\rho(T_1;(\tau_*+\rho)\phi)] \subset [0,l_\gamma]$. Combining this with Lemma 4.6 yields that shrinking happens for (u_ρ,h_ρ). But this contradicts the definition of τ_*.

- Spreading is impossible. Otherwise, there exists $t_1 > 0$ such that $u(t_1,x;\tau_*\phi) > v_\alpha(x)$ for $x \in [0,l_\alpha] \subset [0,h(t_1;\tau_*\phi)]$. Because the solution $u(t,x)$ depends continuously on the initial data, for small $\varepsilon > 0$, we also have $u(t_1,x;(\tau_* - \varepsilon)\phi) > v_\alpha(x)$ for $x \in [0,l_\alpha]$, and $h(t_1;(\tau_* - \epsilon)\phi) > l_\alpha$. By Lemma 4.5, spreading also happens for $u(t+t_1,x;(\tau_* - \varepsilon)\phi)$, which contradicts the definition of τ_*.

(iii) Spreading happens when $\tau \in (\tau^*,+\infty)$. By the comparison principle and the definition of τ^*, we derive that only spreading happens when $\tau \in (\tau^*,+\infty)$. □

The proof of Theorem 1.2 now follows directly from Lemma 4.4.

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